ASP2062 Introduction to Astrophysics

Planet formation I — Daniel Price

Key revision points

1. Angular momentum increases with radius in a Keplerian flow
2. Conservation of angular momentum leads to the formation of accretion discs
3. We can derive a simple diffusion equation describing the evolution of discs from the conservation of mass and angular momentum

2 Protoplanetary discs

Planet formation is a topic being revolutionised by recent observations from the ALMA telescope (Fig 1). It is a hugely exciting time in the field. We will try to cover the basics in ASP2062 to understand what we are now seeing with our own eyes.

2.1 The angular momentum problem

Conservation of angular momentum means that it is actually very difficult to accrete material onto stars. In general, material will start to orbit the star at a given radius R. The orbital speed is given from Newton's law of motion $\mathbf{F} = m\mathbf{a}$, considering the force due to gravity and the acceleration due to uniform circular motion according to

$$-\frac{GMm}{R^2}\hat{\mathbf{r}} = -\frac{mv_\phi^2}{R}\hat{\mathbf{r}}, \tag{1}$$

giving the orbital speed as

$$v_\phi = \pm\sqrt{\frac{GM}{R}}, \tag{2}$$

or, equivalently, the angular speed

$$\Omega(R) \equiv \frac{v_\phi}{R} = \sqrt{\frac{GM}{R^3}}. \tag{3}$$

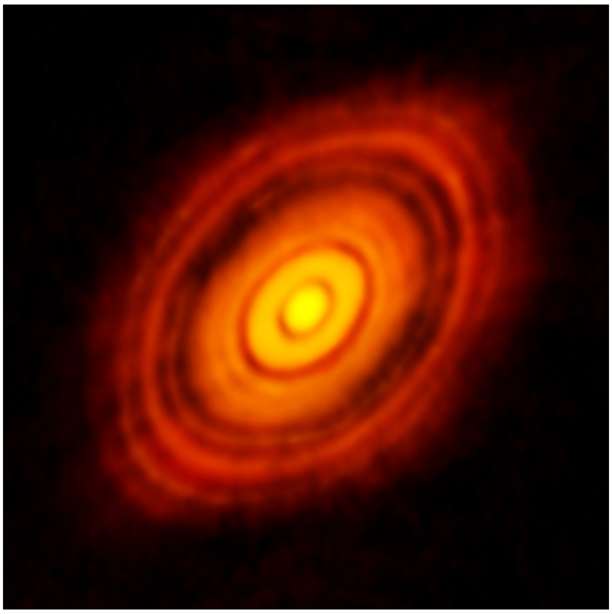

Figure 1: The accretion disc around the young star HL Tau, imaged at mm-wavelengths in Oct 2014 by the ALMA telescope. This is the first ever high resolution image of an accretion disc in astrophysics.

This is known as *Keplerian rotation*. Consider the specific angular momentum L, given by

$$L = Rv_\phi = R^2\Omega(R) = \sqrt{GMR}. \tag{4}$$

We see that angular momentum *increases* with radius, so material must *lose* angular momentum to land on the star. However, angular momentum is conserved in the universe, so cannot be truly lost but only transported onto (given to) different material. Nature achieves this by forming a rotating disc of material known as an *accretion disc*.

Angular velocity and angular momentum

How does the radial dependence of angular velocity compare to that of the specific angular momentum? One of the unusual features of Keplerian flow is that angular momentum increases with radius but angular velocity *decreases*. Thus material on wider orbits moves at a slower speed but carries more angular momentum.

2.2 Accretion discs

Our derivation of the accretion disc equations follows the famous review article by Pringle (1981). It is also covered in the textbooks by Frank et al. (2002) and Armitage (2010).

Since discs are essentially two dimensional, it is useful to discuss disc physics in terms of two dimensional quantities. We define the *surface density* of material in the disc at a given annulus as

$$\Sigma(R,\phi,t) = \int_{-\infty}^{\infty} \rho(R,\phi,z,t)\mathrm{d}z, \tag{5}$$

where $R \equiv \sqrt{x^2+y^2}$ is the radius in cylindrical coordinates. A good way to think about surface density vs. density is that $[\rho] = \mathrm{g/cm^3}$ while $[\Sigma] = \mathrm{g/cm^2}$.

2.2.1 Mass and angular momentum

Consider an annulus of material in a disc between a radius R and $R + \Delta R$. We have:

$$\text{Mass of annulus} = \text{surface density} \times \text{area} = 2\pi R \Delta R \Sigma, \tag{6}$$

$$\text{Angular momentum of annulus} = ML = 2\pi R \Delta R \Sigma R^2 \Omega, \tag{7}$$

where we used the specific angular momentum (4).

