# Dynamic Asset Price Jumps and the Performance of High Frequency Tests and Measures*

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**Abstract**

This paper provides an extensive evaluation of high frequency jump tests and measures, in the context of dynamic models for asset price jumps. Specifically, we investigate: i) the power of alternative tests to detect individual price jumps, including in the presence of volatility jumps; ii) the frequency with which sequences of dynamic jumps are identified; iii) the accuracy with which the magnitude and sign of sequential jumps are estimated; and iv) the robustness of inference about dynamic jumps to test and measure design. Substantial differences are discerned in the performance of alternative methods in certain dimensions, with inference being sensitive to these differences in some cases. Accounting for measurement error when using measures constructed from high frequency data to conduct inference on dynamic jump models would appear to be advisable.

*Keywords*: *Dynamic price jumps; Price jump tests; Nonparametric jump measures; Hawkes process; Discretized jump diffusion model; Bayesian Markov chain Monte Carlo.*

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# 1 Introduction

Extreme movements (or 'jumps') in asset prices play an important role in the tail behaviour of return distributions, with the perceived risk (and, hence, risk premium) associated with this extreme behaviour differing from that associated with small and regular movements (see, Bates, 1996, and Duffie *et al.*, 2000, for early illustrations of this point, and Todorov and Tauchen, 2011, Maneesoonthorn *et al.*, 2012, and Bandi and Reno, 2016, for more recent expositions). Indeed, the modelling of jumps (in both the price itself and its volatility) has been given particular attention in the option pricing literature, where the additional risk factor implied by random jumps has helped explain certain stylized patterns in option-implied volatility (Merton, 1976; Bates, 2000; Duffie *et al.*, Eraker, 2004; Todorov, 2010; Maneesoonthorn *et al.*; Bandi and Reno). Evidence of price jump *clustering* in spot returns - whereby price and/or volatility jumps occur in successive intervals - has also been uncovered, with various approaches having been adopted to model this dynamic behaviour, including the common occurrence of price and volatility jumps over time (Chan and Maheu, 2002; Eraker *et al.*, 2003; Maheu and McCurdy, 2004; Fulop *et al.*, 2014; Aït-Sahalia *et al.*, 2015; Bandi and Reno; Maneesoonthorn *et al.*, 2017).

Coincident with the trend towards more sophisticated models for asset prices, the use of high-frequency intraday data to construct nonparametric measures of asset price variation - including the jump component thereof - has become wide-spread. Multiple alternative methods are now available to practitioners, both for testing for jumps and for measuring price variation in the presence of jumps, with some empirical analyses exploiting such measures in addition to, or as a replacement of, measurements based on end-of-day prices. In some cases, nonparametric measures are used to *directly* represent the relevant latent feature (Andersen *et al.*, 2003; Koopman *et al.*, 2005; Andersen *et al.*, 2007; Bollerslev *et al.* 2009; Corsi, 2009; Martin *et al.*, 2009; Busch *et al.*, 2011; Hansen *et al.*, 2012), whilst in other instances, a state space framework - with its attendant measurement errors - is used to absorb the inaccuracy of the measures induced by the use of a finite number of intraday returns in their construction (Barndorff-Nielsen and Shephard, 2002; Creal, 2008; Takahashi *et al.*, 2009; Dobrev and Szerszen, 2010; Maneesoonthorn *et al.*, 2012; Koopman and Scharth, 2013; Maneesoonthorn *et al.*, 2017).

Whilst this wealth of new measures reaps benefits by allowing more complex processes to be identified and estimated, it also presents challenges. Specifically, the variety of ways in which high frequency observations can be exploited, in particular in constructing jump test statistics and measuring jump variation, has the potential to yield conflicting inferential conclusions. Moreover, with respect to any particular method, failure to detect a true jump (or sequence of jumps), spurious detection of a non-existent jump (or jump sequence), and error in the measurement of jump magnitude and/or sign, may *distort* inference on the process assumed to be driving jumps, including its dynamics.

This paper provides an extensive investigation - in the context of dynamic jump models - of the relative accuracy of the many high frequency jump tests and measures that are on offer, plus an assessment of the robustness of inference to the use of different methods. We begin by providing a comprehensive review of the alternative price jump tests that have been proposed, and the nonparametric price jump measures that each test invokes. We group these methods into four categories: i) those based on the difference between a measure of total (squared) variation and a jump-robust measure of integrated variation (Barndorff-Nielsen and Shephard, 2004, 2006; Huang and Tauchen, 2005; Corsi *et al.*, 2010; Andersen *et al.*, 2012); ii) those that exploit measures of higher-order variation (Aït-Sahalia and Jacod, 2009; Podolskij and Ziggel, 2010); iii) those based on returns, rather than measures of variation (Andersen *et al.*, 2007; Lee and Mykland, 2008); and iv) those that exploit variance swaps (Jiang and Oomen, 2008). Using a simulation design that mimics a realistic empirical setting, the size and power performance of ten distinct tests that span these four categories is documented, including in the presence of volatility jumps. (See also Dumitru and Urga, 2012, for an earlier assessment of the ability of certain high frequency tests to detect individual jumps.) We then document the frequency with which the occurrence of a *sequence* of price jumps is correctly identified by each test, and under a variety of different dynamic jump models that are nested within two specifications for the jump intensity: one based on a Hawkes process (Aït-Sahalia *et al.*, 2015; Fulop *et al.*, 2014; Maneesoonthorn *et al.*, 2017) and another functionally dependent on the volatility (Bates, 1996; Pan, 2002; Eraker, 2004; Maneesoonthorn *et al.*). Stochastic volatility - with the potential for dynamic jumps therein - is also accommodated within the experimental design. Two aspects of price jump *measurement* are next investigated: first, the accuracy with which the average magnitude of the jumps within a sequence is estimated, and second, the extent to which the correct jump sign is pinpointed. Finally, we investigate the use of alternative measures in the Bayesian estimation of a discretized jump diffusion model with self-exciting dynamic jumps.

The four broad categories of price jump tests are presented in Section 2, followed by an outline of how each test can be used to extract various price jump measures in Section 3. In Section 4.1, a series of simulation exercises is used to assess the size and power performance of the various tests, in the context of detecting individual price jumps. The ability of the different methods to identify and accurately measure a sequence of jumps is then documented in Section 4.2. Based on our findings in Section 4, in Section 5 four particular approaches are used to conduct inference about the dynamic characteristics of jumps within the discretized jump diffusion model, again using artificially generated data in which the true characteristics of the process are known. Whilst a reasonable degree of robustness to method is documented, the use of a measure that performs relatively poorly in identifying sequences of jumps is found to yield less precise inference on the parameters controlling the jump dynamics. A further result highlighted therein is that accommodation of measurement error in the measures of price jump occurrence, sign and magnitude errors aids in the

acquisition of accurate inference. Section 6 provides some concluding remarks.

## 2 Review of price jump tests

Defining $p_t = \ln(P_t)$ as the natural log of the asset price, $P_t$ at time $t > 0$, we assume the following jump diffusion process for $p_t$,

$$dp_t = \mu_t dt + \sqrt{V_t} dW_t^p + dJ_t^p, \tag{1}$$

where $W_t^p$ is the Brownian motion, and $dJ_t^p = Z_t^p dN_t^p$, with $Z_t^p$ denoting the random price jump size and $dN_t^p$ the increment of a discrete count process, with $P(dN_t^p = 1) = \delta^p dt$ and $P(dN_t^p = 0) = (1 - \delta^p)\, dt$.

The aim of a price jump test is to detect the *presence* of the discontinuous component, $dJ_t^p$, and to conclude whether or not $dN_t^p$ is non-zero over a particular period. The availability of high-frequency data has enabled various measures of variation - incorporating both the continuous and discontinuous components of (1) - to be constructed over a specified interval of time, e.g. one day, with the statistical properties of such measures established using in-fill asymptotics. The resulting distributional results are then utilized in the construction of a price jump test, where the null hypothesis is usually that the asset price is continuous over the particular interval investigated.

This section provides a review of a range of tests based on the concepts of, respectively, squared variation (Section 2.1), higher-order power variation (Section 2.2), standardized daily returns (Section 2.3) and variance swaps (Section 2.4). Our discussion is not limited to the technical construction of the tests, but also extends to certain issues related to implementation, including the role of tuning parameters. The construction of nonparametric measures of price jump magnitude and sign, as based on each of these preliminary tests, is then outlined in Section 3.

### 2.1 Squared variation

The early literature on price jump testing exploits various measures of the *squared variation* of the asset price process. In the context of a continuous-time price process, as defined in (1), the object of interest is the difference between total quadratic variation over a discrete time period (typically a trading day), $Q\mathcal{V}_{t-1,t} = \int_{t-1}^{t} V_s ds + \sum_{t-1<s\leq t}^{N_t^p} (Z_s^p)^2$, and variation from the continuous component alone, quantified by the integrated variance, $\mathcal{V}_{t-1,t} = \int_{t-1}^{t} V_s ds$. By definition, the difference between these two quantities defines the contribution to price variation of the discontinuous jumps, $\mathcal{J}_{t-1,t}^2 = \sum_{t-1<s\leq t}^{N_t^p} (Z_s^p)^2$, and price jump test statistics can thus be constructed from the difference between *measures* of $Q\mathcal{V}_{t-1,t}$ and $\mathcal{V}_{t-1,t}$. In the following subsections, we briefly outline the classical Barndorff-Nielsen and Shephard (2004, 2006) approach, also exploited by Huang and Tauchen (2005), followed by two alternative tests proposed by Andersen *et al.* (2012) and Corsi *et al.* (2010) respectively, all of which exploit the difference in the squared variation measures. These tests are all one-sided upper-tail tests by construction.

### 2.1.1 Barndorff-Nielsen and Shephard (2004, 2006) and Huang and Tauchen (2005)

The test of Barndorff-Nielsen and Shephard (2004, 2006), denoted hereafter as BNS, is based on the difference between realized variance, $RV_t = \sum_{i=1}^{M} r_{t_i}^2$, and the corresponding bipower variation, $BV_t = \frac{\pi}{2}\left(\frac{M}{M-1}\right)\sum_{i=2}^{M}|r_{t_i}|\left|r_{t_{i-1}}\right|$, where $r_{t_i} = p_{t_i} - p_{t_{i-1}}$ denotes the $i^{th}$ of $M$ equally-spaced returns observed during day $t$. As has become standard knowledge, under (1), $RV_t \xrightarrow{p} Q\mathcal{V}_{t-1,t}$ and $BV_t \xrightarrow{p} \mathcal{V}_{t-1,t}$, as $M \to \infty$ (Barndorff-Nielsen and Shephard, 2002; Andersen *et al.*, 2003; BNS). Thus, $RV_t - BV_t \xrightarrow{p} \mathcal{J}_{t-1,t}^2$ as $M \to \infty$, and the decision regarding the occurrence of a price jump on day $t$ can be based on testing whether or not $RV_t - BV_t$ is significantly larger than zero.

The joint (in-fill) asymptotic distribution of the two measures of squared volatility, $RV_t$ and $BV_t$, appropriately scaled, has been established by BNS to be bivariate normal. Specifically, under the assumption of no price jumps (plus various regularity conditions),

$$\sqrt{\frac{M}{\mathcal{I}Q_{t-1,t}}}\begin{pmatrix} RV_t - \mathcal{V}_{t-1,t} \\ BV_t - \mathcal{V}_{t-1,t} \end{pmatrix} \xrightarrow{D} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & \left(\frac{\pi}{2}\right)^2 + \pi - 3 \end{bmatrix}\right) \text{ as } M \to \infty. \tag{2}$$

The theoretical quantity $\mathcal{I}Q_{t-1,t} = \int_{t-1}^{t} V_s^2 ds$ is referred to as the integrated quarticity, an estimate of which can also be constructed from high-frequency returns data. For the purpose of the numerical investigations conducted in this paper, the following tripower quarticity measure is employed,

$$TP_t = \mu_{4/3}^{-3}\left(\frac{M^2}{M-2}\right)\sum_{i=3}^{M} \mid r_{t_{i-2}} \mid^{4/3} \mid r_{t_{i-1}} \mid^{4/3} \mid r_{t_i} \mid^{4/3},$$

where $\mu_{4/3} = 2^{2/3}\Gamma(7/6)\Gamma(1/2)^{-1}$ and $TP_t \xrightarrow{p} \mathcal{I}Q_{t-1,t}$ as $M \to \infty$.

