Annals of Combinatorics 9 (2005) 335-344
0218-0006/05/030335-10
DOI 10.1007/s00026-005-0261-7

Annals of Combinatorics

On the Number of Latin Squares

Brendan D. McKay and Ian M. Wanless*

Department of Computer Science, Australian National University, Canberra, ACT 0200, Australia
{bdm, imw}@cs.anu.edu.au

Received September 3, 2004

AMS Subject Classification: 05B15, 05C70, 15A15

Abstract. We (1) determine the number of Latin rectangles with 11 columns and each possible number of rows, including the Latin squares of order 11, (2) answer some questions of Alter by showing that the number of reduced Latin squares of order n is divisible by $f!$ where f is a particular integer close to $\frac{1}{2}n$, (3) provide a formula for the number of Latin squares in terms of permanents of $(+1,-1)$-matrices, (4) find the extremal values for the number of 1-factorisations of k-regular bipartite graphs on $2n$ vertices whenever $1 \le k \le n \le 11$, (5) show that the proportion of Latin squares with a non-trivial symmetry group tends quickly to zero as the order increases.

Keywords: Latin square, enumeration, 1-factorisation, permanent, regular bipartite graph

1. Introduction

For $1 \le k \le n$, a $k \times n$ *Latin rectangle* is a $k \times n$ array $L = (\ell_{ij})$ with entries from $\{1, 2, \ldots, n\}$ such that the entries in each row and in each column are distinct. Of course, L is a *Latin square* if $k = n$. We say that L is *reduced* if the first row is $(1, 2, \ldots, n)$ and the first column is $(1, 2, \ldots, k)^T$. If $R_{k,n}$ is the number of reduced $k \times n$ Latin rectangles then $L_{k,n}$, the total number of $k \times n$ Latin rectangles, is $n!\,(n-1)!\,R_{k,n}/(n-k)!$. We will sometimes write $R_{n,n}$ as R_n and $L_{n,n}$ as L_n.

The determination of $R_{k,n}$, especially in the case $k = n$, has been a popular pursuit for a long time. The number of reduced squares up to order 5 was known to Euler [5] and Cayley [4]. McMahon [8] used a different method to find the same numbers, but obtained the wrong value for order 5. The number of reduced squares of order 6 was found by Frolov [6] and later by Tarry [18]. Frolov [6] also gave an incorrect count of reduced squares of order 7. Norton [14] enumerated the Latin squares of order 7 but incompletely, this was completed by Sade [15] and Saxena [16]. The number of reduced squares of order 8 was found by Wells [20], of order 9 by Bammel and Rothstein [2].

The value of R_{10} was found first in 1990 by the amateur mathematician Eric Rogoyski working on his home computer and in the following year by the present first

* School of Engineering and Logistics, Charles Darwin University, Darwin NT 0909, Australia.

author. The resulting joint paper [12] also presented the number of Latin rectangles with up to 10 columns. Before he died in 2002, Rogoyski worked for several years on the squares of order 11 but the computing power available to him was inadequate, despite his approach being sound. Given the advance in computers since then, we can now complete the computations moderately easily.

Several explicit formulas for general n are in the literature ([17], for example). Saxena [16] succeeded in using such a formula to compute R_7. We will give another very simple formula in Section 5. At the time of writing, not even the asymptotic value of R_n is known. In the case of rectangles, the best asymptotic result is for $k = o(n^{6/7})$, by Godsil and McKay [7].

2. Terminology

It can be useful to think of a Latin square of order n as a set of n^2 triples of the form (row, column, symbol). For each Latin square there are six *conjugate* squares obtained by uniformly permuting the coordinates in each of its triples. For example, the transpose of L is obtained by swapping the row and column coordinates in each triple.

An *isotopism* of a Latin square L is a permutation of its rows, permutation of its columns and permutation of its symbols. The resulting square is said to be *isotopic* to L and the set of all squares isotopic to L is called an *isotopy class*. In the special case when the same permutation is applied to the rows, columns and symbols we say that the isotopism is an *isomorphism*. An isotopism that maps L to itself is called an *autotopism* of L and any autotopism that is an isomorphism is called an *automorphism*. The *main class* of L is the set of squares which are isotopic to some conjugate of L. Latin squares belonging to the same main class are said to be *paratopic* and a map which combines an isotopism with conjugation is called a *paratopism*. A paratopism which maps a Latin square to itself is called an *autoparatopism* of the square.

