Two canonical VARMA forms: SCM vis-à-vis Echelon form

George Athanasopoulos
D.S. Poskitt
Farshid Vahid

MONASH University

Outline of presentation

- Motivation for VARMA models
- SCM methodology
  - Tiao and Tsay (1989)
  - Athanasopoulos and Vahid (2006a)
- Echelon form
  - Hannan and Kavalieris (1984)
  - Poskitt (1992)
  - Lütkepohl and Poskitt (1996)
- Comparison of the two forms:
  - Theoretical - Theorem
  - Experimental - Simulations
  - Practical - Forecasting real macroeconomic variables
- Summary of findings and future research

Why VARMA?

- Closer approximations to Wold representation
- Parsimonious representations
  - any invertible VARMA can be represented by an infinite order VAR
  - Athanasopoulos and Vahid (2006b) - VARMA outperformed VAR when forecasting macroeconomic data
- Difficult to Identify
  - *"If univariate ARIMA modelling is difficult then VARMA modelling is even more difficult - some might say impossible!"* - Chatfield

Identification Problem

Consider a bivariate VARMA(1,1)

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix} - \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t-1} \\ \varepsilon_{2,t-1} \end{bmatrix}$$

- $\phi_{21} = \phi_{22} = \theta_{21} = \theta_{22} = 0$
- 2^{nd} row: $y_{2,t} = \varepsilon_{2,t}$
- $y_{2,t-1} = \varepsilon_{2,t-1}$
- ϕ_{12} and θ_{12} are not separately identifiable
- Set either coefficient to zero to achieve identification

1. Redundant parameters
2. Exchangeable models

Definition of a SCM: For a given K-dimensional $VARMA(p, q)$

$$\mathbf{y}_t = \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \ldots + \mathbf{\Phi}_p \mathbf{y}_{t-p} + \eta_t - \mathbf{\Theta}_1 \eta_{t-1} - \ldots - \mathbf{\Theta}_q \eta_{t-q} \quad (1)$$

$$z_{r,t} = {\alpha_r}' \mathbf{y}_t \sim SCM(p_r, q_r)$$

if α_r satisfies

$${\alpha_r}'\mathbf{\Phi}_{p_r} \neq \mathbf{0}^T \text{ where } 0 \leq p_r \leq p$$
$${\alpha_r}'\mathbf{\Phi}_l = \mathbf{0}^T \text{ for } l = p_r + 1, ..., p$$
$${\alpha_r}'\mathbf{\Theta}_{q_r} \neq \mathbf{0}^T \text{ where } 0 \leq q_r \leq q$$
$${\alpha_r}'\mathbf{\Theta}_l = \mathbf{0}^T \text{ for } l = q_r + 1, ..., q$$

SCM Methodology: Find K-linearly independent vectors $\mathbf{A} = (\alpha_1, \ldots, \alpha_K)'$ which transform (1) into

$$\mathbf{A}\mathbf{y}_t = \mathbf{\Phi}_1^* \mathbf{y}_{t-1} + \ldots + \mathbf{\Phi}_p^* \mathbf{y}_{t-p} + \varepsilon_t - \mathbf{\Theta}_1^* \varepsilon_{t-1} - \ldots - \mathbf{\Theta}_q^* \varepsilon_{t-q} \quad (2)$$

where $\mathbf{\Phi}_i^* = \mathbf{A}\mathbf{\Phi}_i, \varepsilon_t = \mathbf{A}\eta_t$ and $\mathbf{\Theta}_i^* = \mathbf{A}\mathbf{\Theta}_i\mathbf{A}^{-1}$

- Series of C/C tests: $E\begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} \begin{bmatrix} y_{1,t} & y_{2,t} \end{bmatrix} [\alpha_1] = \mathbf{0}$

$$\alpha_1' \mathbf{y}_t \sim SCM(0, 0)$$

MONASH University

Example: $K = 3$

$$\begin{aligned} \alpha_1' \mathbf{y}_t &\sim SCM(1,1) \\ \alpha_2' \mathbf{y}_t &\sim SCM(1,0) \\ \alpha_3' \mathbf{y}_t &\sim SCM(0,0) \end{aligned}$$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \mathbf{y}_t = \begin{bmatrix} \phi_{11}^{(1)} & \phi_{12}^{(1)} & \phi_{13}^{(1)} \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} & \phi_{23}^{(1)} \\ 0 & 0 & 0 \end{bmatrix} \mathbf{y}_{t-1} + \varepsilon_t - \begin{bmatrix} \theta_{11}^{(1)} & \theta_{12}^{(1)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \varepsilon_{t-1}$$

