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On Minc's sixth Conjecture

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Let Λ_n^k denote the set of $n \times n$ binary matrices which have each row and column sum equal to k. Minc's Conjecture 6 asserts that $\min_{A\in\Lambda_n^k} \operatorname{per}((1/k)A)$ is monotone decreasing in k. Here, three special cases of this conjecture and also of the corresponding statement for the maximum permanent in Λ_n^k are proved. The three cases are for matrices which are sufficiently (i) small, (ii) sparse or (iii) dense.

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1. Introduction

This note addresses Conjecture 6 in Minc's well-known catalogue [6] of unsolved problems on permanents, which is as follows.

CONJECTURE 1.1 *For a fixed n,*

$$\min\left\{\operatorname{per}\left(\frac{1}{k}A\right) : A \in \Lambda_n^k\right\}$$

is monotone decreasing in k.

The original motivation for Conjecture 1.1 was probably its tangential relationship with the famous van der Waerden conjecture. However, one might equally well ask

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whether the following is true:

Conjecture 1.2 *For a fixed n,*

$$\max\left\{\mathrm{per}\left(\frac{1}{k}A\right) : A \in \Lambda_n^k\right\}$$

is monotone decreasing in k.

In this article three special cases of both questions are established. Namely, Conjecture 1.1 is proved for

- $k < o(n^{1/4})$,
- $k > n - o(n^{6/7})$,
- $n \leq 11$,

and Conjecture 1.2 for

- $k < O(n^{1/3})$,
- $k > n - o(n^{6/7})$,
- $n \leq 11$.

2. Sparse matrices

In this section, Conjectures 1.1 and 1.2 are proved for sufficiently sparse matrices.

Let Δ_n^k denote the set of $n \times n$ matrices of non-negative integers which have each row and column sum equal to k. Let $q = q(n, k)$ denote the probability that a matrix chosen uniformly at random from Δ_n^k actually lies in Λ_n^k. If $k = o(n^{1/2})$, then by a result from [10],

$$\left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n \leq \min_{A \in \Lambda_n^k} \mathrm{per}\, A \leq \frac{k^{2n}}{q\binom{kn}{n}} \tag{1}$$

where q satisfies

$$\frac{1}{q} = \exp\left(\frac{(k-1)^2}{2} + \frac{(k-1)^2 k}{6n} + o(1)\right). \tag{2}$$

One can use this result to prove:

Theorem 2.1 *Conjecture* 1.1 *is true for* $k = o(n^{1/4})$.

Proof Define

$$R_k = \frac{\min\{\mathrm{per}((1/(k+1))A) : A \in \Lambda_n^{k+1}\}}{\min\{\mathrm{per}((1/k)A) : A \in \Lambda_n^k\}}.$$

Suppose $k = o(n^{1/4})$. Then by (1) and (2),

$$\begin{aligned} R_k &\le \left(\frac{k}{k+1}\right)^n \frac{(k+1)^{2n}}{q\binom{nk+n}{n}} \frac{k^{(k-2)n}}{(k-1)^{(k-1)n}} \\ &= \left(\frac{k^{k-1}(k+1)}{(k-1)^{k-1}}\right)^n \frac{n!(nk)!}{(kn+n)!} \exp\bigl(O(k^2)\bigr) \\ &= \left(\frac{k^{2k-1}}{(k-1)^{k-1}(k+1)^k}\right)^n \exp\bigl(o(\sqrt{n})\bigr) \\ &= \exp\Bigl(-\frac{n}{2k^2} + o\Bigl(\frac{n}{k^2}\Bigr)\Bigr). \end{aligned}$$

Hence R_k tends to zero, and the theorem is proved. ■

It is worth remarking that the largest contribution to the error term in the above calculation comes from (2). A significantly stronger result could be deduced if it were first shown that the minimum permanent in Δ_n^k is achieved by a matrix in Λ_n^k (this is Conjecture 24 in [7]).

The proof of the corresponding result for Conjecture 1.2 will make use of the following simple lemma.