The limiting distribution in (2) allows for the construction of a variety of hypothesis tests for price jumps (Huang and Tauchen, 2005). Arguably, the most frequently-used of such tests is based on the relative price jump measure, $RJ_t = (RV_t - BV_t)/RV_t$, with the corresponding price jump test statistic,

$$T_{BNS,t} = \frac{RJ_t}{\sqrt{\left(\left(\frac{\pi}{2}\right)^2 + \pi - 5\right) M^{-1} \max\left(1, \frac{TP_t}{BV_t^2}\right)}}, \tag{3}$$

being standard normal under the null hypothesis of no jumps as $M \to \infty$. Consistent with the subscript in (3), we refer to this test as the BNS test.

### 2.1.2 Corsi, Pirino and Reno (2010)

Corsi, Pirino and Reno (2010), hereafter CPR, propose a test that is also based on the discrepancy between realized variance and a jump-robust estimator of the integrated variance. However, instead of employing the $BV_t$ measure for the latter, and the $TP_t$ measure of $\mathcal{I}Q_{t-1,t}$, they use the threshold

measures,

$$CTBV_t = \frac{\pi}{2}\left(\frac{M}{M-1}\right)\sum_{i=2}^{M}\tau_1\left(r_{t_i},\vartheta_{t_i}\right)\tau_1\left(r_{t_{i-1}},\vartheta_{t_{i-1}}\right) \text{ and}$$

$$CTriPV_t = \mu_{4/3}^{-3}\left(\frac{M^2}{M-2}\right)\sum_{i=3}^{M}\tau_{4/3}\left(r_{t_i},\vartheta_{t_i}\right)\tau_{4/3}\left(r_{t_{i-1}},\vartheta_{t_{i-1}}\right)\tau_{4/3}\left(r_{t_{i-2}},\vartheta_{t_{i-2}}\right),$$

respectively, where $\mu_{4/3}^{-3}$ is as defined in the previous section. The term $\tau_\zeta\left(r_{t_i},\vartheta_{t_i}\right)$ denotes the so-called 'truncating function' for the absolute return raised to the power $\zeta$, and $\vartheta_{t_i}$ denotes the threshold level such that any returns that satisfy $r_{t_i}^2 > \vartheta_{t_i}$ are truncated at $\vartheta_{t_i} = c_\vartheta^2\widehat{V}_{t_i}$, where $\widehat{V}_{t_i}$ denotes a local variance estimator, reflecting the level of spot volatility at time $t_i$, and $c_\vartheta$ is a constant. For the value of $c_\vartheta = 3$ that is recommended, the relevant truncating functions proposed by CPR then take the form,

$$\tau_1\left(r_{t_i},\vartheta_{t_i}\right) = \begin{cases} |r_{t_i}| & r_{t_i}^2 < \vartheta_{t_i} \\ 1.094\sqrt{\vartheta_{t_i}} & r_{t_i}^2 > \vartheta_{t_i} \end{cases}$$

and

$$\tau_{4/3}\left(r_{t_i},\vartheta_{t_i}\right) = \begin{cases} |r_{t_i}|^{4/3} & r_{t_i}^2 < \vartheta_{t_i} \\ 1.129\vartheta_{t_i}^{2/3} & r_{t_i}^2 > \vartheta_{t_i} \end{cases}.$$

The resultant test statistic is defined as

$$T_{CRP,t} = \frac{1-\frac{CTBV_t}{RV_t}}{\sqrt{\left(\left(\frac{\pi}{2}\right)^2+\pi-5\right)M^{-1}\max\left(1,\frac{CTriPV_t}{CTBV_t^2}\right)}},$$

which, under the null hypothesis of no jumps, is also standard normal as $M \to \infty$.[1]

### 2.1.3 Andersen, Dobrev and Schaumburg (2012)

Andersen, Dobrev and Schaumburg (2012), hereafter referred to as ADS, propose alternative measures of $\mathcal{V}_{t-1,t}$ based on the squared variation of the minimum and the median of adjacent absolute returns,

$$MinRV_t = \frac{\pi}{\pi-2}\left(\frac{M}{M-1}\right)\sum_{i=2}^{M}\min\left(|r_{t_i}|,|r_{t_{i-1}}|\right)^2$$

and

$$MedRV_t = \frac{\pi}{\pi+6-4\sqrt{3}}\left(\frac{M}{M-2}\right)\sum_{i=3}^{M}med\left(|r_{t_i}|,|r_{t_{i-1}}|,|r_{t_{i-2}}|\right)^2,$$

respectively, with *large* returns thereby eliminated from the calculation. Taking the minimum and median of adjacent absolute returns also effectively imposes adaptive truncation, with the threshold for the truncation determined by neighbouring returns. Such an adaptive truncation

[1] Note that, in contrast to the formulation of the CPR procedure summarized in Dumitru and Urga (2012), the truncating functions proposed by CPR are seen to differ depending on the power, $\zeta$, to which the absolute return is raised.

scheme is arguably advantageous over the (fixed) threshold approach of CPR, as it avoids the need to make a subjective choice of local variance estimator.

Both $MinRV_t$ and $MedRV_t$ are consistent estimators of $\mathcal{V}_{t-1,t}$, and the resulting asymptotic properties of these estimators can be exploited to construct price jump tests in a similar fashion to those advocated under the BNS and CPR frameworks, with

$$T_{MinRV,t} = \frac{1 - \frac{MinRV_t}{RV_t}}{\sqrt{1.81 M^{-1} \max\left(1, \frac{MinRQ_t}{MinRV_t^2}\right)}} \tag{4}$$

and

$$T_{MedRV,t} = \frac{1 - \frac{MedRV_t}{RV_t}}{\sqrt{0.96 M^{-1} \max\left(1, \frac{MedRQ_t}{MedRV_t^2}\right)}} \tag{5}$$

each converging to standard normal variables as $M \to \infty$ under the null hypothesis of no jumps. The terms $MinRQ_t$ and $MedRQ_t$ denote the corresponding minimum and median measures of $\mathcal{I}Q_{t-1,t}$, constructed, respectively, as $MinRQ_t = \frac{\pi}{3\pi-8}\left(\frac{M^2}{M-1}\right)\sum_{i=2}^{M} \min\left(|r_{t_i}|, |r_{t_{i-1}}|\right)^4$ and $MedRQ_t = \frac{3\pi}{9\pi+72-52\sqrt{3}}\left(\frac{M^2}{M-2}\right)\sum_{i=3}^{M} med\left(|r_{t_i}|, |r_{t_{i-1}}|, |r_{t_{i-2}}|\right)^4$. The tests based on the statistics in (4) and (5) are hereafter referred to as the MINRV and MEDRV tests, respectively.

## 2.2 Higher-order $\mathcal{P}$-power variation

A second class of price jump test exploits the behaviour of higher-order $\mathcal{P}$-power variation, and estimators thereof. Following Barndorff-Nielson and Shephard (2004), let an estimator of the $\mathcal{P}$-power variation of $p_t$ be defined as $\widehat{B}(\mathcal{P}, \Delta_M)_t = \sum_{i=1}^{M} |r_{t_i}|^{\mathcal{P}}$, where $\Delta_M = 1/M$ denotes the common length of the time intervals between consecutive returns, and $\mathcal{P} > 0$. The limiting behaviour of this estimator, for different values of $\mathcal{P}$, sheds light on the different components of the variation in $p_t$. In the case of $\mathcal{P} = 2$, $\widehat{B}(\mathcal{P}, \Delta_M)_t \xrightarrow{p} Q\mathcal{V}_{t-1,t}$ as $M \to \infty$, as is consistent with the distributional result that $RV_t \xrightarrow{p} Q\mathcal{V}_{t-1,t}$ . For $0 < \mathcal{P} < 2$,

$$\frac{\Delta_M^{1-\mathcal{P}/2}}{m_{\mathcal{P}}} \widehat{B}(\mathcal{P}, \Delta_M)_t \xrightarrow{p} A(\mathcal{P})_t \text{ as } M \to \infty, \tag{6}$$

where

$$A(\mathcal{P})_t = \int_{t-1}^{t} \left|V_s^{1/2}\right|^{\mathcal{P}} ds, \tag{7}$$

denotes the $\mathcal{P}$-power *integrated* variation, $m_{\mathcal{P}} = E\left(|U|^{\mathcal{P}}\right) = \pi^{-1/2} 2^{\mathcal{P}/2} \Gamma\left(\frac{\mathcal{P}+1}{2}\right)$ and $U$ denotes a standard normal random variable. In contrast, for $\mathcal{P} > 2$, the increments from the jump component dominate, and the estimator converges in probability to the $\mathcal{P}$-power *jump* variation, $B(\mathcal{P})_t = \sum_{t-1<s\leq t} |dJ_s|^{\mathcal{P}}$ . If the jump component in (1) is not present and $p_t$ is continuous as a consequence, then the limiting result in (6) holds for any $\mathcal{P} > 0$.

These limiting results can be used in a variety of ways to detect jumps. For example, Aït-Sahalia and Jacod (2009) compare $\widehat{B}(\mathcal{P},\Delta_M)_t$ constructed over two different sampling intervals, while Podolskij and Ziggel (2010) rely on the limiting distribution of a modified version of $\widehat{B}(\mathcal{P},\Delta_M)_t$. These two approaches are outlined briefly in the subsequent sections.

### 2.2.1 Aït-Sahalia and Jacod (2009)

The test of Aït-Sahalia and Jacod (2009) (ASJ hereafter) exploits the fact that for any positive integer $k \geq 2$, and on the assumption that $p_t$ is continuous, the ratio

$$\widehat{S}(\mathcal{P},k,\Delta_M)_t = \frac{\widehat{B}(\mathcal{P},k\Delta_M)_t}{\widehat{B}(\mathcal{P},\Delta_M)_t}, \tag{8}$$

converges in probability to $k^{\mathcal{P}/2-1}$. Further they define the standardized test statistic

$$T_{ASJ,t} = \left(\widehat{\Sigma}^c_{M,t}\right)^{-1/2}\left(\widehat{S}(\mathcal{P},k,\Delta_M)_t - k^{\mathcal{P}/2-1}\right), \tag{9}$$

with $\widehat{\Sigma}^c_{M,t} = \Delta_M M(\mathcal{P},k)\,\widehat{A}(2\mathcal{P},\Delta_M)_t/\widehat{A}(\mathcal{P},\Delta_M)_t^2$. The term $\widehat{A}(\mathcal{P},\Delta_M)_t$ denotes the truncated $\mathcal{P}$-power integrated variation estimator,

$$\widehat{A}(\mathcal{P},\Delta_M)_t = \frac{\Delta_M^{1-\mathcal{P}/2}}{m_{\mathcal{P}}}\sum_{i=1}^{M}|r_{t_i}|^{\mathcal{P}}\,\mathbf{1}_{\left\{|r_{t_i}|<\vartheta\Delta_M^{\varpi}\right\}}, \tag{10}$$

and

$$M(\mathcal{P},k) = \frac{1}{m_{\mathcal{P}}^2}\left(k^{\mathcal{P}-2}(1+k)\,m_{2\mathcal{P}} + k^{\mathcal{P}-2}(k-1)\,m_{\mathcal{P}}^2 - 2k^{\mathcal{P}/2-1}m_{k,\mathcal{P}}\right),$$

where $m_{\mathcal{P}}$ is defined following (7), $m_{k,\mathcal{P}} = E\left(|U|^{\mathcal{P}}\left|U+\sqrt{k-1}V\right|^{\mathcal{P}}\right)$, and where $V$ is a standard normal variable independent of $U$. The indicator function $\mathbf{1}_{\left\{|r_{t_i}|<\vartheta(\Delta_M)^{\varpi}\right\}}$ equals one if the absolute return is smaller than the threshold value, $\vartheta\Delta_M^{\varpi}$, serving as a truncation trigger in the estimation of (7). ASJ show that when $\mathcal{P} > 2$, $\vartheta > 0$ and $\varpi \in (1/2 - 1/\mathcal{P}, 1/2)$ then $T_{ASJ,t}$ in (9) is asymptotically standard normal as $M \to \infty$. The critical region for the ASJ test[2] is defined in the lower tail of the limiting distribution, since the limit of the ratio $\widehat{S}(\mathcal{P},k,\Delta_M)_t$ in (8) is one when a jump is present - a quantity that is always lower than $k^{\mathcal{P}/2-1}$.