- Normalise diagonally (test for improper normalisations)
- Reduce parameters of **A**
- Canonical SCM

$$\begin{bmatrix} 1 & 0 & 0 \\ \alpha_{21} & 1 & 0 \\ \alpha_{31} & \alpha_{32} & 1 \end{bmatrix} \mathbf{y}_t = \begin{bmatrix} \phi_{11}^{(1)} & \phi_{12}^{(1)} & \phi_{13}^{(1)} \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} & \phi_{23}^{(1)} \\ 0 & 0 & 0 \end{bmatrix} \mathbf{y}_{t-1} + \varepsilon_t - \begin{bmatrix} \theta_{11}^{(1)} & \theta_{12}^{(1)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \varepsilon_{t-1}$$

Echelon Form Definition: A $K-$dimensional *VARMA* representation

$$(\boldsymbol{\Phi}_0 - \boldsymbol{\Phi}_1 L - \ldots - \boldsymbol{\Phi}_p L^p)\mathbf{y}_t = (\boldsymbol{\Theta}_0 - \boldsymbol{\Theta}_1 L - \ldots - \boldsymbol{\Theta}_q L^q)\varepsilon_t$$

is in Echelon form if $[\boldsymbol{\Phi}(L) : \boldsymbol{\Theta}(L)]$ is left coprime and

1. $\boldsymbol{\Phi}_0 = \boldsymbol{\Theta}_0$ is lower triangular with unit diagonal elements,
2. row r of the polynomial operators $[\boldsymbol{\Phi}(L) : \boldsymbol{\Theta}(L)]$ is of maximum degree k_r,
3. the operators have the form:

$$\phi_{rr}(L) = 1 - \sum_{j=1}^{k_r} \phi_{rr}^{(j)} L^j \quad \text{for} \quad r = 1, \ldots, K,$$

$$\phi_{rc}(L) = - \sum_{j=k_r-k_{rc}+1}^{k_r} \phi_{rc}^{(j)} L^j \quad \text{for} \quad r \neq c,$$

$$\theta_{rc}(L) = \theta_{rc}^{(0)} - \sum_{j=1}^{k_r} \theta_{rc}^{(j)} L^j \text{ with } \theta_{rc}^{(0)} = \phi_{rc}^{(0)} \text{ for } r, c = 1, \ldots, K.$$

where $\phi_{rc}^{(j)}$ specifies the element of $\boldsymbol{\Phi}_j$ in row r and column c.

MONASH University

Echelon Form Definition: A $K-$dimensional *VARMA* representation

$$(\boldsymbol{\Phi}_0 - \boldsymbol{\Phi}_1 L - \ldots - \boldsymbol{\Phi}_p L^p)\mathbf{y}_t = (\boldsymbol{\Theta}_0 - \boldsymbol{\Theta}_1 L - \ldots - \boldsymbol{\Theta}_q L^q)\varepsilon_t$$

is in Echelon form if $[\boldsymbol{\Phi}(L) : \boldsymbol{\Theta}(L)]$ is left coprime and

1. $\boldsymbol{\Phi}_0 = \boldsymbol{\Theta}_0$ is lower triangular with unit diagonal elements,
2. row r of the polynomial operators $[\boldsymbol{\Phi}(L) : \boldsymbol{\Theta}(L)]$ is of maximum degree k_r,
3. the operators have the form:

$$\phi_{rr}(L) = 1 - \sum_{j=1}^{k_r} \phi_{rr}^{(j)} L^j \quad \text{for} \quad r = 1, \ldots, K,$$

$$\phi_{rc}(L) = - \sum_{j=k_r-k_{rc}+1}^{k_r} \phi_{rc}^{(j)} L^j \quad \text{for} \quad r \neq c,$$

$$\theta_{rc}(L) = \theta_{rc}^{(0)} - \sum_{j=1}^{k_r} \theta_{rc}^{(j)} L^j \text{ with } \theta_{rc}^{(0)} = \phi_{rc}^{(0)} \text{ for } r, c = 1, \ldots, K.$$

where $\phi_{rc}^{(j)}$ specifies the element of $\boldsymbol{\Phi}_j$ in row r and column c.

