ASP2062 Introduction to Astrophysics

Planet formation II — Daniel Price

Key revision points

1. Friction between orbiting rings causes them to spread into a disc
2. Accretion discs transport mass inwards and angular momentum outwards, transferring all of the angular momentum to a small amount of mass
3. Discs are in hydrostatic equilibrium in the vertical direction, in general showing a 'flared' profile depending on the radial temperature gradient
4. The 'friction' can be understood as an effective viscosity caused by turbulence

2.1 Ring spreading in discs

In the last lecture we derived the basic equation governing accretion discs in the form

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)\right]. \tag{1}$$

Lynden-Bell and Pringle (1974) famously derived an exact solution to this equation for the simple case of a discrete ring of matter m at some initial radius R_0, i.e.

$$\Sigma(R, t = 0) = \frac{m}{2\pi R_0}\delta(R - R_0), \tag{2}$$

where δ is the Dirac delta function[1].

[1] A mathematical oddity; the delta function is defined according to

$$\delta(x - x_0) \equiv \begin{cases} 1; & x = x_0, \\ 0; & \text{elsewhere.} \end{cases} \tag{3}$$

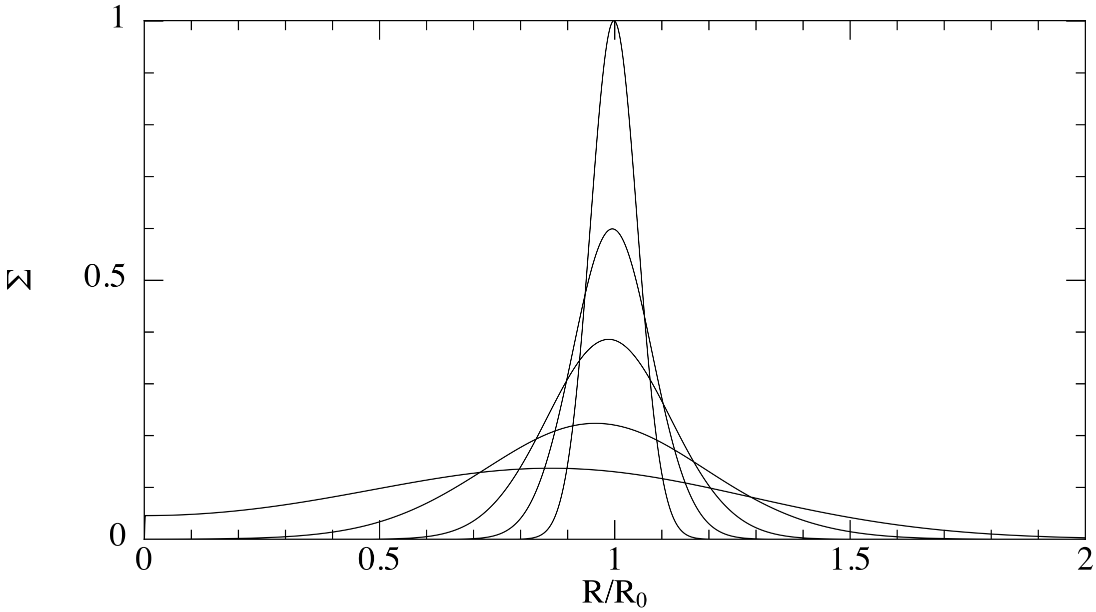

Figure 1: Ring spreading in an accretion disc, according to the analytic solution derived by Lynden-Bell and Pringle (1974). Figure by DJP. Click here for a movie.

What do you mean by a "diffusion equation"?

Equation (1) looks complicated, but it is actually just a form of the well-known heat equation

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial X^2}, \tag{4}$$

where $X \equiv 2R^{1/2}$, $f \equiv 3\Sigma R^{1/2} = \frac{3}{2}\Sigma X$ and $D \equiv 3\nu/R = 12\nu/X^2$.

The analytic solution in terms of $x = R/R_0$ and $\tau = 12\nu t/R_0^2$ for this case is

$$\Sigma(x,\tau) = \frac{m}{\pi R_0^2}\frac{1}{\tau x^{1/4}} \exp\left[\frac{-(1+x)^2}{\tau}\right] I_{1/4}\left(\frac{2x}{\tau}\right), \tag{5}$$

where $I_{1/4}$ is a modified Bessel function[2].

This simple solution *demonstrates all of the key physics of accretion discs.* Figure 1 shows snapshots of surface density as a function of time, according to (5). You can see that the action of viscosity causes the ring to *spread* or *diffuse.* The majority of the mass moves *inwards*, losing energy and transferring angular momentum to a small amount of matter that spreads *outwards* to large R.