The number of isomorphism classes, isotopy classes and main classes has been determined by McKay, Meynert and Myrvold [11] for $n \leq 10$. Our computation does not allow us to extract this information for $n = 11$. However, we do show in Section 7 that $L_n/(6n!^3)$ provides an increasingly accurate estimate of the number of main classes as n grows.

3. The Algorithm

Our approach is essentially that introduced by Sade [15], adapted to the computer by Wells [20,21], and slightly improved by Bammel and Rothstein [2]. It was also used by McKay and Rogoyski [12]. Given a $k \times n$ Latin rectangle L, we can define a bipartite graph $B(L)$ with vertices $C \cup S$, where $C = \{c_1, c_2, \ldots, c_n\}$ represents the columns of L and $S = \{s_1, s_2, \ldots, s_n\}$ represents the symbols. There is an edge from c_i to s_j if and only if the symbol j appears in column i of L. Thus $B(L)$ is regular of degree k. Clearly $B(L)$ does not determine L in general, since it does not record the order of the symbols in each column. For us this is an advantage, since it means there are many fewer graphs than there are Latin rectangles.

Given a regular bipartite graph B on $C \cup S$ of degree k, let $m(B)$ be its number of 1-factorizations, counted without regard to the order of the factors. Obviously $m(B)$

is an invariant of the isomorphism class of B. In speaking of isomorphisms and automorphisms of such bipartite graphs, we will admit the possibilities that C and S are preserved setwise or that they are exchanged. (More complex mixings of C and S would, in principle, be possible in the case of disconnected graphs, but we have chosen to disallow them.) Using this convention, let $\mathrm{Aut}(B)$ be the automorphism group of B and let $\mathcal{B}(k,n)$ be a set consisting of one representative of the isomorphism classes of bipartite graphs B on $C \cup S$ of degree k.

The theoretical basis of our approach is summarized in the following theorem. Parts 1 and 3 were proved in [12] and part 2 can be proved along similar lines.

Theorem 3.1. 1. *The number of reduced $k \times n$ Latin rectangles is given by*

$$R_{k,n} = 2nk!(n-k)! \sum_{B \in \mathcal{B}(k,n)} m(B)|\mathrm{Aut}(B)|^{-1}.$$

2. *The number of reduced Latin squares of order n is given by*

$$R_n = 2nk!(n-k)! \sum_{B \in \mathcal{B}(k,n)} m(B)m(\bar{B})|\mathrm{Aut}(B)|^{-1},$$

where $\bar{B}$ is the bipartite complement (the complement in $K_{n,n}$) of B and k is any integer in the range $0 \le k \le n$.

3. *Let $B \in \mathcal{B}(k,n)$ for $k \ge 1$. Let e be an arbitrary edge of B. Then*

$$m(B) = \sum_F m(B-F),$$

where the sum is over all 1-factors F of B that include e.

For each $k = 1, 2, \ldots, 11$ in turn we found $m(B)$ for all $\mathcal{B}(k,11)$ using part 3 of Theorem 3.1 and were then able to deduce $R_{k,11}$ from part 1 of Theorem 3.1. The number of graphs in $\mathcal{B}(k,11)$ is 1, 14, 4196, 2806508 and 78322916, for $k = 1, \ldots, 5$, respectively. For $k \ge 6$ the graphs in $\mathcal{B}(k,11)$ are the bipartite complements of those in $\mathcal{B}(11-k,11)$. The main practical difficulty was the efficient management of the fairly large amount of data. Two implementations were written in a way that made them independent in all substantial aspects (except for their reliance on nauty [10] to recognise the isomorphism class of some graphs). For example, they used different edges e in applying part 3 of Theorem 3.1, so that generally different subgraphs were encountered. The execution time of each implementation was about 2 years (corrected to 1 GHz Pentium III), but they would have completed in under 2 months if about 3 GB memory had been available on the machines used.