Kronecker Indices (k_r): the maximum row degrees $\mathbf{k} = (k_1, \ldots, k_K)'$ which define the structure of the system, and

$$k_{rc} = \begin{cases} \min(k_r + 1, k_c) & \text{for} \quad r \geq c \\ \min(k_r, k_c) & \text{for} \quad r < c \end{cases}, \qquad (3)$$

for $r, c = 1, \ldots, K$, specifies the number of free parameters in the operator $\phi_{rc}(L)$ for $r \neq c$.

- Search through Kronecker indices for optimal model
- Poskitt(1992) search procedure and BIC
- Canonical Reverse Echelon form representations

Example: $\mathbf{k} = (k_1, k_2, k_3)' = (1, 1, 0)'$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \phi_{31}^{(0)} & \phi_{32}^{(0)} & 1 \end{bmatrix} \mathbf{y}_t = \begin{bmatrix} \phi_{11}^{(1)} & \phi_{12}^{(1)} & \phi_{13}^{(1)} \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} & \phi_{23}^{(1)} \\ 0 & 0 & 0 \end{bmatrix} \mathbf{y}_{t-1} + \mathbf{\Theta}_0 \varepsilon_t - \begin{bmatrix} \theta_{11}^{(1)} & \theta_{12}^{(1)} & 0 \\ \theta_{21}^{(1)} & \theta_{22}^{(1)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \varepsilon_{t-1}.$$

Theorem: Consider a VARMA process $\mathbf{y}_t$, represented in canonical reverse Echelon form with Kronecker indices $\mathbf{k} = (k_1, \ldots, k_K)'$. Now suppose that $\mathbf{y}_t$ is also represented in a canonical SCM representation that consists of $K-$SCMs of orders $s_r = (p_r, q_r)$ for $r = 1, \ldots, K$.

$$\mathbf{k} = (k_1, \ldots, k_K)' = \mathbf{s}^{\max} = (s_1^{\max}, \ldots, s_K^{\max})'$$

where $s_r^{\max} = \max(p_r, q_r)$ for $r = 1, \ldots, K$.

Example 1: Consider $\mathbf{y}_t \sim SCM(1,1), SCM(1,1), SCM(0,0)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_{31} & \alpha_{32} & 1 \end{bmatrix} \mathbf{y}_t = \begin{bmatrix} \phi_{11}^{(1)} & \phi_{12}^{(1)} & \phi_{13}^{(1)} \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} & \phi_{23}^{(1)} \\ 0 & 0 & 0 \end{bmatrix} \mathbf{y}_{t-1} + \varepsilon_t - \begin{bmatrix} \theta_{11}^{(1)} & \theta_{12}^{(1)} & 0 \\ \theta_{21}^{(1)} & \theta_{22}^{(1)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \varepsilon_{t-1}.$$

$$\mathbf{k} = \mathbf{s}^{\max} = (\max(1,1), \max(1,1), \max(0,0))' = (1,1,0)$$

Example 2: Consider $\mathbf{y}_t \sim SCM(1,1), SCM(1,0), SCM(0,0)$

$$\begin{bmatrix} 1 & 0 & 0 \\ \alpha_{21} & 1 & 0 \\ \alpha_{31} & \alpha_{32} & 1 \end{bmatrix} \mathbf{y}_t = \begin{bmatrix} \phi_{11}^{(1)} & \phi_{12}^{(1)} & \phi_{13}^{(1)} \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} & \phi_{23}^{(1)} \\ 0 & 0 & 0 \end{bmatrix} \mathbf{y}_{t-1} + \varepsilon_t - \begin{bmatrix} \theta_{11}^{(1)} & \theta_{12}^{(1)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \varepsilon_{t-1}.$$

$$\mathbf{k} = \mathbf{s}^{\max} = (\max(1,1), \max(1,0), \max(0,0))' = (1,1,0)$$

Theorem: Consider a VARMA process $\mathbf{y}_t$, represented in canonical reverse Echelon form with Kronecker indices $\mathbf{k} = (k_1, \ldots, k_K)'$. Now suppose that $\mathbf{y}_t$ is also represented in a canonical SCM representation that consists of $K-$SCMs of orders $s_r = (p_r, q_r)$ for $r = 1, \ldots, K$.

$$\mathbf{k} = (k_1, \ldots, k_K)' = \mathbf{s}^{\max} = (s_1^{\max}, \ldots, s_K^{\max})'$$

where $s_r^{\max} = \max(p_r, q_r)$ for $r = 1, \ldots, K$.