Now, recall that since the angular momentum $L \propto R^{1/2}$, then for material spreading to $R = \infty$ we have $L \to \infty$. So, while most of the mass is transported *inwards* and is

[2]Don't let this scare you; the point is that there are standard routines to compute these kind of functions e.g. in MATHEMATICA or MATLAB or with a simple Fortran routine.

accreted onto the star, the angular momentum is transported *outwards* by a small amount of matter, being ultimately carried to infinite radius by none of the mass.

2.2 Vertical structure of discs

Perpendicular to the disc, there is essentially no flow. The disc is supported against gravity by a pressure gradient, i.e. it is in *hydrostatic equilibrium*. In cylindrical coordinates we can express this as

$$\frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{\partial \Phi}{\partial z}, \tag{6}$$

where P is the pressure and Φ is the gravitational potential. For a central point mass we have

$$\Phi = -\frac{GM}{r} = -\frac{GM}{\sqrt{R^2 + z^2}}, \tag{7}$$

where we use $r \equiv \sqrt{x^2 + y^2 + z^2}$ to denote the spherical radius and $R \equiv \sqrt{x^2 + y^2}$ to denote the radius in cylindrical coordinates. Using (7) in (6) we have

$$\frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{GMz}{(R^2 + z^2)^{\frac{3}{2}}}. \tag{8}$$

If we make the assumption of a *thin disc*, i.e. that $z \ll R$, then we have

$$\frac{1}{\rho}\frac{\partial P}{\partial z} \approx -\Omega^2(R) z, \tag{9}$$

where $\Omega(R) = (GM/R)^{3/2}$ is the Keplerian angular speed. If we make the further assumption that the disc is *vertically isothermal*, i.e. that temperature depends only on R and not z, such that $P = c_{\rm s}^2(R)\rho$, where $c_{\rm s}^2(R) \equiv k_{\rm B}T(R)/\mu m_{\rm H}$ then we have

$$c_{\rm s}^2 \frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\Omega^2 z, \tag{10}$$

giving

$$\frac{\mathrm{d}(\ln\rho)}{\mathrm{d}z} = -\frac{\Omega^2 z}{c_{\rm s}^2}. \tag{11}$$

Integrating both sides, assuming $\rho = \rho_c$ at $z = 0$ (i.e. at the midplane), we have

$$\rho = \rho_c \exp\left(\frac{-\Omega^2 z^2}{2c_{\rm s}^2}\right) = \rho_c \exp\left(\frac{-z^2}{2H^2}\right), \tag{12}$$

where we define H as the *pressure scale height*, i.e.

$$H \equiv \frac{c_s}{\Omega}. \tag{13}$$

Hence the vertical disc structure is just a Gaussian in the z direction, but with an aspect ratio H/R that depends on radius.

How to define the size of a continuous distribution?

The pressure scale height gives the "size" of the disc in the vertical direction. But the disc does not have a sharp edge. Our definition uses the standard deviation of the density profile in the z direction to define the 'height'. This is similar to definitions used elsewhere in Astronomy, e.g. the full-width-at-half-maximum used to define the 'width' of spectral lines.

2.3 Disc flaring

For a Keplerian disc, $\Omega(R) \propto R^{-3/2}$. If we assume that sound speed, and hence temperature, depends on R to some power q, i.e. $c_s(R) \propto R^{-q}$ so that $T(R) \propto R^{-2q}$ then we have

$$H \propto R^{\frac{3}{2}-q}, \tag{14}$$

and therefore that the disc *aspect ratio* H/R scales as

$$\frac{H}{R} \propto R^{\frac{1}{2}-q}. \tag{15}$$

This means the disc will "flare" (aspect ratio increases with radius) if $q < \frac{1}{2}$, i.e. if $T(R) \propto R^{-1}$ (temperature inversely proportional to radius) or shallower. Most circumstellar discs satisfy this requirement and hence appear flared (see Figure 2).

