The Jeans Instability

1.1 Stability analysis

Starting with the equations of fluid dynamics expressing mass and momentum conservation for a compressible, self-gravitating gas,

\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{v}, \tag{1}
\]

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla P}{\rho} - \nabla \Phi, \tag{2}
\]

together with the Poisson equation for the gravitational field

\[
\nabla^2 \Phi = 4\pi G \rho, \tag{3}
\]

show that the dispersion relation for linear perturbations is given by

\[
\omega^2 = c_s^2(k^2 - k_J^2), \tag{8}
\]

1Hint: Assume that there are small perturbations of the form

\[
\mathbf{v} = \mathbf{v}_0 + \delta \mathbf{v}, \tag{4}
\]

\[
\rho = \rho_0 + \delta \rho, \tag{5}
\]

\[
\delta P = c_s^2 \delta \rho, \tag{6}
\]

and linearise the equations to first order in the perturbations (i.e., neglect terms that are second order and higher). Take \(\frac{\partial}{\partial t}(1)\) and \(\nabla \cdot (2)\) and, assuming \(\rho_0 = \) constant, combine these equations with (3) to obtain the equation

\[
\frac{\partial^2}{\partial t^2}(\delta \rho) = c_{s,0}^2 \nabla^2 (\delta \rho) + \rho_0 4\pi G \delta \rho. \tag{7}
\]

Now, assume perturbations of the form \(\delta \rho = D \rho_{0} e^{i(k \cdot r - \omega t)}\), and substitute into (7) to obtain the dispersion relation.
where the Jeans wavenumber is given by

\[ k_J^2 = \frac{4\pi G \rho_0}{c_s^2}. \]  

(9)

1.2 Jeans length and Jeans mass

- What is the critical wavenumber at which perturbations will become unstable (grow exponentially)?
- Express this critical wavenumber as a wavelength. Evaluate this length scale for a typical molecular cloud density and sound speed.
- Express the critical wavenumber in terms of a critical mass for gravitational stability. Evaluate this typical mass given typical molecular cloud densities and sound speed.
- What is the wavelength for maximum growth in the Jeans instability?
- Discuss the implications of the dependence of the Jeans length and mass on density and temperature. What do you expect would be the effect of heating on star formation?