The ASJ test statistic requires five tuning components to be selected: the order of power variation, $\mathcal{P}$; the time interval length, $\Delta_M$; the integer defining the multiples of the time interval, $k$; the truncation scale factor, $\vartheta$; and the truncation root, $\varpi$. The authors suggest using $\mathcal{P} = 4$, which is a conservative choice aimed at striking a balance between the power of the test to detect jumps and the quality of approximation of the (null) sampling distribution. They also advise

[2] Aït-Sahalia and Jacod also derive a test using the limiting distribution of $\widehat{S}(\mathcal{P},k,\Delta_M)_t$ under the null hypothesis that $p_t$ contains jumps. However, one of the key assumptions needed to derive this asymptotic result is that jumps in $p_t$ may not occur simultaneously with any jumps in the diffusive variance process, $V_t$. As such an assumption is rather restrictive, and given the large number of tests that are already under consideration in this paper, we have chosen to focus solely on the test derived under the assumption of continuity.

that $k$ should be chosen such that it is 'not too big', with no significant differences in (their) simulation experiments detected for $k = 2, 3, 4$, and a value of $k = 2$ deemed to be reasonable. The choice of truncation level, $\vartheta \Delta_M^{\varpi}$, determines the point at which to discriminate between continuous movements and price jumps, and is key to the computation of (10). With the asymptotic theory restricting the range of the truncation root to $\varpi \in (1/2 - 1/\mathcal{P}, 1/2)$, ASJ suggest taking $\varpi$ close to the upper bound, specifically 0.48. They further advise that the truncation scale factor, $\vartheta$, should be between 3 to 5 times the average value of (an estimate of) the diffusive variance, $\left(\int_0^t V_s ds\right)^{1/2}$. All such values of the tuning components are employed in all implementations of the AJS test herein, along with a truncation scheme that is 3 times the diffusive volatility estimate, $\sqrt{BV_t}$ .

### 2.2.2 Podolskij and Ziggel (2010)

Poldolskij and Ziggel (2010) (PZ hereafter) propose a price jump test based on the standardized statistic

$$T_{PZ,t} = \frac{\hat{B}_T(\mathcal{P}, \Delta_M)}{\sqrt{Var(\eta_i)\, \hat{B}_T(2\mathcal{P}, \Delta_M)}},$$

where

$$\hat{B}_T(\mathcal{P}, \Delta_M) = M^{\frac{\mathcal{P}-1}{2}} \sum_{i=1}^{M} |r_{t_i}|^{\mathcal{P}} \left(1 - \eta_i \mathbf{1}_{\left\{|r_{t_i}| < \vartheta(\Delta_M)^{\varpi}\right\}}\right). \tag{11}$$

PZ demonstrate that for $\mathcal{P} \geq 2$, $T_{PZ,t}$ is asymptotically standard normal as $M \to \infty$, under the null hypothesis of no price jumps. Note that the test statistic involves the generation of an auxiliary independent and identically distributed random variable, $\eta_i$, from a distribution that is symmetric around 1, with $E(\eta_i) = 1$, $Var(\eta_i) < \infty$ and $E\left(|\eta_i|^{2+d}\right) > 0$ for some $d > 0$. The authors recommend simulating $\eta_i$ from the distribution $P^{\eta} = \frac{1}{2}(\varsigma_{1-\tau} + \varsigma_{1+\tau})$, where $\varsigma$ is the Dirac measure and $\tau$ is a small constant with a value of 0.1 or 0.05. They also suggest truncation values of $\vartheta = 2.3\sqrt{BV_t}$, and $\varpi = 0.4$ respectively, with the value of 2.3 defining the $99^{th}$ percentile of the standard normal distribution and, thus, serving as a basis for discriminating between continuous movements (Gaussian) and jumps (non-Gaussian). As with the ASJ test, the power term $\varpi$ is bounded between 0 and $\frac{1}{2}$, with larger values of $\varpi$ ensuring a faster convergence of the threshold estimator in (11) to $A(\mathcal{P})_t$. However, PZ argue that if the value of $\varpi$ is too large, there is a higher probability of large but still continuous increments being identified as jumps; hence PZ propose the use of a smaller value ($\varpi = 0.4$) than do ASJ ($\varpi = 0.48$). In this paper, we implement two versions of the PZ test under the two power settings, $\mathcal{P} = 2$ and $\mathcal{P} = 4$, denoted respectively by PZ2 and PZ4, and each with $\tau$ set at 0.05.

## 2.3 Standardized returns

An alternative framework to one in which price jump test statistics are constructed from various measures of variation, is one that considers the behaviour of (appropriately standardized) returns

themselves. In brief, based on the assumption of Brownian motion for the asset price, the return computed over an arbitrarily chosen interval length and scaled by the square root of a consistent estimator of the corresponding integrated variance, should be asymptotically standard normal if price jumps are absent. Two tests that exploit this property are discussed in this section.

### 2.3.1 Andersen, Bollerslev and Dobrev (2007)

Andersen, Bollerslev and Dobrev (2007), ABD hereafter, propose a simple method for extracting information about price jumps, making use of both intraday returns and estimates of integrated variance. The key to detecting price jumps in this framework is to recognize that intraday returns, standardized by a consistent estimator of the integrated volatility, given by

$$T_{ABD,t_i} = \frac{r_{t_i}}{\sqrt{M^{-1}BV_t}},$$

is approximately standard normal when jumps are absent and under the assumption that volatility is locally constant over the trading day. Price jumps can thus be detected using the characteristics of standard normal distributions, with a price jump deemed to be present if $|T_{ABD,t_i}| > \Phi^{-1}\left(1 - \frac{\alpha^*}{2}\right)$, where $\alpha^* = 1 - (1-\alpha)^{1/M}$ denotes the Bonferroni adjusted significance level, given the (daily) significance level $\alpha$, and $\Phi^{-1}(\cdot)$ denotes the inverse of the standard normal cumulative distribution function. The use of the Bonferroni adjustment addresses the issue of multiple testing, as this test is conducted $M$ times throughout the trading day. ABD suggest that a conservative choice should be used for $\alpha$, and document that the test performs satisfactorily in terms of producing the nominal size, along with sufficient power, at the significance level of $\alpha = 1e^{-5}$ (or 0.001%).

### 2.3.2 Lee and Mykland (2008)

The approach of Lee and Mykland (2008), LM hereafter, was derived independently from that of ABD, but is based on the same idea of diffusive returns being conditionally Gaussian. LM propose a test statistic based on the maximum of the standardized returns, and characterize the distribution of the test statistic using extreme value theory. They argue that the use of normal quantiles, even with a Bonferroni style of adjustment, is too permissive, resulting in over-rejection of the null.

The LM test statistic is defined as

$$T_{LM,t} = \frac{\left(\max\left(\widetilde{T}_{LM,t_i}\right) - C_M\right)}{S_M},$$

where $\widetilde{T}_{LM,t_i} = \frac{|r_{t_i}|}{\sqrt{\hat{V}_{t_i}}}$, $C_M = \frac{(2\log M)^{1/2}}{0.8} - \frac{\log\pi + \log(\log M)}{1.6(2\log\pi)^{1/2}}$, $S_M = \frac{1}{0.6(2\log\pi)^{1/2}}$, $\hat{V}_{t_i}$ denotes the local variance estimate and $\max\left(\widetilde{T}_{LM,t_i}\right)$ is defined over the $M$ intra-period returns. LM demonstrate that under the null of no jumps, $T_{LM,t}$ converges to the standardized Gumbel distribution, $G$, as $M \to \infty$, with the null hypothesis of no jump rejected if the test statistic falls above the upper tail critical value.

LM suggest that the point-in-time variance estimate be defined as $\hat{V}_{t_i} = \frac{BV_{t_i}}{K-2}$, where $BV_{t_i} = \frac{\pi}{2}\left(\frac{K}{K-1}\right)\sum_{j=i-K+2}^{i}\left|r_{t_j}\right|\left|r_{t_{j-1}}\right|$ is the bipower variation estimated over the window size $K$ up to and including time $t_i$. We use $K = 10$ in the numerical work below.

### 2.4 Variance swaps - Jiang and Oomen (2008)

Variance swaps are instruments made up of financial assets and/or derivatives and are used as tools to hedge against volatility risk. The payoff of a variance swap can be replicated by taking a short position in the so-called "log contract" and a long position in the underlying asset, with the long position being continuously re-balanced (see Neuberger, 1994). The payoff of such a replicating strategy, computed as the accumulated difference between proportional returns and continuously compounded logarithmic returns, equates to half of the integrated variance under the assumption of no price jump. When a jump is present, the replication error is completely determined by the realized jump.

Jiang and Oomen (2008), JO hereafter, exploit this concept by considering a function of the difference between the $i^{th}$ arithmetic return, $R_{t_i}$, and the $i^{th}$ log return, $r_{t_i}$,

$$SwV_t = 2\sum_{i=1}^{M}\left(R_{t_i} - r_{t_i}\right), \tag{12}$$

with associated probability limit,

$$p\lim\left(SwV_t - RV_t\right) = \begin{cases} 0 & \text{when } p_t \in \Omega_t^c \\ 2\int_{t-1}^{t}\left(\exp\left(Z_s\right) - Z_s - 1\right)dN_s^p - \int_{t-1}^{t} Z_s^2 dN_s^p & \text{when } p_t \in \Omega_t^j \end{cases}. \tag{13}$$

The set $\Omega_t^j$ is that which contains all non-continuous (jump) price processes and $\Omega_t^c$ is the set containing all continuous price processes. JO propose a number of test statistics based on (12), but here we consider only

$$T_{JO,t} = \frac{BV_t}{M^{-1}\sqrt{\widehat{\Omega}_{SwV}}}\left(1 - \frac{RV_t}{SwV_t}\right),$$

where $\widehat{\Omega}_{SwV} = 3.05\frac{M^3}{M-3}\sum_{i=1}^{M}\prod_{k=0}^{3}\left|r_{t_{i-k}}\right|^{3/2}$ is a consistent estimator of $\int_{t-1}^{t} V_s^3 ds$. Under the null of no jumps, $T_{JO,t}$ converges to a standard normal distribution as $M \to \infty$. Since $SwV_t - RV_t$ tends to be positive when the price jump is positive and negative when price jump is negative, the test is conducted as a two-sided test in which the direction of the jump is detected in addition to its presence.

## 3 Extracting measures of jump occurrence, size and sign

In addition to the detection (or otherwise) of price jumps, we note that the ten tests outlined throughout Section 2 enable nonparametric measures of price jumps - both the occurrence and magnitude thereof - to be extracted. These nonparametric measures, in turn, contain information

about the dynamics of price jumps which, when allied with a theoretical model, can be used to draw empirical conclusions about the dynamic behaviour of extreme price movements. Prior to outlining the methods of measure extraction, and to aid the reader, we first summarize the approaches to testing described above, including the abbreviations used to reference all:

i. *Tests based on squared variation*: [1] the BNS test of Barndorff-Nielsen and Shephard (2004, 2006) and Huang and Tauchen (2005); [2] the CPR test of Corsi, Pirino and Reno (2010); and the [3] MINRV and [4] MEDRV tests of Andersen, Dobrev and Schaumburg (2012);

ii. *Tests based on higher-order $\mathcal{P}$-power variation*: [5] the ASJ test of Aït-Sahalia and Jacod (2009); and the [6] PZ2 and [7] PZ4 tests of Podolskij and Ziggel (2010);

iii. *Tests based on standardized returns*: [8] the ABD test of Andersen, Bollerslev and Dobrev (2007); and [9] the LM test of Lee and Mykland (2008); and

iv. *Test based on variance swaps*: [10] the JO test of Jiang and Oomen (2008).

For all ten tests, the price jump *occurrence* indicator, $I^p_{*,t}$, is defined as

$$I^p_{*,t} = \mathbf{1}\left(T_{*,t} \in C_*\left(\alpha\right)\right), \tag{14}$$

where $C_*\left(\alpha\right)$ denotes the critical region of test '$*$' (given by the relevant abbreviation), defined by significance level $\alpha$, and with associated statistic $T_{*,t}$. The indicator thus equals one when the relevant test concludes in favour of a jump, and equals zero otherwise.

The method of extracting the price jump *size*, $\widetilde{Z}^p_{*,t}$, depends however on the particular test being used. Tauchen and Zhou (2011), for example, use the BNS approach to define

$$\widetilde{Z}^p_{BNS,t} = sign\left(r_t\right) \times \sqrt{\max\left(RV_t - BV_t, 0\right)}, \tag{15}$$

with the assumption adopted that the *sign* of the price jump is equivalent to the sign of the daily return over the day. Indeed, for all tests that utilize squared and higher-order $\mathcal{P}$-power variation, this same approach to (signed) measurement can be adopted, with the appropriate measure of integrated variance simply replacing $BV_t$ in (15). For example, the (signed) jump size for the CPR approach can be computed as

$$\widetilde{Z}^p_{CPR,t} = sign\left(r_t\right) \times \sqrt{\max\left(RV_t - CTBV_t, 0\right)},$$

while the $MedRV_t$ and $MinRV_t$ measures can replace $BV_t$ under the ADS approach. Similarly, the truncated power variation for $\mathcal{P} = 2$ can be used to extract signed jump sizes associated with the ASJ and PZ(2 and 4) tests, in conjunction with the truncation schemes used in constructing the test statistics themselves.