Example 1: Consider $\mathbf{y}_t \sim SCM(1,1)$, SCM(1,1), $SCM(0,0)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_{31} & \alpha_{32} & 1 \end{bmatrix} \mathbf{y}_t = \begin{bmatrix} \phi_{11}^{(1)} & \phi_{12}^{(1)} & \phi_{13}^{(1)} \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} & \phi_{23}^{(1)} \\ 0 & 0 & 0 \end{bmatrix} \mathbf{y}_{t-1} + \varepsilon_t - \begin{bmatrix} \theta_{11}^{(1)} & \theta_{12}^{(1)} & 0 \\ \theta_{21}^{(1)} & \theta_{22}^{(1)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \varepsilon_{t-1}.$$

$$\mathbf{k} = \mathbf{s}^{\max} = (\max(1,1), \max(1,1), \max(0,0))' = (1,1,0)$$

Example 2: Consider $\mathbf{y}_t \sim SCM(1,1)$, SCM(1,0), $SCM(0,0)$

$$\begin{bmatrix} 1 & 0 & 0 \\ \alpha_{21} & 1 & 0 \\ \alpha_{31} & \alpha_{32} & 1 \end{bmatrix} \mathbf{y}_t = \begin{bmatrix} \phi_{11}^{(1)} & \phi_{12}^{(1)} & \phi_{13}^{(1)} \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} & \phi_{23}^{(1)} \\ 0 & 0 & 0 \end{bmatrix} \mathbf{y}_{t-1} + \varepsilon_t - \begin{bmatrix} \theta_{11}^{(1)} & \theta_{12}^{(1)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \varepsilon_{t-1}.$$

$$\mathbf{k} = \mathbf{s}^{\max} = (\max(1,1), \max(1,0), \max(0,0))' = (1,1,0)$$

Identifying VARMA(p,q): SCM methodology

p	q 0	1	2	3
0	?			
1				
2				
3				

Identifying VARMA(p,q): SCM methodology

	q			
p	0	1	2	3
0	✓	?		
1	?			
2				
3				

Identifying VARMA(p,q): SCM methodology

p	q 0	1	2	3
0	✓	✓	?	
1	✓	?		
2	?			
3				

Identifying VARMA(p,q): SCM methodology

p \ q	0	1	2	3
0	✓	✓	✓	?
1	✓	✓	?	
2	✓	?		
3	?			

Identifying VARMA(p,q): SCM methodology

p	q 0	1	2	3
0	✓	✓	✓	?
1	✓	✓	?	
2	✓	?		
3	?			

Identifying VARMA(p,q): Echelon form

p	q 0	1	2	3
0	?			
1				
2				
3				

Identifying VARMA(p,q): SCM methodology

p \ q	0	1	2	3
0	✓	✓	✓	?
1	✓	✓	?	
2	✓	?		
3	?			

Identifying VARMA(p,q): Echelon form

p \ q	0	1	2	3
0	✓			
1		?		
2				
3				

MONASH University

Identifying VARMA(p,q): SCM methodology

p \ q	0	1	2	3
0	✓	✓	✓	?
1	✓	✓	?	
2	✓	?		
3	?			

Identifying VARMA(p,q): Echelon form

p \ q	0	1	2	3
0	✓			
1		✓		
2			?	
3				

MONASH University

Identifying VARMA(p,q): SCM methodology

p \ q	0	1	2	3
0	✓	✓	✓	?
1	✓	✓	?	
2	✓	?		
3	?			

Identifying VARMA(p,q): Echelon form

p \ q	0	1	2	3
0	✓			
1		✓		
2			✓	
3				?

Monte Carlo Simulations: Identify some prespecified VARMA processes

PANEL A: *DGP of equation* (22)

N	SCM			Echelon	
	M.O.	E.O.	$\mathbf{k}_{SCM}$	M.O.	E.O.
100	-	-	-	-	-
150	-	-	-	-	-
200	100	96	100	100	100
400	-	-	-	-	-

SCMs - (1,0)(1,0)(1,0)

PANEL D: *DGP of equation* (25)

N	SCM			Echelon	
	M.O.	E.O.	$\mathbf{k}_{SCM}$	M.O.	E.O.
100	88	52	88	97	49
150	94	78	94	99	82
200	96	94	96	100	95
400	100	86	100	100	100

SCMs - (1,1)(1,0)(0,0)

PANEL G: *DGP of equation* (28)

N	SCM			Echelon	
	M.O.	E.O.	$\mathbf{k}_{SCM}$	M.O.	E.O.
100	96	10	96	93	88
150	92	18	92	94	94
200	98	20	98	97	97
400	94	62	94	97	97