We also ran the computations for $n \le 10$ and obtained the same results as reported in [12]. We repeat those results, and include the new results, in Table 1. It is unlikely that R_{12} will be computable by the same method for some time, since the number of regular bipartite graphs of order 24 and degree 6 is more than 10^{11}.

Note that our value of R_{11} agrees precisely with the numerical estimate given in [12], where estimates of R_n were given for $11 \le n \le 15$.

n	k	$R_{k,n}$
1	1	1
2	1	1
	2	1
3	1	1
	2	1
	3	1
4	1	1
	2	3
	3	4
	4	4
5	1	1
	2	11
	3	46
	4	56
	5	56
6	1	1
	2	53
	3	1064
	4	6552
	5	9408
	6	9408
7	1	1
	2	309
	3	35792
	4	1293216
	5	11270400
	6	16942080
	7	16942080
8	1	1
	2	2119
	3	1673792
	4	4209 09504
	5	27206 658048
	6	33 53901 89568
	7	53 52814 01856
	8	53 52814 01856
9	1	1
	2	16687
	3	1034 43808
	4	20 76245 60256
	5	11268 16430 83776
	6	12 95260 54043 81184
	7	224 38296 79166 91456
	8	377 59757 09642 58816
	9	377 59757 09642 58816
10	1	1
	2	1 48329
	3	81549 99232
	4	14717 45210 59584
	5	746 98838 30762 86464
	6	8 70735 40559 10037 09440
	7	1771 44296 98305 41859 22560
	8	42920 39421 59185 42730 03520
	9	75807 21483 16013 28114 89280
	10	75807 21483 16013 28114 89280
11	1	1
	2	14 68457
	3	79 80304 83328
	4	143 96888 00784 66048
	5	75 33492 32304 79020 93312
	6	9 62995 52373 29250 51587 78880
	7	24012 32164 75173 51550 21735 52640
	8	86 10820 43577 87266 78085 83437 51680
	9	2905 99031 00338 82693 11398 90275 94240
	10	5363 93777 32773 71298 11967 35407 71840
	11	5363 93777 32773 71298 11967 35407 71840

Table 1: Reduced Latin rectangles.

4. Some Divisibility Properties of R_n

Despite obtaining the same value repeatedly for R_{11} by applying Theorem 3.1(2) for different k in two independent computations, we sought to check our answer further by determining its value modulo some small prime powers. By means of the algorithms described in [11], we computed representatives L of all the isotopy classes of Latin squares of order 11 for which the order of the autotopism group $\mathrm{Is}(L)$ is divisible by 5, 7, or 11. The numbers of such isotopy classes are listed in Table 2. Since the number of reduced squares in the isotopy class of L is $n\,n!/|\mathrm{Is}(L)|$, these counts imply that R_{11} equals 8515 modulo 21175, in agreement with our computations.

$\lvert\mathrm{Is}(L)\rvert$	isotopy classes
5	55621
7	8065
10	359
11	24
14	160
20	102
21	45
22	12
55	6
60	3
1210	1

Table 2: Isotopy classes with certain group sizes.

We also have the following simple divisibility properties.

Theorem 4.1. *For each integer $n \geq 1$,*

1. *R_{2n+1} is divisible by* $\gcd(n!\,(n-1)!\,R_n, (n+1)!)$.
2. *R_{2n} is divisible by $n!$.*

Proof. Consider R_{2n+1} first. We define an equivalence relation on reduced Latin squares of order $2n+1$ such that each equivalence class has size either $n!\,(n-1)!\,R_n$ or $(n+1)!$. Let A be the leading principal minor of $L = (\ell_{ij})$ of order n.