2.4 Disc viscosity

We have not yet discussed what *actually* makes accretion discs accrete. That is, what *causes* the friction between rings, i.e. the disc viscosity ν? We can estimate the timescale for the disc to drain onto the star, known as the *viscous timescale* using dimensional analysis, and compare this to observed disc lifetimes ($\sim$ 10 Myr). Since $[\nu] = L^2/T$, to get a time we need an area divided by ν, i.e.

$$t_\nu \sim \frac{R^2}{\nu}, \tag{16}$$

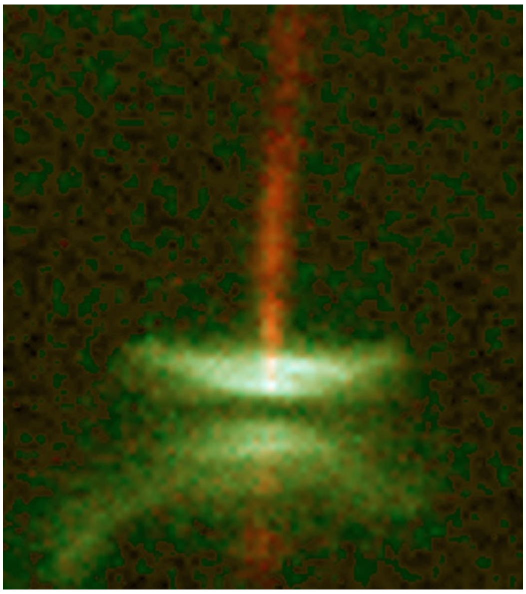

Figure 2: Disc and jet around the young stellar object HH30, imaged with the Hubble Space Telescope. The disc itself is obscured by dust, the green emission shows the starlight scattered off the disc surface. The flared profile of the disc is seen in silhouette.

If we assume $R = 10\text{AU}$ and an estimate of the microscopic viscosity in the gas (see box),

Are discs like honey?

If ν was an actual, microscopic viscosity then to order of magnitude we'd have

$$\nu \sim \lambda c_{\rm s}, \tag{17}$$

where λ is the mean free path of the gas and $c_{\rm s}$ is the sound speed (around 500 m/s in typical protostellar discs). We can estimate λ using

$$\lambda = \frac{1}{n\sigma_{\rm mol}}, \tag{18}$$

where n is the number density of molecules ($\rho \approx 10^{-13}$ g/cm^3 gives $n \approx 10^{10}$ cm^{-3}) and $\sigma_{\rm mol}$ is the collision cross section between the molecules (around 2×10^{-15}cm^2 for H_2). Hence $\lambda \approx 5 \times 10^4$ cm (i.e. 500m) and $\nu \approx 2 \times 10^9$ cm^2/s.

we have

$$t_\nu \sim \frac{(10\text{AU} \times 1.5 \times 10^{13}\text{cm/AU})^2}{2 \times 10^9\text{cm/s}} = 3 \times 10^{11}\text{yr}, \tag{19}$$

which is longer than the age of the universe! Hence whatever is causing the disc to accrete cannot be *actual* viscosity. The low microscopic viscosity provides a clue, since it implies a large *Reynolds number*

$$R_{\rm e} \equiv \frac{LV}{\nu}, \tag{20}$$

where L is a typical length scale and V is a typical velocity. Using $L = H \sim 0.05 \times 1\text{AU}$ and $V = c_\text{s} \sim 5 \times 10^4$ cm/s, we have

$$R_\text{e} \approx \frac{0.05 \times 1.5 \times 10^{13} \times 5 \times 10^4}{2.5 \times 10^9} \approx 1.5 \times 10^7. \tag{21}$$

High Reynolds numbers are usually an indication of *turbulence*. Then we can understand ν as an *effective turbulent viscosity*.

2.4.1 The Shakura-Sunyaev prescription

For turbulence in disc we would expect motions with $V \lesssim c_\text{s}$ and 'eddies' that are smaller than H in size. Hence it makes sense to express ν according to

$$\nu = \alpha c_\text{s} H, \tag{22}$$

where $\alpha \in [0, 1]$ is a dimensionless parameter. This is known as the *Shakura-Sunyaev prescription*, or simply the *alpha model* after Shakura and Sunyaev (1973) and is the bedrock of much of our theoretical understanding of accretion discs.

What causes turbulence in discs?

Despite the high Reynolds numbers, hydrodynamical Keplerian discs are stable to small perturbations and hence do not spontaneously become turbulent. Our current understanding is that turbulence arises in most accretion discs due to an instability caused by magnetic fields in differentially rotating flows, the *magneto-rotational instability* (Balbus and Hawley, 1991). However this is problematic in protoplanetary discs due to the low ionisation fraction. How protostellar discs actually accrete is therefore an open question.

References

Balbus, S. A. and J. F. Hawley: 1991, 'A powerful local shear instability in weakly magnetized disks. I - Linear analysis. II - Nonlinear evolution'. ApJ **376**, 214–233.

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