Smoothed Particle Hydrodynamics: When you should, when you shouldn’t (or: things I wish my mother taught me)
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SPH starts here...

What is the density?
The SPH density estimate

Kernel-weighted sum:

\[ \rho(r) = \sum_{j=1}^{N} m_j W(|r - r_j|, h) \]

e.g. \[ W = \frac{\sigma}{h^3} e^{-r^2/h^2} \]

Resolution follows mass

Grid

SPH

\[ \frac{dx}{dt} = \mathbf{v} \]
From density to hydrodynamics

\[ L_{sph} = \sum_j m_j \left[ \frac{1}{2} v_j^2 - u_j(\rho_j, s_j) \right] \quad \text{Lagrangian} \]

\[ du = \frac{P}{\rho^2} d\rho \quad \text{1st law of thermodynamics} \]

\[ \nabla \rho_i = \sum_j m_j \nabla W_{ij}(h) \quad \text{density sum} \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) - \frac{\partial L}{\partial r} = 0 \quad \text{Euler-Lagrange equations} \]

\[ \frac{dv_i}{dt} = -\sum_j m_j \left( \frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla_i W_{ij}(h) \quad \text{equations of motion!} \]

What this gives us: Advantages of SPH

- An exact solution to the continuity equation
- Resolution follows mass
- ZERO dissipation
- Advection done perfectly
- EXACT conservation of mass, momentum, angular momentum, energy and entropy
- A guaranteed minimum energy state
Zero dissipation

Zero dissipation - Example I.

Propagation of a circularly polarised Alfvén wave
Zero dissipation - II. Advection of a current loop

1000 crossings (Rosswog & Price 2007)  
2 crossings (Gardiner & Stone 2005)

SPH grid

Zero dissipation...
Zero dissipation... so we have to add some

Must treat EVERY discontinuity

Viscosity + Conductivity
But must treat discontinuities properly...

This issue has NOTHING to do with the Kelvin-Helmholtz instability

Richtmyer-Meshkov Instability

Exploding blob (Børve & Price 2010)
dissipation terms need to be explicitly added
The key is a good switch

Figure 2. As Fig. 1, but for SPH with standard ($\alpha = 1$) or Morris & Monaghan (1997) artificial viscosity, as well as our new method (only every fifth particle is plotted). Also shown are the undamped wave (solid) and lower-amplitude sinusoids (dashed). Only with our method the wave propagates undamped, very much like SPH without any viscosity, as in Fig. 1.

6 Lee Cullen & Walter Dehnen

Figure 6. Steepening of a 1D sound wave: velocity and viscosity parameter vs. position for standard SPH, the M&M method, our new scheme, and Godunov particle hydrodynamics of first and second order (GPH, Cha & Whitworth 2003), each using 100 particles per wavelength. The solid curve in the top panel is the solution obtained with a high-resolution grid code.

Cullen & Dehnen (2010)

Exact conservation
Exact conservation: Advantages

Orbits are orbits... even when they’re not aligned with any symmetry axis.

Lodato & Price (2010)

Exact conservation: Disadvantages

- Calculations keep going, even when they’re screwed up...

In grid codes, “screwing it up” => CRASH

In SPH, “screwing it up” => NOISE

How to fix this

```c
if (particles_are_noisy()) {
    die();
}
```

```c
if (particles_are_noisy())
    stop
endif
```

```c
if (particles ^ AnyofP("noise") ):
    die('sorry, your SPH code crashed, we are not AMUSEd')
```

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The minimum energy state

The “grid” in SPH...