2.2.2 Conservation of mass

The rate of change of mass is equal to the net flow of material in and out from neighbouring annuli, i.e.

$$\frac{\partial}{\partial t}\left(2\pi R \Delta R \Sigma\right) = 2\pi R \Sigma(R,t) v_R(R,t) - 2\pi(R+\Delta R)\Sigma(R+\Delta R,t)v_R(R+\Delta R,t), \tag{8}$$

where v_R is the radial component of velocity and we assume an axisymmetric disc such that Σ does not depend on ϕ, i.e. $\Sigma = \Sigma(R,t)$. We can turn this into a differential equation by expanding $v_R(R+\Delta R)$ and $\Sigma(R+\Delta R)$ in a Taylor series about R, i.e.

$$v_R(R+\Delta R,t) = v_R(R,t) + \Delta R \frac{\partial v_R}{\partial R}(R,t) + \mathcal{O}\left(\Delta R^2\right), \tag{9}$$

$$\Sigma(R+\Delta R,t) = \Sigma(R,t) + \Delta R \frac{\partial \Sigma}{\partial R}(R,t) + \mathcal{O}\left(\Delta R^2\right). \tag{10}$$

We assume that ΔR is small, neglecting terms of order ΔR^2 to give the term in (8) as

$$\Sigma(R+\Delta R,t)v_R(R+\Delta R,t) = v_R\Sigma(R) + \Delta R\left[\Sigma\frac{\partial v_R}{\partial R} + v_R\frac{\partial \Sigma}{\partial R}\right] + \mathcal{O}(\Delta R^2), \tag{11}$$

$$= v_R\Sigma + \Delta R\frac{\partial}{\partial R}\left(\Sigma v_R\right). \tag{12}$$

From (8) we then have

$$R\frac{\partial \Sigma}{\partial t} = -v_R\Sigma - R\frac{\partial}{\partial R}\left(\Sigma v_R\right), \tag{13}$$

giving our final differential equation for mass conservation in the form

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\Sigma v_R\right) = 0. \tag{14}$$

2.2.3 Conservation of angular momentum

Following the same procedure for the conservation of angular momentum between annuli, we end up with a differential equation of the form

$$R\frac{\partial}{\partial t}\left(\Sigma R^2 \Omega\right) + \frac{\partial}{\partial R}\left(R\Sigma v_R R^2 \Omega\right) = \frac{1}{2\pi}\frac{\partial G}{\partial R}, \tag{15}$$

where $G(R,t)$ is the torque of an outer annulus acting on a neighbouring inner one at radius R. The physical interpretation of the terms in (15) is that the *rate of change of angular momentum* (first term) is determined by the *change in surface density due to radial flow* (2nd term) and by the *difference in torque applied by stresses at the inner and outer edge* (right hand side; $\partial G/\partial R$).

If we suppose that neighbouring annuli exert "friction" on each other, the torque G is

$$G = \underbrace{2\pi R}_{\text{circumference}} \times \underbrace{\nu \Sigma R \frac{\mathrm{d}\Omega}{\mathrm{d}R}}_{\text{viscous force per unit length}} \times \underbrace{R}_{\text{lever arm}}, \tag{16}$$

where ν is the viscosity coefficient, with dimensions of area per unit time. Then (15) becomes

$$\frac{\partial}{\partial t}\left(\Sigma R^2 \Omega\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(R\Sigma v_R R^2 \Omega\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu R^3 \Sigma \frac{\mathrm{d}\Omega}{\mathrm{d}R}\right). \tag{17}$$

Now here's the magic: We can combine (17) and (14) into a single equation for the time evolution of Σ by eliminating v_R. We can write (17) as

$$R^2\Omega\left[\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\Sigma v_R\right)\right] + \Sigma v_R \frac{\partial}{\partial R}\left(R^2\Omega\right) = \text{RHS}, \tag{18}$$

from which we can eliminate the first term using (8), giving

$$v_R = \frac{\frac{\partial}{\partial R}\left(\nu R^3 \Sigma \frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)}{R\Sigma \frac{\partial}{\partial R}(R^2\Omega)}. \tag{19}$$

If the disc is Keplerian then $\mathrm{d}\Omega/\mathrm{d}R = -\frac{3}{2}(GM)^{1/2}R^{-5/2}$ and $\partial/\partial R(R^2\Omega) = \frac{1}{2}(GM)^{1/2}R^{-1/2}$ and we have

$$v_R = \frac{-3}{\Sigma R^{1/2}}\frac{\partial}{\partial R}\left(\nu\Sigma R^{1/2}\right). \tag{20}$$

Finally, substituting this in (8) we have

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(\nu\Sigma R^{1/2}\right)\right]. \tag{21}$$

We will unpack the physics of this equation — by solving it — in the next lecture.

References

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