LEMMA 2.2 *For integers $d > c \ge 1$,*

$$\frac{d!^c}{c!^d} \le \left(\frac{d}{c}\right)^{cd}.$$

This lemma is a one line corollary of an inequality in [4, p. 60] and can also easily be proved with the aid of Stirling's formula. The details have been omitted here, opting instead to give intuition for why the result should be true. If p pebbles are tossed independently and uniformly at random into b buckets, then the probability that each bucket receives the same number of pebbles presumably decreases as b increases (subject to the condition that b divides p). Applying this principle when $p = cd$ and either $b = d$ or $b = c$ gives that

$$\frac{(p!/c!^d)}{d^p} \le \frac{(p!/d!^c)}{c^p},$$

which is equivalent to the inequality in the lemma.

THEOREM 2.3 *Conjecture 1.2 is true for $n \ge k^3 - k$.*

Proof: Let $n = tk + r$ for $0 \le r < k$ and $t \ge k^2 - 1$. By [9] it is known that $\max\{\operatorname{per} A : A \in \Lambda_n^k\} \ge k!^t r!$. Also, by Brègman's theorem [1], it is known that $\max\{\operatorname{per} A : A \in \Lambda_n^{k+1}\} \le (k+1)!^{n/(k+1)}$. Thus, if

$$Q_k = \frac{\max\bigl\{\operatorname{per}((1/(k+1)A)) : A \in \Lambda_n^{k+1}\bigr\}}{\max\bigl\{\operatorname{per}((1/k)A) : A \in \Lambda_n^k\bigr\}}$$

then,

$$Q_k \le \left(\frac{k}{k+1}\right)^n \frac{(k+1)!^{n/(k+1)}}{k!^t r!}$$
$$= \left(\frac{k}{k+1}\right)^r \frac{(k+1)!^{r/(k+1)}}{r!} \left[\left(\frac{k}{k+1}\right)^k \frac{(k+1)!^{k/(k+1)}}{k!}\right]^t. \qquad (3)$$

Now,

$$\begin{aligned}\left(\frac{k}{k+1}\right)^{k(k+1)} \frac{(k+1)!^k}{k!^{k+1}} &= \frac{k^k}{k!}\left(\frac{k}{k+1}\right)^{k^2} \\ &< \frac{e^k}{\sqrt{2\pi k}} \exp\left(k^2 \log\left(1 - \frac{1}{k+1}\right)\right) \\ &< \frac{1}{\sqrt{2\pi k}} \exp\left(k + k^2\left(-\frac{1}{k+1} - \frac{1}{(k+1)^2}\right)\right) \\ &= \frac{1}{\sqrt{2\pi k}} \exp\left(\frac{k^2+2k}{2k^2+4k+2}\right) \\ &< \sqrt{\frac{e}{2\pi k}}.\end{aligned}$$

As this is less than 1 and $t \ge k^2 - 1 = (k+1)(k-1)$ it can be concluded that

$$\left[\left(\frac{k}{k+1}\right)^k \frac{(k+1)!^{k/(k+1)}}{k!}\right]^t < \left(\frac{e}{2\pi k}\right)^{(k-1)/2}.$$

By inspecting (3), it is now obvious that $Q_k < 1$ when $r = 0$, so it may be assumed that $k > r \ge 1$. In that case, one can use Lemma 2.2 and elementary calculus to deduce that

$$\left(\frac{k}{k+1}\right)^r \frac{(k+1)!^{r/(k+1)}}{r!} \le \left(\frac{k}{k+1}\right)^r \left(\frac{k+1}{r}\right)^r = \left(\frac{k}{r}\right)^r < e^{k/e}.$$

Therefore, by (3),

$$Q_k \le e^{k/e}\left(\frac{e}{2\pi k}\right)^{(k-1)/2} < 1$$

for all $k \ge 2$. The result follows immediately. ■

3. Dense matrices

This will be achieved using a result proved by Godsil and McKay [3] in the context of extensions to Latin rectangles. They showed that when $k = n - r$ for