Extracting $\widetilde{Z}^p_{*,t}$ within the ABD and LM frameworks requires a different approach. ABD suggest that the aggregated price jumps over a trading day be computed as

$$\widetilde{Z}^p_{ABD,t} = \sum_{i=1}^{M} r_{t_i} \mathbf{1}\left(|T_{ABD,t_i}| > \Phi^{-1}\left(1 - \frac{\alpha^*}{2}\right)\right).$$

The sign of $\widetilde{Z}^p_{ABD,t}$ then depends on the sign of the sum of the returns that contribute to the aggregation itself. LM have not entertained this, but since the principle that underlies the construction of their test is identical to that of ABD, we note that a comparable approach (but with $\widetilde{T}_{LM,t_i}$ replacing $T_{ABD,t_i}$) is available.

The JO approach provides yet another avenue for extracting information about price jump size. Building on the relationship given in (13), with a daily discretization scheme, the price jump size $\widetilde{Z}^p_{JO,t}$, *including its sign*, is one that satisfies

$$SwV_t - RV_t = 2\left(\exp\left(\widetilde{Z}^p_{JO,t}\right) - \widetilde{Z}^p_{JO,t} - 1\right) I^p_{JO,t} - \left(\widetilde{Z}^p_{JO,t}\right)^2 I^p_{JO,t}. \tag{16}$$

Thus, when a jump is detected under this approach, $\widetilde{Z}^p_{JO,t}$ is obtained as a solution to the above nonlinear relation.

The time series of observed measures, $I^p_{*,t}$ and $\widetilde{Z}^p_{*,t}$, can be used to gain insight into the dynamics of price jumps and/or the impact of such dynamics on returns. For example, Andersen *et al.* (2010) use variants of the BNS-based measures ($I^p_{BNS,t}$ and $\widetilde{Z}^p_{BNS,t}$) to produce appropriate standardization of daily returns on the Dow Jones Industrial Average stocks. Tauchen and Zhou (2011) also use the BNS-based measures, but to evaluate the price jump dynamics of the S&P500, the 10-year US Treasury bond, and the US dollar/Japanese Yen exchange rate. Finally, Maneesoonthorn *et al.* (2017) use these same jump measures to supplement daily return and nonparametric volatility measures in a multivariate state space model for the S&P500 market index, treating these measures as being observed with error.

To date however, we are not aware of any study that provides a comprehensive assessment of both the full range of preliminary jump tests (as outlined above), and the associated set of measurements of jump occurrence, size and sign, including the accuracy with which these measurements pinpoint respectively the presence and magnitude of a sequence of dependent price jumps. We are also not aware of any study in which the impact on inference regarding price and/or volatility jump dynamics of using *different* measures is documented. It is these tasks that we now undertake in the remainder of the paper.

# 4 Assessment of test and measurement accuracy

We assess the performance of each jump detection and measurement method in an artificial data scenario. In Section 4.1 we document the power of each method to detect individual price jumps. In particular, this simulation exercise sheds light on the robustness of the alternative price jump

tests in the presence of a discontinuous volatility process, something that has not, to our knowledge, been documented elsewhere, for any of the tests discussed. In Section 4.2 we then document the frequency with which a *sequence* of price jumps is correctly detected and measured, under two alternative dynamic specifications for the jump intensity. The accuracy of detection is assessed in Section 4.2.1, and Section 4.2.2 documents two aspects of price jump measurement, namely: i) the accuracy of price jump *magnitude* measurement; and ii) the extent to which the correct *sign* of any price jump is found.

## 4.1 Detection of individual jumps: empirical size and power

In this section, we assess the ability of the ten jump tests (BNS, CPR, MINRV, MEDRV, ASJ, PZ2, PZ4, ABD, LM and JO) to correctly detect price jumps, including in the presence of volatility jumps. With reference to (1), reproduced and re-numbered here for convenience,

$$dp_t = \mu_t dt + \sqrt{V_t} dW_t^p + dJ_t^p, \tag{17}$$

we now specify

$$\mu_t = \mu + \gamma V_t, \tag{18}$$

and define a jump diffusion process for $V_t$,

$$dV_t = \kappa \left(V_t - \theta\right) + \sigma_v \sqrt{V_t} dW_t^v + dJ_t^v, \tag{19}$$

where $dW_t^v$ is assumed to be uncorrelated with $dW_t^p$, and $dJ_t^v = Z_t^v dN_t^v$, with $Z_t^v$ denoting the random volatility jump size and $dN_t^v$ the increment of a discrete count process, with $P(dN_t^v = 1) = \delta^v dt$ and $P(dN_t^v = 0) = (1 - \delta^v)\, dt$. For the purpose of this first simulation exercise, we set $\delta^p = \delta^v = 1$, which implies that both $dN_t^p = 1$ and $dN_t^v = 1$, for all $t$. Price and volatility jump sizes are then generated over a $100 \times 100$ grid of paired values given by $Z_t^p \in \left[-10\sqrt{\theta}, +10\sqrt{\theta}\right]$ and $Z_t^v \in [0, 20\theta]$. Non-zero values of $Z_t^p$ and $Z_t^v$ thus imply, with unit probability, that a jump in the price and volatility process, respectively, has in fact occurred, whilst zero values for $Z_t^p$ and $Z_t^v$ imply otherwise. The data is generated using true parameter values: $\mu = 0.2$, $\gamma = -7.9$, $\kappa = 0.03$, $\theta = 0.02$ and $\sigma_v = 0.02$ (adhering to the theoretical restriction $2\kappa\theta \geqslant \sigma_v^2$), and with the diffusive variance process initialized at $\theta$.

For each scenario, a very fine Euler discretization is employed to simulate high-frequency observations, with 720 observations created per trading day, equivalent to generating price observations every 30 seconds. The price jump test statistics are then constructed using every $10^{th}$ observation over the daily interval, equivalent to the five-minute sampling frequency that is the consensus choice in the literature. We then compute, over 1000 independent Monte Carlo iterations, the proportion of times that a test detects a price jump. Note that this proportion equates to the empirical size of the test when $Z_t^p = 0$ for all $t$, and the empirical *power* of the test otherwise. The nominal

size of each test is set at 1%. The results are summarized graphically in Figure 1, with the price jump magnitude ($Z_t^p$) on one axis, the simultaneous volatility jump magnitude on the other, and the proportion of times that the jump is detected using each method recorded on the vertical axis. Note that the results for PZ4 were very similar to those for PZ2 and have thus been omitted from the graphical display.

As can be observed from Panel E in Figure 1, the ASJ test has low power to detect sizeable price jumps, even when a volatility jump is absent ($Z_t^v = 0$). Once a volatility jump is also present, even one that is small in magnitude, the power of this test drops essentially to zero. The power curve of the ABD test in Panel G exhibits qualitatively similar behaviour, if much less extreme. Whilst the poor power properties of these two tests are in line with certain simulation results reported in Dumitru and Urga (2012), we are not aware of any other study that has documented this full range of results, including the impact of volatility jumps.

In contrast to the behaviour of the ASJ and ABD tests, the remaining seven tests display a much less extreme loss in power (for detecting price jumps) in the presence of volatility jumps. Of these, both the PZ2 and LM tests display a minimal change in the shape of their power curves (across the $Z_t^p$ axis) as $Z_t^v$ increases, although at the cost of quite severe size distortion (Panels F and H respectively). That is, these two procedures incorrectly reject the true null of no price jump with higher and higher probability as the magnitude of the volatility jump increases. Noting that powers are computed as raw proportions, it is perhaps not surprising that these two procedures have the highest power of all nine shown, when variance jumps are present. As the size distortion changes with the magnitude of the volatility jump, full knowledge of this magnitude is required for any size correction to be performed. Given that this is not feasible in practical settings, we have chosen not to record size-adjusted powers. The remaining five tests, BNS, CPR, MINRV, MEDRV and JO (Panels A, B, C, D and I respectively), all display relatively robust size properties in the presence of volatility jumps; however they all exhibit, to varying degrees, power curves that flatten near the origin (over the $Z_t^p$ axis) as the magnitude of the volatility jump increases.

### 4.2 Detection and measurement of a sequence of jumps

We now generate artificial data over a period of $T = 2000$ sequential 'days' using the model in (17)-(19) augmented by the following specifications. First, price and volatility jump occurrences are now defined by the dynamic Poisson processes,

$$\Pr\left(dN_t^p = 1\right) = \delta_t^p dt \tag{20}$$

and

$$\Pr\left(dN_t^v = 1\right) = \delta_t^v dt \tag{21}$$

respectively. Only a single price and volatility jump is allowed to occur on each day, with $\delta_t^p$ and $\delta_t^v$ determining the probability of each of those jumps occurring. Two alternative specifications for

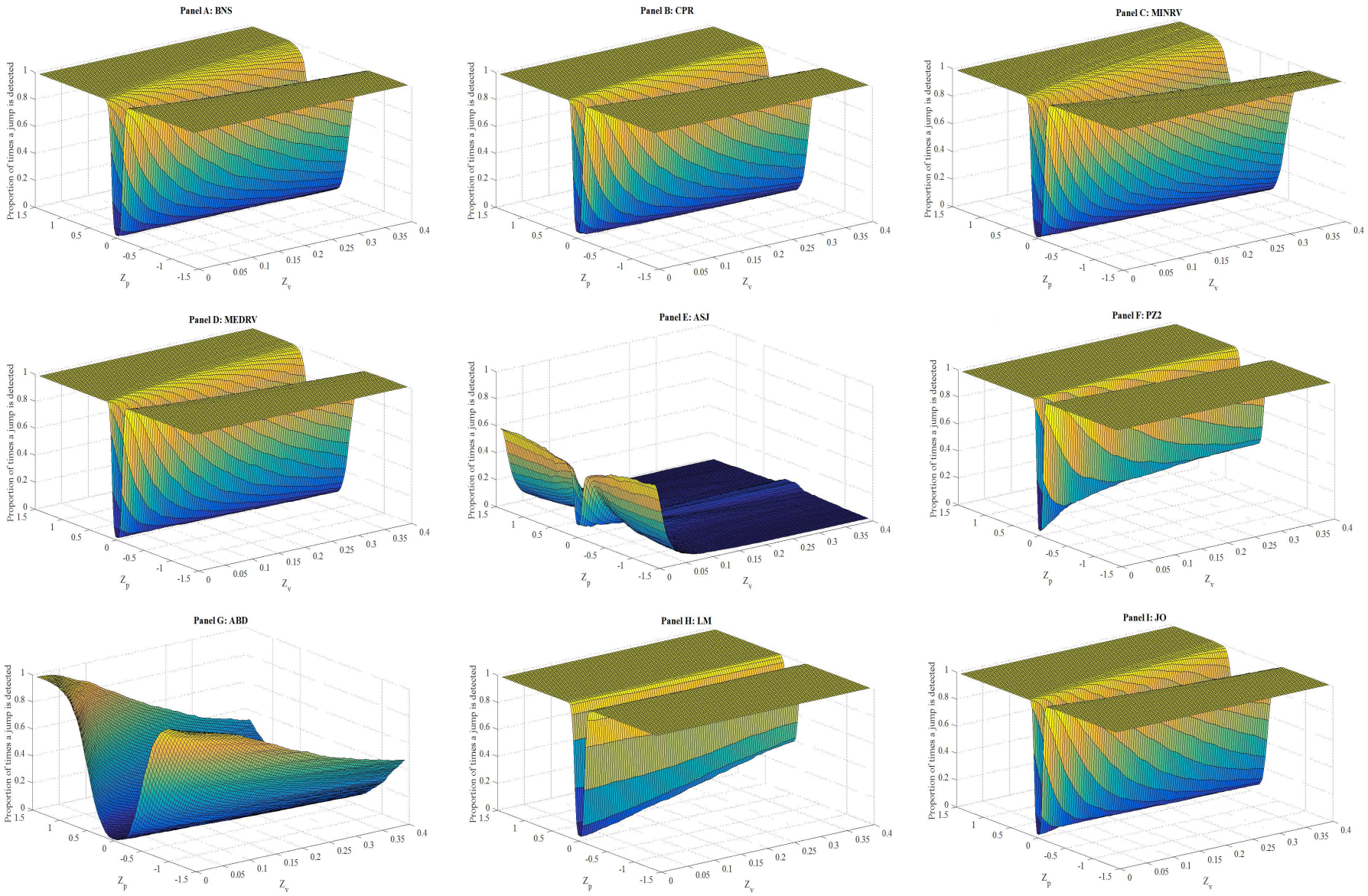


Figure 1: Power curves over changing price and volatility jump sizes ($Z_p$ and $Z_v$ respectively).