If A is a (reduced) Latin subsquare, then the squares equivalent to L are those obtainable by possibly replacing A by another reduced subsquare, permuting the n partial rows $(\ell_{i,n+1}, \ell_{i,n+2}, \ldots, \ell_{i,2n+1})$ for $1 \leq i \leq n$, permuting the $n-1$ partial columns $(\ell_{n+1,j}, \ell_{n+2,j}, \ldots, \ell_{2n+1,j})$ for $2 \leq j \leq n$ then permuting columns $n+1, n+2, \ldots, 2n+1$ to put the first row into natural order. These $n!\,(n-1)!\,R_n$ operations are closed under composition and give different reduced Latin squares, so each equivalence class has size $n!\,(n-1)!\,R_n$.

If A is not a Latin subsquare, the squares equivalent to L are those obtainable by applying one of the $(n+1)!$ isomorphisms in which the underlying permutation fixes each of the points $1, 2, \ldots, n$. No isomorphism of this form can be an automorphism of a square in which A is not a subsquare (see [11, Theorem 1]). Hence the squares obtained are different and the equivalence class has $(n+1)!$ elements.

The case of R_{2n} is the same except the second argument gives $n!$ instead of $(n+1)!$. ∎

Corollary 4.2. *If $n = 2p - 1$ for some prime p, then R_n is divisible by $\lfloor (n-1)/2 \rfloor !$. Otherwise, R_n is divisible by $\lfloor (n+1)/2 \rfloor !$.*

Proof. This follows from Table 1 for $n \leq 8$. For $n \geq 9$, note that $m \mid (m-2)!$ for $m > 4$ unless m is prime. ∎

Note that, for $n \geq 12$, the corollary gives the best divisor that can be inferred from Table 1 and Theorem 4.1, except that R_{13} is divisible by 7! and not merely by 6!.

Alter [1] (see also Mullen [9]) asked whether an increasing power of two divides R_n as n increases and whether R_n is divisible by 3 for all $n \geq 6$. Theorem 4.1 answers both these questions in the affirmative. Indeed it shows much more — that for any integer $m > 1$ the power of m dividing R_n grows at least linearly in n. That is, for each m there exists $\lambda = \lambda(m) > 0$ such that R_n is divisible by $m^{\lfloor \lambda n \rfloor}$ for all n.

Alter also asked for the highest power of two dividing R_n, and here we must admit our ignorance. It seems from the evidence in Table 3 that the power grows faster than linearly, but we were unable to prove this.

n	Prime factorisation of R_n
2	1
3	1
4	2^2
5	$2^3 \cdot 7$
6	$2^6 \cdot 3 \cdot 7^2$
7	$2^{10} \cdot 3 \cdot 5 \cdot 1103$
8	$2^{17} \cdot 3 \cdot 1361291$
9	$2^{21} \cdot 3^2 \cdot 5231 \cdot 3824477$
10	$2^{28} \cdot 3^2 \cdot 5 \cdot 31 \cdot 37 \cdot 547135293937$
11	$2^{35} \cdot 3^4 \cdot 5 \cdot 2801 \cdot 2206499 \cdot 62368028479$

Table 3: Prime factorisations of R_n for $n \leq 11$.

5. A Formula for R_n

The literature contains quite a few exact formulas for R_n, but none of them appear very efficient for explicit computation (though Saxena [16] managed to compute R_7 using such a formula).

Perhaps the simplest formulas are those in [17], which relate R_n to the permanents of all 0-1 matrices of order n. Here we give one that is very similar but uses ± 1 matrices instead. Unlike the inclusion-exclusion proof of [17], we give a simple analytic proof.

Theorem 5.1. *Let $p(z)$ be any monic polynomial of degree n and let $\mathcal{M}_n$ be the family of all $n \times n$ matrices over $\{-1, +1\}$. Then*

$$L_n = 2^{-n^2} \sum_{X \in \mathcal{M}_n} p(\operatorname{Per} X)\, \pi(X),$$

where Per X *is the permanent of* X *and* $\pi(X)$ *is the product of the entries of* X.