$0 \le r = o(n^{6/7})$ as $n \to \infty$ then,

$$\mathrm{per}\left(\frac{1}{n-r}A\right) = \frac{n!}{n^n}\exp\left(\frac{r}{2n}+\frac{r^2}{2n^2}+\frac{r^3}{2n^3}+\frac{r^4}{4n^4}-\frac{r^5}{2n^5}-\frac{13r^6}{6n^6}+f(A)+o\left(\frac{1}{n}\right)\right) \tag{4}$$

uniformly over $A \in \Lambda_n^k$. Here, $f(A)$ is a function which is discussed below but which obeys the uniform bound $f(A) = O(k^3/n^3)$. If $r = o(n^{2/3})$, then $f(A) = o(1/n)$ so (4) shows that Conjectures 1.1 and 1.2 are true for sufficiently dense matrices. Indeed something stronger is true!

THEOREM 3.1 *If* $k = n - o(n^{2/3})$ *then*

$$\min_{A\in\Lambda_n^k} \mathrm{per}\left(\frac{1}{k}A\right) > \max_{B\in\Lambda_n^{k+1}} \mathrm{per}\left(\frac{1}{k+1}B\right).$$

In the next result $\mathcal{G}(M)$ is used to denote the usual bipartite graph associated with a binary matrix M. The two vertex classes of $\mathcal{G}(M)$ correspond respectively to the rows and columns of M, and each 1 in M corresponds to an edge in $\mathcal{G}(M)$.

Notation $A \le B$ is used for matrices A and B of the same dimensions, to denote the fact that no entry in A exceeds the corresponding entry in B. Since the case of present interest is when A and B are binary matrices, this is equivalent to saying that B can be formed by changing some (or none) of the zeroes of A to ones. In other words $\mathcal{G}(A)$ is a subgraph of $\mathcal{G}(B)$.

The function $f(A)$ in (4) can be written as

$$f(A) = \varepsilon_4(A)\left(\frac{1}{4n^4}-\frac{r}{n^5}+\frac{6r^2}{4n^6}-\frac{r^3}{n^7}+\frac{\varepsilon_4(A)}{32n^8}\right)+\varepsilon_5(A)\left(\frac{1}{5n^5}-\frac{r}{n^6}+\frac{2r^2}{n^7}\right)$$
$$+\varepsilon_6(A)\left(\frac{1}{6n^6}-\frac{r}{n^7}\right)+\frac{\varepsilon_7(A)}{7n^7}$$

where $\varepsilon_i(A)$ denotes the number of a certain type of walk of length $2i$ in $\mathcal{G}(J-A)$, where J denotes the all 1 matrix of the same order as A (see [3] for the full definition of ε_i). Now suppose that $A \le B$ so that $\mathcal{G}(J-B)$ is a subgraph of $\mathcal{G}(J-A)$, it follows from the definition that $\varepsilon_i(A) \ge \varepsilon_i(B)$. So by inspection, $f(A) \ge f(B)$ for $r = o(n)$ and the next result follows immediately.

THEOREM 3.2 *If* $k = n - o(n^{6/7})$ *and* $A \le B$ *where* $A \in \Lambda_n^k$ *and* $B \in \Lambda_n^{k+1}$ *then*

$$\mathrm{per}\left(\frac{1}{k}A\right) > \mathrm{per}\left(\frac{1}{k+1}B\right). \tag{5}$$

If $k<n$ then every $A \in \Lambda_n^k$ has some $B \in \Lambda_n^{k+1}$ satisfying $A \le B$. Likewise, every $B \in \Lambda_n^{k+1}$ has some $A \in \Lambda_n^k$ satisfying $A \le B$. In particular these statements

hold if A minimises the permanent in Λ_n^k or B maximises the permanent in Λ_n^{k+1}. Thus,

COROLLARY 3.3 *Conjectures* 1.1 *and* 1.2 *are both true when* $k = n - o(n^{6/7})$.