$\delta_t^p$ are entertained, given by

$$\text{Hawkes (H): } d\delta_t^p = \alpha_p \left(\delta_\infty^p - \delta_t^p\right) dt + \beta_p dN_t^p \tag{22}$$

$$\text{State Dependent (SD): } \delta_t^p = \beta_{p0} + \beta_{p1} V_t. \tag{23}$$

The parameter $\delta_\infty^p$ appearing in (22) is the steady state level of $\delta_t^p$ to which the price jump intensity reverts once the impact of 'self-excitation' (via $dN_t^p$) dissipates; see Hawkes (1971a,b). The unconditional jump intensity implied by the Hawkes process is $\delta_0^p = \frac{\alpha_p \delta_\infty^p}{\alpha_p - \beta_p}$, while that of the state dependent intensity is $\delta_0^p = \beta_{p0} + \beta_{p1} E\left(V_t\right)$.

One or other of the specifications in (22) and (23) have been used elsewhere to model dynamic price (and volatility) jump intensity (see, for example, Bates, 1996, Pan, 2002, Eraker, 2004, Aït-Sahalia *et al.*, 2015, and Maneesoonthorn *et al.*, 2017), and the two are employed here in order to examine the robustness (or otherwise) of the jump detection methods to different assumptions adopted for $\delta_t^p$. In particular, the state dependent specification in (23) induces a smoother intensity process, following the pattern of the latent volatility process quite closely, while the Hawkes model in (22) typically produces sharp spikes following each jump event. Plots of representative examples of each process are presented and discussed in Section 4.2.3 below, and the link between the distinctive features of each process and the accuracy with which sequences of jumps generated from each process are detected, also documented therein.

Associated with each of the two specifications for $\delta_t^p$, we consider three different scenarios for volatility jumps: 1) volatility jumps are absent ($\delta_t^v = 0$ for all $t$) (corresponding to models labelled hereafter as SD1 and H1 respectively); 2) volatility jumps have a constant intensity ($\delta_t^v = \delta_0^v$) (Models SD2 and H2); and 3) the volatility jump intensity takes the same form as the price jump intensity ($\delta_t^v = \delta_t^p$), but with volatility and price jumps occurring independently one of the other (i.e. with $dN_t^p$ and $dN_t^v$ assumed to be independent random variables) (Models SD3 and H3). The parameters in (22) and (23) are chosen to ensure that the unconditional price jump intensity, $\delta_0^p$, is the same for both models, with the parameter settings for all six scenarios described here recorded in Panel A of Table 1. Note that the parameter values are chosen to be in line with those estimated using S&P500 data in the empirical analysis in Maneesoonthorn *et al.* (2017).

In contrast to the settings adopted in Section 4.1 for price and volatility jump size ($Z_t^p$ and $Z_t^v$ respectively) we now model these two latent variables explicitly as follows. The price jump size is specified as

$$Z_t^p = S_t^p \exp\left(M_t^p\right), \tag{24}$$

with the sign of the jump defined as

$$S_t^p = \begin{cases} -1 & \text{with probability } \pi_p \\ 1 & \text{with probability } 1 - \pi_p \end{cases}, \tag{25}$$

and the logarithmic magnitude as

$$M_t^p \sim N(\mu_p, \sigma_p^2). \tag{26}$$

The (positive) volatility jump size is modelled as

$$Z_t^v \sim Exp\left(\mu_v\right), \tag{27}$$

where $\mu_v = 1.5 \times \theta$, with $\theta$ as appearing in (19). The parameter values used in specifying (24) to (27) are given in Panel B of Table 1, again with reference to empirical estimates recorded in Maneesoonthorn *et al.* (2017).

Table 1: True parameter values for the model components specified in (17)-(27). SD1 and H1 impose no volatility jumps. SD2 and H2 assume constant intensity volatility jumps independent of price jumps. SD3 and H3 also assume independent price and volatility jumps but with each sharing the same jump intensity process.

| Panel A: Jump intensity settings | | | | | | |
|---|---|---|---|---|---|---|
| | SD1 | SD2 | SD3 | H1 | H2 | H3 |
| $\delta_0^p$ | 0.105 | 0.105 | 0.105 | 0.105 | 0.105 | 0.105 |
| $\beta_{p0}$ | 0.100 | 0.100 | 0.100 | – | – | – |
| $\beta_{p1}$ | 0.500 | 0.500 | 0.500 | – | – | – |
| $\alpha_p$ | – | – | – | 0.094 | 0.094 | 0.094 |
| $\beta_p$ | – | – | – | 0.059 | 0.059 | 0.059 |
| $\delta_0^v$ | 0 | 0.105 | 0.105 | 0 | 0.105 | 0.105 |
| $\beta_{v0}$ | – | – | 0.100 | – | – | – |
| $\beta_{v1}$ | – | – | 0.500 | – | – | – |
| $\alpha_v$ | – | – | – | – | – | 0.094 |
| $\beta_v$ | – | – | – | – | – | 0.059 |
| **Panel B: Jump size settings** | | | | | | |
| $\mu_p$ | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| $\sigma_p$ | 0.50 | 0.50 | 0.50 | 0.50 | 0.50 | 0.50 |
| $\pi_p$ | 0.50 | 0.50 | 0.50 | 0.50 | 0.50 | 0.50 |
| $\mu_v = 1.5 \times \theta$ | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 |

Using the simulated five-minute intraday data for each day, a sequence of $T = 2000$ price jump tests are conducted and the associated sequences of price jump occurrence ($I_{*,t}^p$) and size ($\widetilde{Z}_{*,t}^p$) computed for each test '$*$'. The intraday data are simulated as in Section 4.1, with occurrences of both price and volatility jumps restricted to one per day, as noted above. The accuracy metrics associated with all measures are computed as averages over the $N = 1000$ Monte Carlo iterations of the simulated price sequence. Evaluation metrics pertaining to the *detection* of the sequence price jumps are outlined in Section 4.2.1, whilst those related to the measurement of *magnitude* and *sign* are given in Section 4.2.2. Discussion of all numerical accuracy results then follows in

Section 4.2.3.[3]

### 4.2.1 Accuracy of sequential jump detection

For the purpose of documenting the accuracy of the price jump indicator measure, $I^p_{*,t}$ defined in (14), we estimate the probability of correctly detecting a price jump, $DJ^* = \Pr\left(I^p_{*,t} = 1 | \Delta N^p_t = 1\right)$, over the sequence of $T = 2000$ days. Specifically, for Monte Carlo iteration $i$, this conditional probability is estimated by $DJ^*_i = \sum_{t=1}^{T} \mathbf{1}\left(I^p_{*,t} = 1 \text{ and } \Delta N^p_t = 1\right) / \sum_{t=1}^{T} \Delta N^p_t$, with the average over the $N$ Monte Carlo iterations then recorded, as

$$\overline{DJ^*} = \frac{1}{N} \sum\nolimits_{i=1}^{N} DJ^*_i.$$

We next perform the comparable calculation, estimating the probability of *not* detecting a jump, conditional on the actual absence of a jump, $NDJ^* = \Pr\left(I^p_{*,t} = 0 | \Delta N^p_t = 0\right)$, with

$$\overline{NDJ^*} = \frac{1}{N} \sum\nolimits_{i=1}^{N} NDJ^*_i$$

recorded, where $NDJ^*_i = \sum_{t=1}^{T} \mathbf{1}\left(I^p_{*,t} = 0 \text{ and } \Delta N^p_t = 0\right) / \sum_{t=1}^{T} \mathbf{1}(\Delta N^p_t = 0)$.

Frameworks in which $\overline{DJ^*}$ and $\overline{NDJ^*}$ are close to one are deemed to produce accurate price jump occurrence measurement. Ideally, however, the error sequence should also be independent over time so that an error at one time point does not alter the rate of error in subsequent periods. To this end we define the sequence of errors,

$$Err_{*,t} = \mathbf{1}\left(I^p_{*,t} \neq \Delta N^p_t\right),$$

for each method and, in the spirit of Christoffersen (1998), conduct a test of the null hypothesis that $Err_{*,t}$ is a sequence of independent Bernoulli draws for each Monte Carlo iteration of the sequence. We then record (over the Monte Carlo replications) the proportion of times that the null hypothesis is rejected, using the abbreviation $SDE^*$ to denote the proportion of rejections in favour of the alternative of serially dependent errors. The approach with $SDE^*$ closest to zero is preferred.

### 4.2.2 Accuracy of sequential jump measurement

As noted from (24), the latent price jump size at time $t$ comprises two components: magnitude ($\exp M^p_t$) and sign ($S^p_t$), and the accuracy with which each component is measured, across the sequence of $T = 2000$ days, is considered separately. For all but the JO approach, the measured jump sign coincides exactly with the sign of the returns (see relevant details in Section 3).

[3]Note that the detection of multiple jumps and the identification of jump times could also, in principle, be performed using modified versions of all jump tests discussed here; see, for example, Andersen *et al.* (2010), who use the BNS test to perform these tasks. Given the broad scope of the paper as it stands, such extensions have not been considered here.

The accuracy with which the true price jump *magnitude* is estimated is measured by the mean squared error computed from the distance between the absolute values of the measurement, $\left|\widetilde{Z}^p_{*,t}\right|$, and the simulated price jump size $|Z^p_t|$. Specifically, the mean squared error is defined as

$$MSE^* = \frac{\sum_{t=1}^{T}\left(\left|\widetilde{Z}^p_{*,t}\right| - |Z^p_t|\right)^2 \Delta N^p_t}{\sum_{t=1}^{T}\Delta N^p_t},$$

with the approach producing the smallest $MSE^*$ preferred. The accuracy with which the price jump *sign* is pinpointed is measured by an estimate of the probability that the correct sign is identified, conditional on the occurrence of a jump

$$SCD^* = \Pr\left(sign\left(\widetilde{Z}^p_{*,t}\right) = sign\left(Z^p_t\right)|\Delta N^p_t = 1\right),$$

with this estimate given by

$$\overline{SCD^*} = \frac{1}{N}\sum_{i=1}^{N} SCD^*_i,$$

where $SCD^*_i = \frac{\sum_{t=1}^{T}\Delta N^p_t \mathbf{1}\left(sign\left(\widetilde{Z}^p_{*,t}\right) = sign\left(Z^p_t\right)\right)}{\sum_{t=1}^{T}\Delta N^p_t}$ provides the corresponding estimated probability from the $i^{th}$ Monte Carlo replicated sample. Large values of $\overline{SCD^*}$ are desired.

### 4.2.3 Numerical accuracy results

Tables 2 and 3 record the five accuracy metrics ($\overline{DJ^*}$, $\overline{NDJ^*}$, $SDE^*$, $MSE^*$ and $\overline{SCD^*}$) for all ten approaches, under the Hawkes (22) and the state dependent (23) specifications, respectively. The figures that represent the largest values of $\overline{DJ^*}$, $\overline{NDJ^*}$ and $\overline{SCD^*}$, and the smallest values of $SDE^*$ and $MSE^*$, under any particular data generating process, are highlighted in bold.

First, we note that when volatility jumps are introduced in specifications H2, H3, SD2 and SD3, all approaches exhibit a decline in accuracy according to all five measures. This is consistent with the findings discussed in Section 4.1 regarding the robustness of price jump detection, with all relevant power curves being impacted in some way by an increase in the volatility jump size. Further, we find that the introduction of dynamics in the volatility jumps via the Hawkes specification (e.g. H3 versus H2) does not produce too marked an affect on performance, but that the move from SD2 to SD3, under the state dependent framework, results in a more notable reduction in accuracy.

A comparison of the corresponding metrics in Tables 2 and 3 highlights the way in which the nature of the dynamic structure of the price jump intensity ($\delta^p_t$) affects the accuracy with which the different aspects of the jump process are measured. Under the Hawkes process in (22), and for all settings for $\delta^v_t$, all ten frameworks estimate the magnitude and sign of the price jump more accurately (i.e. $MSE^*$ is lower and $\overline{SCD^*}$ is higher) than they do under the state dependent process in (23). Correct detection of price jumps ($\overline{DJ^*}$) also tends to be higher under the Hawkes process, whilst the (correct) detection of non-jump days ($\overline{NDJ^*}$) is very similar across the two tables. However, the errors associated with detecting price jumps, as indicated by $SDE^*$, tend

to be more serially correlated under the Hawkes process. To understand why the latter may be the case, we produce a single path (using the parameter settings given in Table 1) of the two jump intensity processes in Figure 2. As can be observed, despite the unconditional intensity for both processes being identical, the simulated dynamic paths are quite distinct. Movements in the Hawkes intensity (Panel A) are far more erratic, with sharp rises in the intensity induced by an occurrence of a past jump. In contrast, movements in the state dependent intensity (Panel B) are much less severe, reflecting the characteristics of the Brownian motion that drives the stochastic volatility process on which $\delta_t^p$ depends. As a result, the Hawkes intensity generates a greater degree of jump clustering than does the state dependent model, which is more difficult to discern, and with a greater degree of autocorrelation being found in the detection errors as a consequence.