What happens to a random particle arrangement?

\[
\frac{dv_i}{dt} = -\sum_j m_j \left( \frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla_i W_{ij}
\]

SPH particles know how to stay regular
Why “rpSPH” (Morris 1996, Abel 2010) is a bad idea

\[
\frac{dv_i}{dt} = \sum_j m_j \left( \frac{P_i - P_j}{\rho_j^2} \right) \nabla_i W_{ij}
\]

Improving the gradient operator leads to WORSE results

Corollary: Better to use a worse but conservative gradient operator

Corollary: Need positive pressures
Compromise approach gives stability

Subtract $-\mathbf{B}(\nabla \cdot \mathbf{B})$ from MHD force:

Stable but nonconservative

2D shock tube

- intrinsic “remeshing” of particles

![Graphs showing pressure, velocity, etc. over time and space.](image)
Why you cannot use “more neighbours” (or: How to halve your resolution)

\[ N_{\text{neigh}} \text{ should NOT be a free parameter!} \]

i.e., should not change the ratio of smoothing length to particle spacing

pairing occurs for > 65 neighbours for the cubic spline in 3D

2D shock tube

- use smoother quintic kernel - truncated at 3h instead of 2h (NOT the same as “more neighbours” with the cubic spline)
Grid vs. SPH: Turbulence

Turbulence in the Interstellar Medium

- highly supersonic, Mach numbers \( \sim 5-20 \)
- isothermal to good approximation
- unknown driving mechanism, but “large scale”
- super-Alfvenic - magnetic fields mildly important
- statistics of turbulence may determine statistics of star formation (e.g. Padoan & Nordlund 2002, Hennebelle & Chabrier 2008)

\[ \text{Larson (1981)} \]

\[ \text{Goldsmith et al. (2008)} \]
GRID vs. SPH

Padoan et al. (2007), commenting on Ballesteros-Paredes et al. (2006):

“...The complete absence of an inertial range with a reasonable slope, or with a reasonable dependence of the slope on the Mach number, makes their SPH simulations totally inadequate for testing the turbulent fragmentation model...”

...but low resolution SPH (58^3)

Fig. 8.—Power spectra compensated for the slope of the Stagger code HD run, $\beta = 1.9$. The TVD and SPH power spectra are the same as in Fig. 2 of Ballesteros-Paredes et al. (2006) for the Mach numbers 3 and 6.

Price & Federrath (2010): Comparison of driven turbulence
Particle penetration and high Mach number shocks

Take care with viscosity at high Mach numbers!

**TURBULENCE: Theory**

- Kolmogorov (1941):
  \[ \dot{E} = \frac{\eta v^3}{L} = \text{const} \]
  \[ v_L \propto L^{1/3} \]
  \[ E_{\text{kin}} \propto v^2_L \propto L^{2/3} \propto k^{-2/3} \]
  \[ E(k) = \frac{dE_{\text{kin}}}{dk} \propto k^{-5/3} \]
  (for incompressible turbulence)

- Kritsuk et al. (2007):
  \[ \dot{E} = \frac{\eta \rho v^3}{L} = \text{const} \]
  \[ \rho^{1/3} v_L \propto L^{1/3} \]
  \[ (\rho^{1/3} v_L)^2 \propto L^{2/3} \propto k^{-2/3} \]
  \[ E(k) = \frac{d(\rho^{1/3} v_L)^2}{dk} \propto k^{-5/3} \]
  (for compressible and supersonic turbulence)
Kinetic energy spectra (time averaged)

Burgers-like $k^{-2}$ spectrum in the kinetic energy for Mach 10 hydro

Price & Federrath (2010)

Density-weighted energy spectra ($\rho^{1/3}v$)

Confirms Kritsuk et al. (2007) suggestion of Kolmogorov-like $k^{-5/3}$ spectrum in this variable

Price & Federrath (2010)
You get what you pay for (i.e., need high resolution in any method)

But SPH resolution is in density field

Price & Federrath (2010)
Density PDFs:

Summary: Advantages and disadvantages of SPH

Advantages:
- Resolution follows mass
- Zero dissipation until explicitly added
- Exact and simultaneous conservation of all physical quantities is possible
- Intrinsic remeshing procedure
- Does not crash

Disadvantages:
- Resolution follows mass
- Dissipation terms must be explicitly added to treat discontinuities
  - methods can be crude (need a good switch)
- Exact conservation no guarantee of accuracy
- Need to be careful with effects from particle remeshing
- Screw-ups indicated by noise rather than code crash

But remember: You get what you pay for!