The above approach will not work for sparse matrices. By way of example, let P denote the permutation matrix corresponding to the full cycle permutation $(1\ 2\ 3 \ldots n)$ for some $n \geq 9$. Define circulant matrices $A = P^0 + P^1$ and $B = P^0 + P^1 + P^2$. Then, by [6, p. 46],

$$\mathrm{per}\left(\frac{1}{2}A\right) = 2^{1-n} \quad \text{and} \quad \mathrm{per}\left(\frac{1}{3}B\right) = \left(\frac{1+\sqrt{5}}{6}\right)^n + \left(\frac{1-\sqrt{5}}{6}\right)^n + \frac{2}{3^n}.$$

Hence (5) fails to hold, even though $A \leq B$.

4. Small matrices

The enumeration of Latin squares of orders up to 11 by the author and McKay [5] involved finding the permanent of one representative of every equivalence class (modulo transposition and permutation of the rows and columns) in Λ_n^k for $1 \leq k \leq n \leq 11$. These permanents were found in the process of solving a much harder problem equivalent to finding the number of ways each representative can be written as a sum of permutation matrices. As a result of these exhaustive computations it can be reported that both Conjectures 1.1 and 1.2 hold for $n \leq 11$. The precise values of the extremal permanents are given in tables 1 and 2.

In tables 1 and 2 a prime ($'$) is used to mark values which are not achieved by any circulant matrix in the appropriate class. For $n \leq 11$, this answers Problems 11 and 12 from [7] which ask when the extremal values of the permanent in Λ_n^k are achieved by circulant matrices. It is known [8] that the maximum permanent is only rarely achieved by a circulant, at least when considering sufficiently sparse or sufficiently dense matrices. No such information is known about the minimum permanent.

Finally, it is noted that two of the values reported in table 1 confirm particular cases of Conjecture 5 in [6]. This conjecture, due to Ryser, states that if Λ_v^k contains

Table 1. Minimum values of per(A) for $A \in \Lambda_n^k$ for $n \leq 11$.

k	$n=2$	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2
3	–	6	9	12′	17	24	33	42′	60′	83′
4	–	–	24	44	80	144	248′	440′	764′	1316′
5	–	–	–	120	265	578′	1249	2681	5713	12105
6	–	–	–	–	720	1854	4738	12000′	30240′	75510
7	–	–	–	–	–	5040	14833	43386′	126117′	364503
8	–	–	–	–	–	–	40320	133496	439792	1441788′
9	–	–	–	–	–	–	–	362880	1334961	4890740′
10	–	–	–	–	–	–	–	–	3628800	14684570
11	–	–	–	–	–	–	–	–	–	39916800

Table 2. Maximum values of per(A) for $A \in \Lambda_n^k$ for $n \le 11$.

k	$n=2$	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1
2	2	2	4	$4'$	8	$8'$	16	$16'$	32	$32'$
3	–	6	9	13	36	$54'$	81	216	$324'$	$486'$
4	–	–	24	44	82	$148'$	576	$1056'$	$1968'$	$3608'$
5	–	–	–	120	265	$580'$	1313	$2916'$	14400	$31800'$
6	–	–	–	–	720	1854	4752	$12108'$	32826	$86400'$
7	–	–	–	–	–	5040	14833	$43424'$	$127044'$	$373208'$
8	–	–	–	–	–	–	40320	133496	440192	$1448640'$
9	–	–	–	–	–	–	–	362880	1334961	$4893072'$
10	–	–	–	–	–	–	–	–	3628800	14684570
11	–	–	–	–	–	–	–	–	–	39916800

incidence matrices of (v, k, λ)-configurations, then the permanent takes its minimum in Λ_v^k at one of these incidence matrices. Both Λ_{11}^5 and Λ_{11}^6 contain such incidence matrices, which do indeed (uniquely) minimise the permanent over these sets. The next smallest test cases for Ryser's conjecture are Λ_{13}^4 and Λ_{13}^7.

For a progress report on all of Minc's open problems, see [2].

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