We now turn to a discussion of each of the five accuracy metrics in turn - $\overline{DJ^*}$, $\overline{NDJ^*}$, $SDE^*$, $MSE^*$ and $\overline{SCD^*}$ - and to which of the ten frameworks, if any, is the most accurate according to a specific metric. Both tables suggest that the PZ2 approach is most accurate in terms of *detecting* price jumps, with its $\overline{DJ}^*$ measure being the highest (and notably so in some instances) in all but one case (H1), where the PZ2 approach is a close second behind the JO approach. The $\overline{NDJ^*}$ measure, indicating the ability of an approach to correctly identify the absence of a price jump, is close to one for *all* approaches and for all data generating processes. Hence, although the top ranking alternates between the ASJ and PZ4 methods, the magnitudes of this recorded measure across all instances are so close that it could be argued that all approaches are equally accurate in discerning the absence of jumps.

The $SDE^*$ measure, for which low values are indicative of independence in the jump detection errors, is seen - as noted above - to be sensitive to the nature of the dynamic structure of the price jump itself. Whilst all approaches perform worse according to this measure under the Hawkes specification, the ABD and ASJ approaches perform particularly poorly. Of all ten approaches both PZ2 and PZ4 prove to be the most robust across the six different price jump specifications (H1 to SD3), in terms of retaining reasonably small (and similar) values for $SDE^*$.

With regard to the accuracy with which price jump magnitude is estimated, the ASJ approach produces the smallest value of $MSE^*$ in the absence of volatility jumps (specifications H1 and SD1), while the MEDRV approach is most accurate when volatility jumps (whether dynamic or not) are present. Due to the reliance on the sign of the daily return in the construction of the price jump sign measure ($\overline{SCD^*}$), the sign measure for all but one of the approaches is identical for any given data generating process. The JO approach, which solves for the price jump size (including sign) from the nonlinear relation in (16), is the most accurate in detecting the sign when volatility jumps are absent (H1 and SD1), while the sign of the daily return serves better otherwise.

From these results we can deduce that no single approach produces the most accurate measurement across all five metrics. An approach that does well at detecting jumps may not perform well at measuring price jump magnitude, and vice versa. However, one *can* conclude that it is the CPR

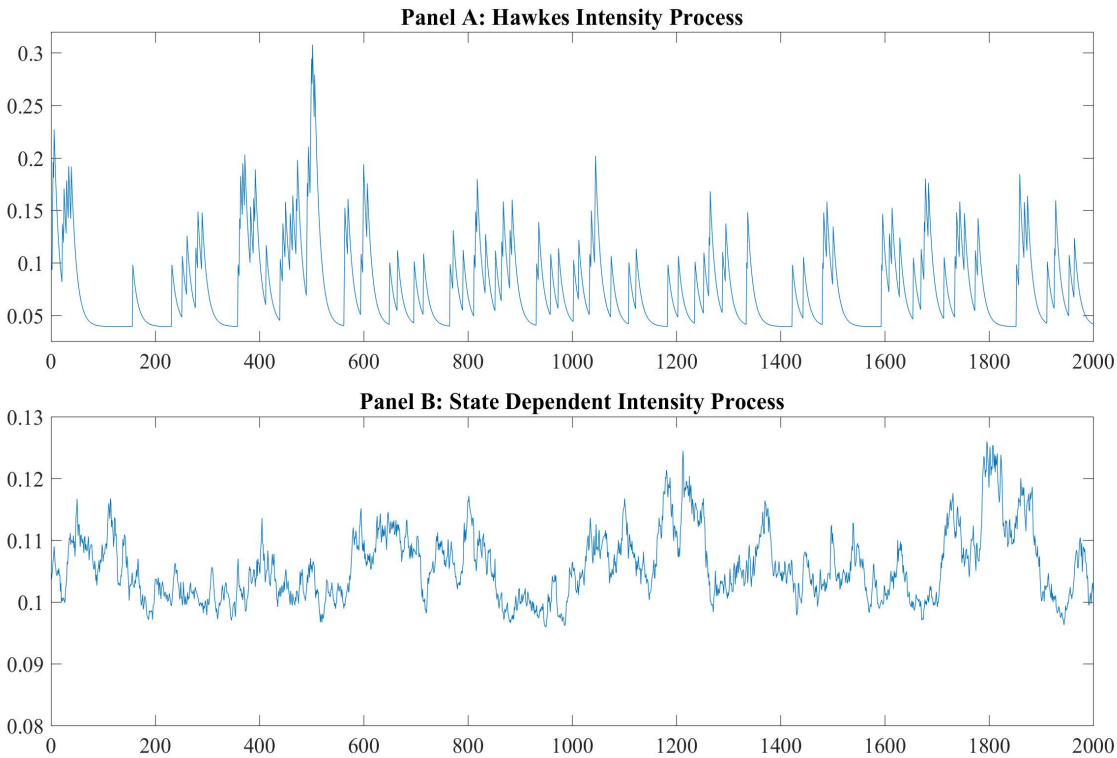


Figure 2: Panel A: the price jump intensity process implied by the Hawkes specification in (22); Panel B: the price jump intensity implied by the state dependent specification in (23). The realizations reproduced in both panels are based on the parameter values recorded in Table 1.

and MEDRV approaches that are most robust across both metric and price jump specification. These two approaches perform reasonably well at both detecting price jumps when present and not detecting absent jumps, in terms of accurately measuring the magnitude of the price jump, and in terms of avoiding dependence in price jump detection errors. The BNS, which is arguably the most popular approach adopted in the literature, is also reasonably robust, but with accuracy measures that are consistently below those of CPR and MEDRV.[4]

# 5 Implied inference about price jump dynamics

## 5.1 Data generation and model estimation

The implications for *inference* - and in particular inference about the nature of price jumps - when using different testing/measurement approaches to produce the relevant jump measures, is now assessed in a simulation setting. We perform this assessment by generating data from the bivariate jump diffusion defined by (17) to (19), with the price jump intensity ($\delta_t^p$) following the dynamic

[4]We also conducted this Monte Carlo assessment under different parameter settings from those documented in Table 1, including cases where the leverage effect is present, with $corr(dW_t^p, dW_t^v) < 0$. The magnitude of the accuracy metrics do change, with accuracy tending to reduce with an increase in the average volatility jump size, as well as with a decrease in the signal-to-noise ratio. Nonetheless, the ranking amongst the approaches remains robust to the changes in parameter values.

Table 2: Accuracy metrics for the price jump measurement framework under the bivariate jump diffusion for price and volatility, with the Hawkes process in (22) for the price jump intensity. In each column, the figure that is deemed the most favourable accuracy metric, under each of the three versions of the data generating process, is highlighted in bold. Large values are favoured for $\overline{DJ^*}$, $\overline{NDJ^*}$ and $\overline{SCD^*}$; small values are favoured for $SDE^*$ and $MSE^*$.

| | | $\overline{DJ^*}$ | $\overline{NDJ^*}$ | $SDE^*$ | $MSE^*$ | $\overline{SCD^*}$ |
|---|---|---|---|---|---|---|
| | BNS | 0.8158 | 0.9862 | 0.0890 | 0.4106 | 0.8851 |
| | CPR | 0.8801 | 0.9543 | **0.0460** | 0.2879 | 0.8851 |
| | MINRV | 0.7706 | 0.9933 | 0.1550 | 0.2852 | 0.8851 |
| | MEDRV | 0.8531 | 0.9884 | 0.0690 | 0.2675 | 0.8851 |
| H1 | ASJ | 0.0430 | 0.9978 | 0.7490 | **0.2546** | 0.8851 |
| | PZ2 | 0.9519 | 0.9759 | 0.0700 | 12.082 | 0.8851 |
| | PZ4 | 0.8609 | **0.9999** | 0.0740 | 12.082 | 0.8851 |
| | ABD | 0.0622 | 0.9946 | 0.7330 | 11.855 | 0.8851 |
| | LM | 0.7336 | 0.9691 | 0.0790 | 10.854 | 0.8851 |
| | JO | **0.9801** | 0.9732 | 0.0970 | 4.5916 | **0.9079** |
| | BNS | 0.3385 | 0.9864 | 0.4170 | 1.7777 | **0.7180** |
| | CPR | 0.4523 | 0.9547 | 0.1860 | 1.5226 | **0.7180** |
| | MINRV | 0.2680 | 0.9932 | 0.5220 | 1.5608 | **0.7180** |
| | MEDRV | 0.3794 | 0.9884 | 0.4110 | **1.3555** | **0.7180** |
| H2 | ASJ | 0.0328 | **0.9978** | 0.7700 | 1.4475 | **0.7180** |
| | PZ2 | **0.6521** | 0.9600 | **0.1220** | 17.099 | **0.7180** |
| | PZ4 | 0.6012 | 0.9853 | 0.2430 | 17.099 | **0.7180** |
| | ABD | 0.0097 | 0.9965 | 0.7710 | 12.082 | **0.7180** |
| | LM | 0.3413 | 0.9688 | 0.3230 | 11.137 | **0.7180** |
| | JO | 0.4980 | 0.9726 | 0.2310 | 6.0486 | 0.4964 |
| | BNS | 0.4033 | 0.9863 | 0.4210 | 1.6197 | **0.7396** |
| | CPR | 0.5128 | 0.9548 | 0.1860 | 1.3792 | **0.7396** |
| | MINRV | 0.3348 | 0.9933 | 0.5580 | 1.4234 | **0.7396** |
| | MEDRV | 0.4455 | 0.9883 | 0.3990 | **1.2338** | **0.7396** |
| H3 | ASJ | 0.0347 | **0.9978** | 0.7600 | 1.3223 | **0.7396** |
| | PZ2 | **0.6961** | 0.9608 | **0.1280** | 16.919 | **0.7396** |
| | PZ4 | 0.6344 | 0.9874 | 0.2880 | 16.919 | **0.7396** |
| | ABD | 0.0144 | 0.9963 | 0.7660 | 12.143 | **0.7396** |
| | LM | 0.3927 | 0.9689 | 0.2880 | 11.176 | **0.7396** |
| | JO | 0.5588 | 0.9727 | 0.2280 | 5.8774 | 0.5573 |

Table 3: Accuracy metrics for the price jump measurement framework under the bivariate jump diffusion for price and volatility, with the state dependent process in (23) for the price jump intensity. In each column, the figure that is deemed the most favourable accuracy metric, under each of the three versions of the data generating process, is highlighted in bold. Large values are favoured for $\overline{DJ^*}$, $\overline{NDJ^*}$ and $\overline{SCD^*}$; small values are favoured for $SDE^*$ and $MSE^*$.

| | | $\overline{DJ^*}$ | $\overline{NDJ^*}$ | $SDE^*$ | $MSE^*$ | $\overline{SCD^*}$ |
|---|---|---|---|---|---|---|
| | BNS | 0.8103 | 0.9862 | 0.0730 | 0.4187 | 0.8818 |
| | CPR | 0.8755 | 0.9542 | 0.0460 | 0.2946 | 0.8818 |
| | MINRV | 0.7646 | 0.9932 | 0.0690 | 0.2918 | 0.8818 |
| | MEDRV | 0.8476 | 0.9883 | 0.0360 | 0.2718 | 0.8818 |
| SD1 | ASJ | 0.0438 | 0.9977 | 0.0570 | **0.2587** | 0.8818 |
| | PZ2 | **0.9495** | 0.9760 | 0.0640 | 12.163 | 0.8818 |
| | PZ4 | 0.8587 | **1.0000** | **0.0220** | 12.163 | 0.8818 |
| | ABD | 0.0607 | 0.9946 | 0.0520 | 11.931 | 0.8818 |
| | LM | 0.7317 | 0.9691 | 0.0670 | 10.924 | 0.8818 |
| | JO | 0.9036 | 0.9732 | 0.0830 | 4.6269 | **0.9034** |
| | BNS | 0.3108 | 0.9864 | 0.0780 | 1.9171 | **0.7101** |
| | CPR | 0.4249 | 0.9550 | 0.0700 | 1.6644 | **0.7101** |
| | MINRV | 0.2414 | 0.9933 | 0.0830 | 1.7030 | **0.7101** |
| | MEDRV | 0.3491 | 0.9884 | 0.0850 | **1.4782** | **0.7101** |
| SD2 | ASJ | 0.0307 | **0.9977** | **0.0540** | 1.6018 | **0.7101** |
| | PZ2 | **0.6252** | 0.9602 | 0.0830 | 18.048 | **0.7101** |
| | PZ4 | 0.5815 | 0.9854 | 0.1400 | 18.048 | **0.7101** |
| | ABD | 0.0082 | 0.9965 | 0.0560 | 12.116 | **0.7101** |
| | LM | 0.3161 | 0.9690 | 0.0740 | 11.192 | **0.7101** |
| | JO | 0.4692 | 0.9728 | 0.0960 | 6.2097 | 0.4675 |
| | BNS | 0.2058 | 0.9863 | 0.1520 | 2.7674 | **0.6711** |
| | CPR | 0.3049 | 0.9551 | 0.1400 | 2.5948 | **0.6711** |
| | MINRV | 0.1502 | 0.9932 | 0.1490 | 2.7321 | **0.6711** |
| | MEDRV | 0.2343 | 0.9884 | 0.1810 | **2.3481** | **0.6711** |
| SD3 | ASJ | 0.0230 | **0.9978** | 0.0740 | 2.7183 | **0.6711** |
| | PZ2 | **0.4858** | 0.9601 | 0.1560 | 26.932 | **0.6711** |
| | PZ4 | 0.4663 | 0.9753 | 0.2440 | 26.932 | **0.6711** |
| | ABD | 0.0057 | 0.9970 | **0.0720** | 12.176 | **0.6711** |
| | LM | 0.2222 | 0.9687 | 0.1170 | 11.385 | **0.6711** |
| | JO | 0.3366 | 0.9723 | 0.1450 | 7.1741 | 0.3338 |

