1 Bonnor-Ebert spheres

1.1 Hydrostatic equilibrium

Starting with the equations of motion for a compressible, self-gravitating gas,

\[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \frac{P}{\rho} - \nabla \Phi, \]

(1)

together with the Poisson equation for the gravitational field

\[ \nabla^2 \Phi = 4\pi G \rho, \]

(2)

show that, in hydrostatic equilibrium (\( \mathbf{v} = 0 \)) and for an isothermal equation of state

\[ P = c_s^2 \rho, \]

(3)

where \( c_s = \text{const} \), the equations reduce to the form (in spherical coordinates)

\[ \frac{d \ln \rho}{dr} = -\frac{GM(r)}{c_s^2 r^2}, \]

(4)

\[ \frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \left[ M(r) = \int_0^r 4\pi \rho(r') r'^2 dr' \right] \]

(5)

1.2 Numerical solution

1.2.1 Solutions for \( \rho_c = \text{const} \)

Write a short program to integrate Eqs. (4) and (5) numerically (using any standard method for integrating ODEs), assuming a given central density \( \rho(0) = \rho_c \).
• Plot the resultant density profile as a function of radius (in parsecs), using a central
density reasonable for molecular cloud cores (e.g. $\rho_c \sim 10^{-17}$ g/cm$^3$) and a typical
sound speed (e.g. $c_s = 2 \times 10^4$ cm/s).

• Plot the resulting density profiles for a range of central densities (e.g. $\rho_c = 10^{-19} -
-10^{-14}$ g/cm$^3$). Plot all the solutions on the same graph using logarithmic axes
for both the x and y axes. How does the solution change as the central density is
increased?

• Plot (with log axes) pressure as a function of radius for the above solutions. Define
an outer radius based on when the pressure falls below a threshold value and compare
the size of the sphere to typical sizes inferred for molecular cloud “cores” ($\sim$ 0.1 pc).
Discuss how an outer boundary condition on the pressure could be used instead of
giving the central density in order to obtain a solution.

• Integrate each of the solutions from $r = 0$ to your outer radius to find the mass of
the sphere (in solar masses) in each case.

• Discuss (briefly) how one might compare the density profile of a Bonnor-Ebert sphere
to observations that measure only the column density.

1.2.2 The singular isothermal sphere

Verify (by substitution into the equations) that the ‘Singular Isothermal Sphere’, given by

$$\rho(r) = \frac{c_s^2}{2\pi G} \frac{1}{r^2},$$

(6)
is a solution to eqs. (4) and (5). What is the central density for this solution? Plot (6)
on a log-log plot alongside the solutions obtained previously given the same sound speed.
Discuss whether or not you expect the Singular Isothermal Sphere to be a stable or unstable
equilibrium.