Proof. If $X = (x_{ij})$ is an $n \times n$ matrix of indeterminates, then by definition Per $X = \sum_{\sigma \in S_n} T_\sigma$ where S_n is the symmetric group and $T_\sigma = x_{1\sigma(1)} x_{2\sigma(2)} \cdots x_{n\sigma(n)}$. If the polynomial $p(\text{Per } X)$ is expanded in terms of monomials in the x_{ij}, then the only monomial involving every x_{ij} comes from products $T_{\sigma_1} T_{\sigma_2} \cdots T_{\sigma_n}$ where the permutations $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the rows of a Latin square. That is, the coefficient of the only monomial with each x_{ij} having odd degree is the number of Latin squares. Multiplying by $\pi(X)$ turns the required monomial into the only one that has even degree in each x_{ij}. Now summing over $X \in \mathcal{M}_n$ causes this monomial to be multiplied by $|\mathcal{M}_n| = 2^{n^2}$ while all the other monomials cancel out. ∎

6. Extremal Graphs with Respect to $m(B)$

In our computations we learned the values of $m(B)$ for each graph $B \in \mathcal{B}(k,n)$ for $n \leq 11$. In Table 4 we record the maximum and minimum values, and the number of graphs (in the column headed "#") that achieve the minimum. The maximum is achieved uniquely in all cases. Of course, for $k \leq 1$ the result is trivial and when $k \geq n-1$ the unique graph has $m(B) = R_n$, so we omit these cases.

In most cases, the graphs maximizing $m(B)$ are the same as those with the maximum number of perfect matchings, as listed in [13]. The only exceptions are as follows, where the notation is that used in [13]:

- For $n = 7, k = 5$ the graph maximising $m(B)$ is $\overline{2J_2 \oplus D_3}$;
- For $n = 9, k = 6$ the graph maximising $m(B)$ is $\overline{3J_3}$;
- For $n = 10, k = 4$ the graph maximising $m(B)$ is $J_4 \oplus \overline{3J_2}$;
- For $n = 11, k = 4$ the graph maximising $m(B)$ is $J_4 \oplus \overline{J_3 \oplus D_4}$.

In the first of these cases the cited graph does, according to [13], maximise the number of perfect matchings, but does not do so uniquely.

7. Proportion of Latin Squares with Symmetry

In this section we prove that the proportion of order n Latin squares which have a non-trivial symmetry tends very quickly to zero as $n \to \infty$.

Theorem 7.1. *The proportion of Latin squares of order n which have a non-trivial autoparatopy group is no more than*

$$n^{-3n^2/8+o(n^2)}. \tag{7.1}$$

Proof. Suppose that a Latin square $L = (\ell_{ij})$ of order n has a non-trivial autoparatopy group. Then by Lemma 4 in [11], L has a autoparatopism α which fixes (pointwise) no more than one quarter of the triples of L.

The number of possibilities for α is less than $6n!^3 = o(n^{3n})$. Given α, we can construct each possible L row by row. Each entry is determined either by α and a previous entry, or can be chosen in at most n ways. The latter possibility occurs once

n	k	$\min m(B)$	#	$\max m(B)$
4	2	1	1	2
5	2	1	1	2
	3	4	1	6
6	2	1	1	4
	3	8	4	24
	4	168	1	224
7	2	1	1	4
	3	8	3	48
	4	456	2	576
	5	54528	1	55296
8	2	1	1	8
	3	16	18	96
	4	1120	1	13824
	5	3 06432	1	4 02432
	6	2518 94784	1	2583 92064
9	2	1	1	8
	3	16	7	288
	4	2720	1	32256
	5	17 18784	1	23 12192
	6	35859 25120	1	37975 08096
	7	2260 68542 91456	1	2271 05054 39232
10	2	1	1	16
	3	24	2	576
	4	6992	1	1 29024
	5	94 57472	1	2167 60320
	6	4 97127 34208	1	7 10221 82400
	7	92007 32190 63808	1	96252 56413 10208
	8	51072 82902 02843 87328	1	51411 31576 53646 54080
11	2	1	1	16
	3	32	25	1152
	4	17040	1	3 31776
	5	494 49728	1	15173 22240
	6	65 69929 07264	1	127 45506 81600
	7	36 18408 76780 25728	1	41 31218 87443 35360
	8	66 74288 35273 45400 70912	1	69 04895 67877 90499 02080
	9	365 09897 56490 71060 26179 78880	1	366 51069 03315 59851 95097 12896