Hawkes process in (22) over $T = 2000$ trading days. To keep the exercise manageable, we impose $\delta^v = 0$, such that the volatility process has no jump component. The primary aim of the exercise is to gauge the robustness, or otherwise, of inference to the use of different high frequency measures (and associated preliminary tests) and to link - where feasible - inaccuracy in test outcomes and/or subsequent measurement to inferential inaccuracy.

As per the description in Section 4.1, data is simulated using a fine Euler discretization, with 720 observations created per trading day, equivalent to generating price observations every 30 seconds; the difference here being that the (discretized) Hawkes process for $\delta_t^p$ also plays a role in data generation, and volatility jumps are excluded. The price jump test statistics, and associated jump measures of occurrence, magnitude and sign, plus measures of integrated variance, are then constructed using every $10^{th}$ observation over the daily interval, equivalent to a five-minute sampling frequency. We focus on four approaches only at this point: CPR and MEDRV, as the most robust approaches, overall, according to the simulation results reported in Section 4.2.3; BNS, as the approach most commonly used in empirical studies; and ASJ, which is most accurate in terms of (sequential) jump magnitude measurement, but arguably the least accurate in terms of (sequential) jump detection.

The continuous-time model assumed to underlie the observed data is then discretized to the daily level, with only a single price jump allowed to occur on one day. This discrete-time model is estimated using a Bayesian simulation scheme, with marginal posterior means (MPMs) and 95% highest probability density (HPD) intervals compared with the true values for all parameters, as a method of gauging inferential accuracy. Particular attention is given to the accuracy with which the jump-related parameters are estimated, and the impact on that accuracy of either allowing for measurement error in the jump measures, or not doing so. As the model to be estimated is a version of the discrete state space model used in the empirical analysis in Maneesoonthorn *et al.* (2017), in which volatility jumps (that were a focus of that paper) are omitted and an additional measurement equation related to the sign of the price jump is introduced, the Markov chain Monte Carlo (MCMC) algorithm used therein is simply adapted to cater for these modifications, and we refer readers to that paper for details of the base algorithm.

With the time interval, $\Delta t = 1$, denoting a trading day, and $r_t = \ln P_{t+1} - \ln P_t$ defined as the log return over the day, the model to be estimated here comprises a collection of five measurement

equations:

$$r_t = \mu + \gamma V_t + \sqrt{V_t}\xi_t^p + Z_t^p \Delta N_t^p \tag{28}$$

$$\ln \widehat{IV}_{*,t} = \psi_0 + \psi_1 \ln V_t + \sigma_{BV}\xi_t^{BV} \tag{29}$$

$$I_{*,t}^p = \begin{cases} Bernoulli\,(\beta) & \text{if } \Delta N_t^p = 1 \\ Bernoulli\,(\alpha) & \text{if } \Delta N_t^p = 0 \end{cases} \tag{30}$$

$$\widetilde{M}_{*,t}^p = M_t^p + \sigma_{M_p}\xi_t^{M_p} \qquad \text{for } RV_t > \widehat{IV}_{*,t} \tag{31}$$

$$\widetilde{S}_{*,t}^p = \begin{cases} \{-1,+1\} & \text{with probabilities } \{s_n, 1-s_n\} \text{ if } S_t^p = -1 \\ \{-1,+1\} & \text{with probabilities } \{s_p, 1-s_p\} \text{ if } S_t^p = +1 \end{cases} \tag{32}$$

five stochastic state equations:

$$V_{t+1} = \kappa\theta + (1-\kappa)\,V_t + \sigma_v\rho\,(r_t - Z_t^p\Delta N_t^p - \mu - \gamma V_t) + \sigma_v\sqrt{(1-\rho^2)\,V_t}\xi_t^v \tag{33}$$

$$\Delta N_t^p \sim Bernoulli(\delta_t^p) \tag{34}$$

$$Z_t^p = S_t^p \exp\left(M_t^p\right) \tag{35}$$

$$S_t^p = \begin{cases} -1 & \text{with probability } \pi_p \\ +1 & \text{with probability } (1-\pi_p) \end{cases} \tag{36}$$

$$M_t^p \sim N\left(\mu_p, \sigma_p^2\right) \tag{37}$$

with $\left(\xi_t^p, \xi_t^{BV}, \xi_t^{M_p}, \xi_t^v\right)' \sim N(0, I_{4\times4})$, and a single conditionally deterministic state equation:

$$\delta_t^p = \alpha_p\delta_\infty^p + (1-\alpha_p)\,\delta_{t-1}^p + \beta_p\Delta N_{t-1}^p \tag{38}$$

that is a discretized version of (22).

The two measures, $\widehat{IV}_{*,t}$ and $I_{*,t}^p$, in (29) and (30) respectively, are constructed from five-minute (simulated) returns, and linked to the relevant latent processes, as follows. The estimate of integrated variance, $\widehat{IV}_{*,t}$, is given by $BV_t$ for the BNS approach, $CTBV_t$ for the CPR approach, $MedRV_t$ for the MEDRV approach, and by the truncated integrated variation in (10) with $\mathcal{P} = 2$ for the ASJ approach. From (29) it is seen that $\ln \widehat{IV}_{*,t}$ is treated as a noisy and potentially biased estimate of log integrated variance, with the latter represented by the logarithm of the end-of-day latent variance $V_t$; see also Koopman and Scharth (2013) and Maneesoonthorn *et al.* (2012, 2017). The jump occurrence measure, $I_{*,t}^p$, is computed as per (14), with $T_{*,t}$ and $C_*\,(\alpha)$ defined according to each of the four approaches. $I_{*,t}^p$ is viewed as a noisy measure of the latent price jump indicator in (34), with constant probabilities $\alpha$ and $\beta$ to be estimated from the data.

The logarithmic jump magnitude and jump sign measures, $\widetilde{M}_{*,t}^p$ and $\widetilde{S}_{*,t}^p$, (also constructed from the five-minute returns) are linked to the latent processes as follows. From (35) to (37) (which are, of course a restatement of (24) to (26), but with the subscript $t$ now explicitly representing day $t$) the latent jump size, $Z_t^p$, is comprised of two components, the magnitude, $\exp\left(M_t^p\right)$, and the sign, $S_t^p$. The logarithm of the magnitude, $M_t^p$, is assumed to be Gaussian, with noisy measure $\widetilde{M}_{*,t}^p$. This measure is computed by taking the logarithm of the absolute value of the *signed* jump size measure,

$\widetilde{Z}^p_{*,t}$, so that the sign (equated for all four of these approaches to the sign of the daily return) is removed.[5] This then allows the observed sign of the daily return, $\widetilde{S}^p_{*,t}$, to be viewed (separately) as a noisy measure of the true, but latent sign, $S^p_t$, as per (32). The measure $\widetilde{S}^p_{*,t}$ correctly detects a negative sign with probability $s_n$, and correctly detects a positive sign with probability $1 - s_p$. We also investigate the implications of using $I^p_{*,t}$, $\widetilde{M}^p_{*,t}$ and $\widetilde{S}^p_{*,t}$, as *exact* representations of $\Delta N^p_t$, $M^p_t$ and $S^p_t$, respectively; i.e. by re-estimating the model with an MCMC algorithm that treats these three latent jump components as being observed without error.

Finally, we note that, again in common with the empirical analysis undertaken in Maneesoonthorn *et al.* (2017), a combination of noninformative and weakly informative priors for the various unknown parameters are adopted. Certain details can be found in Appendix A of that paper, with the full prior specification available from the authors on request. Importantly, whilst this has not been documented in a formal way here, some preliminary investigations of the impact of the prior specification on the numerical results recorded below has been undertaken, with reasonable modifications to the base priors not found to exert any qualitative impact on the results.

## 5.2 Numerical results

Table 4 records the MPMs and HPD intervals for all parameters of the model specified in equations (28)-(38), with the true values recorded in the first column of the table. Due to the nature of the exercise, with the four nonparametric measures, $\widehat{IV}_{*,t}$, $I^p_{*,t}$, $\widetilde{M}^p_{*,t}$ and $\widetilde{S}^p_{*,t}$, having been constructed directly from the high frequency data, and then used as 'observed' measures in the relevant measurement equations, (29) to (32), the parameters of these four equations, $\psi_0$, $\psi_1$, $\sigma_{BV}$, $\beta$, $\alpha$, $\sigma_{M_p}$, $s_n$ and $s_p$, are simply estimated from the data, with there being no reference to 'true values'.

As is clear, the MPMs are all quite close to the corresponding true parameters and the latter fall within the 95% HPD intervals in most cases. One could deduce from this overall result that inference - in this form of model at least - is reasonably robust to the use of different high frequency measures/tests: an encouraging result for practitioners! Nevertheless, there *are* certain qualifications to this overall conclusion. For a start, the estimates (both point and interval) of the average jump size ($\mu_p$) produced using the CPR and ASJ measures, substantially *under*estimate the true value. Simultaneously, these two measures produce point estimates that *over*estimate the unconditional mean for the (discretized) Hawkes intensity process, $\delta^p_0 = \frac{\alpha_p \delta^p_\infty}{\alpha_p - \beta_p}$. The HPD intervals produced using the ASJ measures, for both of these parameters, are also qualitatively larger than those produced using the BNS, CPR and MEDRV measures, as is that for the parameter governing self-intensity, $\beta_p$, while the point estimate of the latter yielded by the ASJ measures is markedly smaller than the true value. The estimates of the parameters in the lower panel of Table 4 provide information regarding aspects of the measurement error implicit in each of the four nonparametric

[5] Note that when the relevant $\widetilde{Z}^p_{*,t}$ equals zero, which happens when $RV_t$ is less than or equal to the relevant measure of integrated variance, we do not view the data as providing any information about price jump size, and with $\widetilde{M}^p_{*,t}$ being undefined in this case.

measures, and the extent to which this varies across the four approaches. Notably, for the BNS, CPR and MEDRV measures, $\ln \widehat{IV}_{*,t}$ is seen to be a relatively accurate measure of $\ln V_t$,, as the corresponding estimates of $\psi_0$ and $\psi_1$ are reasonably close to 0 and 1, respectively, although the BNS interval for $\psi_1$ technically just misses the mark. In contrast, the ASJ version of $\ln \widehat{IV}_{*,t}$ is far from its 'true' value, and also results in larger estimates of the measurement error standard deviation, $\sigma_{BV}$, (in terms of the magnitude of the point estimate and the upper bound of the interval estimate) than do the other three versions. On the other hand, the accuracy with which the ASJ version of $\widetilde{M}^p_{*,t}$ measures the logarithmic price jump size, $M^p_t$, is better than for the other three approaches. Most notably however, viewing $\beta$ as the power with which $I^p_{*,t}$ detects jumps, the estimate of this parameter is markedly lower under the ASJ approach, than it is for the other three approaches. These results regarding the relative performance of the ASJ method are in line with the reported accuracy measures in Table 2, in which this approach was seen to perform poorly in terms of detecting price jumps under a Hawkes process (i.e. its value of $\overline{DJ^*}$ is very low), but well in terms of measuring the magnitude of jumps that are correctly detected (i.e. its values of $MSE^*$ are relatively small).