SCMs - (1,1)(1,0)(1,0)

PANEL H: *DGP of equation* (29)

N	SCM			Echelon	
	M.O.	E.O.	$\mathbf{k}_{SCM}$	M.O.	E.O.
100	80	2	80	86	86
150	94	2	94	91	91
200	96	-	96	93	93
400	98	2	98	97	97

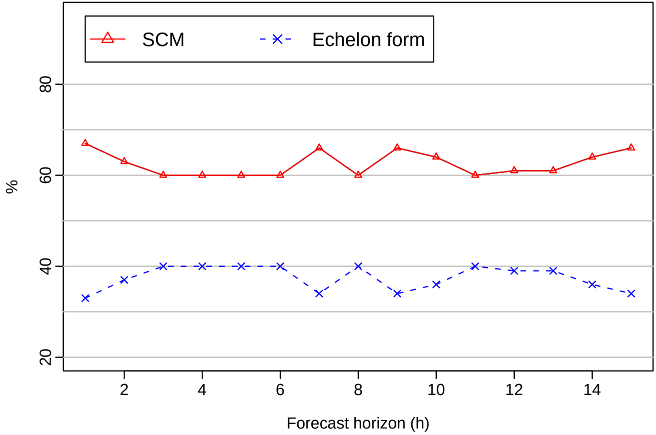
SCMs - (1,1)(1,1)(1,1)

Forecasting: 40 monthly macroeconomic variables from 8 general categories of economic activity, 1959:1-1998:12 (N=480)

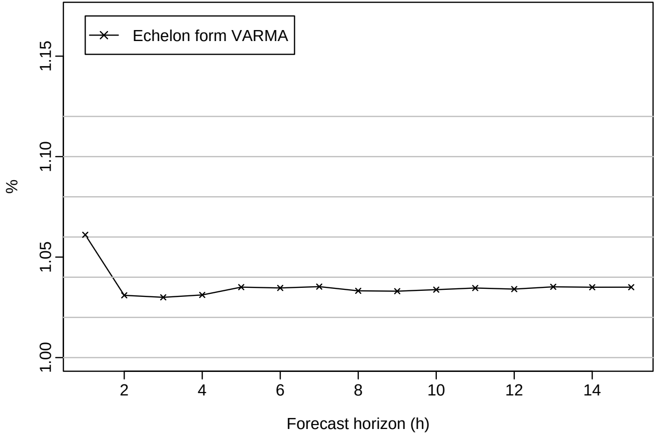
- 70×3 variable systems
- Test sample: $N_1 = 300$
  - Estimated canonical SCM VARMA
  - Estimated canonical reverse Echelon form VARMA
- Hold-out sample: $N_2 = 180$
  - Produced $N_2 - h + 1$ out-of-sample forecasts for each h=1 to 15
  - Forecast error measures: $|MSFE|$ $(tr(MSFE))$
  - Percentage Better: PB_h
  - Relative Ratios:
    - $\overline{RRdMSFE}_h = \frac{1}{70} \sum_{i=1}^{70} \frac{|MSFE_i^{ECH}|}{|MSFE_i^{SCM}|}$

MONASH University

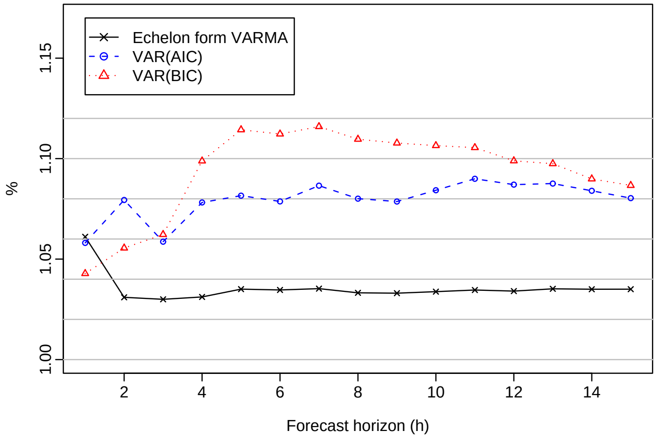
Percentage Better:



Relative Ratios:



Relative Ratios:



Summary:

1. Theoretical: SCM more flexible than Echelon form
2. Simulations: both perform well
3. Forecasting: SCM outperformed Echelon form
    - However both outperformed VARs

Future Research:

1. Closer to automation via Echelon form than SCM
2. Find automatic process to refine the model