Table 4: Minimum and maximum values of $m(B)$.

per orbit of α, and since α fixes at most $\frac{1}{4}$ of the triples of L, the number of orbits is at most $(\frac{1}{4}+\frac{3}{4}\cdot\frac{1}{2})n^2 = \frac{5}{8}n^2$. In total we find that there are most

$$o(n^{3n})n^{5n^2/8}$$

Latin squares with non-trivial autoparatopy group. Our result now follows immediately from the well known lower bound for L_n (see, for example, [19, Thmeorem 17.2]) that says that

$$L_n \geq (n!)^{2n} n^{-n^2} \geq n^{n^2-o(n^2)}. \qquad \blacksquare$$

As a corollary to this last result, we see that the proportion of main classes or of isotopy classes or of isomorphism classes whose members have a non-trivial autoparatopy group is also bounded by (7.1). This is because each such class has somewhere between 1 and $6n!^3 = o(n^{3n})$ members.

Another corollary is that the number of isomorphism classes, isotopy classes and main classes of Latin squares of order n will be asymptotic to $L_n/n!$, $L_n/n!^3$ and $L_n/(6n!^3)$, respectively.

References

1. R. Alter, How many Latin squares are there? Amer. Math. Monthly **82** (1975) 632–634.
2. S.E. Bammel and J. Rothstein, The number of 9×9 Latin squares, Discrete Math. **11** (1975) 93–95.
3. J.W. Brown, Enumeration of Latin squares with application to order 8, J. Combin. Theory **5** (1968) 177–184.
4. A. Cayley, On Latin squares, Oxf. Camb. Dublin Messenger Math. **19** (1890) 135–137.
5. L. Euler, Recherches sur une nouvelle espèce de quarrés magiques, Verh. Zeeuwsch Gennot. Weten Vliss **9** (1782) 85–239.
6. M. Frolov, Sur les permutations carrés, J. de Math. spéc. **IV** (1890) 8–11, 25–30.
7. C.D. Godsil and B.D. McKay, Asymptotic enumeration of Latin rectangles, J. Combin. Theory Ser. B **48** (1990) 19–44.
8. P.A. MacMahon, Combinatory Analysis, Cambridge, 1915.
9. G.L. Mullen, How many *i*-*j* reduced Latin squares are there? Amer. Math. Monthly **85** (1978) 751–752.
10. B.D. McKay, *Nauty* graph isomorphic software, available at: `http://cs.anu.edu.au/~bdm/nauty`.
11. B.D. McKay, A. Meynert, and W. Myrvold, Small Latin squares, quasigroups and loops, submitted.
12. B.D. McKay and E. Rogoyski, Latin squares of order 10, Electron. J. Combin. **2** (1995) #N3 4 pp.
13. B.D. McKay and I.M. Wanless, Maximising the permanent of (0, 1)-matrices and the number of extensions of Latin rectangles, Electron. J. Combin. **5** (1998) #R11 20 pp.
14. H.W. Norton, The 7×7 squares, Ann. Eugenics **9** (1939) 269–307.
15. A. Sade, Enumération des carrés latins. Application au 7ème ordre. Conjectures pour les ordres supérieurs, privately published, Marseille, 1948, 8 pp.
16. P.N. Saxena, A simplified method of enumerating Latin squares by MacMahon's differential operators; II. The 7×7 Latin squares, J. Indian Soc. Agricultural Statist. **3** (1951) 24–79.

17. J.Y. Shao and W.D. Wei, A formula for the number of Latin squares, Discrete Math. **110** (1992) 293–296.
18. G. Tarry, Le problème des 36 officiers, Ass. Franç. Paris **29** (1900) 170–203.
19. J.H. van Lint and R.M. Wilson, A Course in Combinatorics, Cambridge University Press, 1992.
20. M.B. Wells, The number of Latin squares of order eight, J. Combin. Theory **3** (1967) 98–99.
21. M.B. Wells, Elements of Combinatorial Computing, Pergamon Press, Oxford-New York-Toronto, 1971.