Table 5 records the MPMs and 95% HPD intervals when inference is undertaken assuming $\Delta N^p_t = I^p_{*,t}$, $M^p_t = \widetilde{M}^p_{*,t}$ and $S^p_t = \widetilde{S}^p_{*,t}$, for $t = 1, ..., T$; i.e. when no measurement error is accommodated in the measurement of (latent) jump occurrence, (log) magnitude and sign. The most notable contrast with the corresponding results in Table 4 is that in all cases except for CPR, the measures pick up fewer jumps than they should, resulting in point estimates of the true value of the unconditional price jump intensity $\delta^p_0$ that are too small, and interval estimates whose upper bound is also less than the true value. Only MEDRV manages to report an HPD for $\mu_p$ that contains its true value. The ASJ approach, in particular, produces both point and interval estimates of $\delta^p_0$ and $\mu_p$ that are acutely inaccurate, as well as yielding HPD intervals for the other jump parameters, $\mu_p$, $\sigma_p$ and $\pi_p$, that are considerably wider than is the case for the other three measures, as well as being wider than the corresponding intervals in Table 4. We also note that under the CPR and MEDRV approaches, the apparent accuracy of $\ln \widehat{IV}_{*,t}$ as a measure of $\ln V_t$, remains qualitatively unaffected by the change (between Tables 4 and 5) in the treatment of the jump measures, whereas no such robustness is observed for the ASJ approach, and to a lesser degree using the BNS measures, according to the point and interval estimates of $\psi_0$ and $\psi_1$.

With due consideration taken of the limited nature of this exercise (i.e. as based on a particular data generating process, and a single 'empirical' sample generated therefrom) it seems clear that not all approaches used for price jump detection and measurement result in equally reliable inference, with the ASJ method possibly the least reliable overall of the four methods considered. We further remark that despite the overall robustness of the results to the assumed presence, or not, of noise in price jump detection and measurement, the results do confirm that explicit accommodation of measurement error in the price jump measures (as well as in the volatility measure)

*does improve* inferential accuracy, in particular as it pertains to the price jump process itself. The incorporation of measurement error also, of course, accommodates the impact of microstructure noise on the nonparametric measures. Despite various methods having been proposed to offset the influence of microstructure noise on the construction of realized measures (see Hansen and Lunde, 2006, and Bandi and Russell, 2008, amongst others), such methods still rely on in-fill asymptotics. In a practical setting in which the measures are constructed using a finite number of intraday observations, the effect of such noise is unavoidable, and the adoption of a state space structure with measurement error would seem to be a sensible modelling decision.

# 6 Conclusions

Inferential work undertaken in empirical finance settings is often aimed at characterizing the nature and magnitude of the various risks - including price jump risk - thought to play a role in the dynamics of financial markets. Understanding the potential for each price jump test, and associated jump measure, to distort inferential conclusions is thus vital. This paper provides an extensive evaluation of the multitude of price jump tests, and corresponding jump size (and sign) measures, that are now available to practitioners. Robustness to volatility jumps, the ability to detect and measure sequences of dynamic jumps, and the implications for subsequent inference of adopting different test and measurement approaches, have been the primary focus. Our simulation experiments reveal that the power of some price jump tests is not robust to the presence of volatility jumps, while other tests suffer serious size distortions in the presence of confounding jumps in the variance process. The accuracy of the various aspects of price jump occurrence, magnitude and sign measurement, are investigated, with the CPR test of Corsi, Pirino and Reno (2010) and the MEDRV test of Andersen, Dobrev and Schaumburg (2012) identified as being the most robust overall. The issues that can arise when inaccurate measurements are used in an inferential setting are also investigated in a simulation scenario, by comparing the (true) parameters of an assumed state space process with Bayesian posterior point and interval estimates. In particular, we find that accounting for measurement error in jump measures is important when attempting to accurately infer the features of dynamic price jumps, with the use of such measures to *directly* represent the relevant latent quantities without error leading to less precise estimates of certain parameters.

Table 4: Parameter inference when treating jump measures as observed with error. Under each form of measure (BNS, CPR, MEDRV and ASJ), we report the marginal posterior mean (MPM) and 95% highest posterior density (HPD) credible interval for each of the model parameters. The true parameter values used to generate the data are also reported. As noted in the text, true values for $\alpha$, $\beta$, $s_n$, $s_p$, $\sigma_{M_p}$, $\psi_0$, $\psi_1$ and $\sigma_{BV}$ are not specified.

| | | BNS | | CPR | | MEDRV | | ASJ | |
|---|---|---|---|---|---|---|---|---|---|
| | True | MPM | 95% HPD int | MPM | 95% HPD int | MPM | 95% HPD int | MPM | 95% HPD int |
| $\mu$ | 0.2 | 0.231 | (0.126,0.334) | 0.226 | (0.116,0.328) | 0.244 | (0.139,0.345) | 0.217 | (0.101,0.333) |
| $\gamma$ | -7.9 | -5.903 | (-9.771,-0.794) | -6.163 | (-9.797,-0.820) | -6.450 | (-9.826,-1.086) | -6.062 | (-9.814,-0.837) |
| $\rho$ | 0.00 | -0.018 | (-0.086,0.046) | -0.013 | (-0.079,0.060) | 0.008 | (-0.066,0.080) | 0.008 | (-0.062,0.080) |
| $\kappa$ | 0.03 | 0.023 | (0.013,0.033) | 0.023 | (0.013,0.033) | 0.022 | (0.012,0.032) | 0.034 | (0.010,0.144) |
| $\theta$ | 0.02 | 0.021 | (0.016,0.028) | 0.021 | (0.016,0.027) | 0.021 | (0.016,0.029) | 0.022 | (0.014,0.030) |
| $\sigma_v$ | 0.02 | 0.019 | (0.017,0.020) | 0.019 | (0.017,0.022) | 0.018 | (0.017,0.021) | 0.023 | (0.018,0.043) |
| $\mu_p$ | 1.0 | 0.950 | (0.783,1.099) | 0.698 | (0.526,0.842) | 1.035 | (0.874,1.178) | 0.607 | (0.399,0.911) |
| $\sigma_p$ | 0.5 | 0.457 | (0.353,0.581) | 0.531 | (0.436,0.637) | 0.449 | (0.349,0.560) | 0.586 | (0.479,0.553) |
| $\pi_p$ | 0.5 | 0.497 | (0.418,0.576) | 0.474 | (0.408,0.541) | 0.487 | (0.407,0.567) | 0.471 | (0.379,0.553) |
| $\delta_0^p$ | 0.105 | 0.092 | (0.073,0.117) | 0.131 | (0.107,0.167) | 0.086 | (0.069,0.108) | 0.203 | (0.069,0.491) |
| $\alpha_p$ | 0.094 | 0.131 | (0.061,0.219) | 0.159 | (0.069,0.233) | 0.155 | (0.074,0.234) | 0.132 | (0.028,0.219) |
| $\beta_p$ | 0.059 | 0.047 | (0.021,0.077) | 0.043 | (0.017,0.072) | 0.050 | (0.022,0.083) | 0.013 | ($4.8e^{-4}$,0.077) |
| $\psi_0$ | – | -0.135 | (-0.296,-0.005) | -0.116 | (-0.265,0.008) | -0.264 | (-0.420,-0.118) | -0.797 | (-2.382,-0.114) |
| $\psi_1$ | – | 0.957 | (0.916,0.993) | 0.967 | (0.929,1.000) | 0.931 | (0.887,0.971) | 0.798 | (0.424,0.995) |
| $\sigma_{BV}$ | – | 0.205 | (0.196,0.214) | 0.208 | (0.198,0.218) | 0.202 | (0.191,0.211) | 0.234 | (0.163,0.624) |
| $\beta$ | – | 0.821 | (0.657,0.950) | 0.823 | (0.646,0.948) | 0.823 | (0.667,0.949) | 0.041 | ($9.4e^{-3}$,0.097) |
| $\alpha$ | – | $7.7e^{-4}$ | ($1.8e^{-5}$,$2.9e^{-3}$) | $7.7e^{-4}$ | ($1.8e^{-5}$,$3.0e^{-3}$) | $7.7e^{-4}$ | ($1.9e^{-5}$,$2.8e^{-3}$) | $5.4e^{-4}$ | ($1.6e^{-5}$,$2.0e^{-3}$) |
| $\sigma_{M_p}$ | – | 1.612 | (1.434,1.773) | 1.212 | (1.029,1.372) | 1.600 | (1.425,1.759) | 0.371 | (0.269,0.573) |
| $s_n$ | – | 0.937 | (0.873,0.979) | 0.947 | (0.895,0.983) | 0.936 | (0.871,0.978) | 0.954 | (0.879,0.992) |
| $s_p$ | – | 0.059 | (0.019 ,0.118) | 0.044 | (0.015,0.091) | 0.059 | (0.020,0.117) | 0.040 | ($7.1e^{-3}$,0.358) |

Table 5: Parameter inference when treating jump measures as observed without error. Under each form of measure (BNS, CPR, MEDRV and ASJ), we report the marginal posterior mean (MPM) and 95% highest posterior density (HPD) credible interval for each of the model parameters. The true parameter values used to generate the data are also reported. Note that in this case, the parameters $\alpha$, $\beta$, $s_n$, $s_p$, and $\sigma_{M_p}$ are irrelevant as the price jump measures are treated as observed without error. As in Table 4, true values of $\psi_0$, $\psi_1$ and $\sigma_{BV}$ are not specified.

| | | BNS | | CPR | | MEDRV | | ASJ | |
|---|---|---|---|---|---|---|---|---|---|
| | True | MPM | 95% HPD int | MPM | 95% HPD int | MPM | 95% HPD int | MPM | 95% HPD int |
| $\mu$ | 0.2 | 0.237 | (0.131,0.335) | 0.233 | (0.131,0.333) | 0.249 | (0.146,0.345) | 0.260 | (0.137,0.376) |
| $\gamma$ | -7.9 | -6.087 | (-9.782,-0.957) | -6.378 | (-9.811,-1.050) | -6.830 | (-9.879,-1.497) | -5.945 | (-9.763,-0.723) |
| $\rho$ | 0.00 | -0.028 | (-0.097,0.043) | -0.018 | (-0.091,0.055) | -0.004 | (-0.074,0.063) | -0.002 | (-0.066,0.061) |
| $\kappa$ | 0.03 | 0.022 | (0.013,0.032) | 0.023 | (0.014,0.034) | 0.022 | (0.013,0.032) | 0.023 | (0.014,0.033) |
| $\theta$ | 0.02 | 0.021 | (0.017,0.028) | 0.021 | (0.016,0.027) | 0.020 | (0.016,0.026) | 0.024 | (0.019,0.031) |
| $\sigma_v$ | 0.02 | 0.019 | (0.018,0.020) | 0.019 | (0.017,0.020) | 0.017 | (0.016,0.019) | 0.020 | (0.018,0.021) |
| $\mu_p$ | 1.0 | 0.819 | (0.738,0.901) | 0.644 | (0.565,0.721) | 0.969 | (0.894,1.045) | 0.514 | (-0.010,1.028) |
| $\sigma_p$ | 0.5 | 0.508 | (0.456,0.570) | 0.597 | (0.545,0.657) | 0.455 | (0.405,0.511) | 0.621 | (0.399,1.016) |
| $\pi_p$ | 0.5 | 0.500 | (0.422,0.578) | 0.473 | (0.409,0.538) | 0.489 | (0.411,0.566) | 0.560 | (0.323,0.784) |
| $\delta_0^p$ | 0.105 | 0.076 | (0.065,0.088) | 0.108 | (0.095,0.122) | 0.072 | (0.061,0.083) | 0.004 | (0.001,0.007) |
| $\alpha_p$ | 0.094 | 0.134 | (0.066,0.242) | 0.170 | (0.073,0.235) | 0.173 | (0.077,0.258) | 0.193 | (0.053,0.250) |
| $\beta_p$ | 0.059 | 0.048 | (0.024,0.076) | 0.045 | (0.019,0.074) | 0.052 | (0.024,0.087) | 0.029 | (0.001 ,0.099) |
| $\psi_0$ | – | -0.195 | (-0.321,-0.045) | -0.107 | (-0.310,0.040) | 0.066 | (-0.133,0.255) | -0.152 | (-0.287,0.0175) |
| $\psi_1$ | – | 0.950 | (0.915,0.991) | 0.964 | (0.907,1.003) | 1.002 | (0.948,1.054) | 1.001 | (0.963,1.049) |
| $\sigma_{BV}$ | – | 0.206 | (0.197,0.215) | 0.206 | (0.197,0.216) | 0.202 | (0.192,0.212) | 0.168 | (0.159,0.177) |

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