SPH +/- MHD

The state of the art in the dark arts

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Gingold & Monaghan (1977):
Magnetic polytropes with SPH
Phillips & Monaghan (1985): SPH with MHD in conservative form is unstable when beta < 1

Morris (1996): Stability analysis: a compromise solution to stabilise the force equation with “almost-conservative” form
Dolag, Bartelmann & Lesch (1999): SPH+MHD applied to galaxy clusters (beta >> 1)

Børve, Omang & Trulsen (2001): Regularised SPH: very nice MHD shocks by "remeshing" of particles

Screw conservative form: Gradient Particle Magnetohydrodynamics (Maron & Howes 2003)

Hosking & Whitworth (2004): use non-conservative formulation but first to implement two fluid (ion/neutral)

Price & Monaghan 2004a (paper I): dissipative terms for MHD shocks
Price & Monaghan 2004b (paper II): we could do strong shocks (variable smoothing length formulation)

Price & Monaghan 2005 (paper III): How to handle the divergence constraint (using IGNORE or CLEAN approach)

advection of a current loop (Gardiner & Stone 2006, Rosswog & Price 2007)

good results on test problems...

Orszag-Tang vortex problem (PM05, Rosswog & Price 2007)

Magnetic rotor problem (PM05)

Price & Monaghan 2005 (paper III): How to handle the divergence constraint (using IGNORE or CLEAN approach)

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \quad \nabla \cdot \mathbf{B} = 0 \]
...but didn’t work so well for star formation

\[ \mathbf{B} = \nabla \alpha \times \nabla \beta \]

(satisfies \( \nabla \cdot \mathbf{B} = 0 \) by construction: the PREVENT approach)

Enter the Euler potentials
(cf. Phillips & Monaghan 1985)

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \]
\[ \frac{d\alpha}{dt} = 0 \]
\[ \frac{d\beta}{dt} = 0 \]

(advection of magnetic field lines by Lagrangian particles)
Price & Bate (2007): Effect of magnetic fields on single and binary star formation (all with supercritical field strengths)

... problem forming discs in the presence of magnetic fields?
see also Hennebelle & Fromang (2008), Hennebelle & Ciardi (2009), Mellon & Li (2009), Duffin & Pudritz (2009)

Price & Bate (2007): Effect on binary formation

... a fragmentation crisis?

cf. also Hennebelle & Teyssier (2009), Mellon & Li (2008), Machida et al. (2008) and others

Price (2008): We got unhelpfully distracted by a discussion on Kelvin-Helmholtz instabilities...
Price & Bate (2009): Effect of magnetic fields and radiation on star cluster formation

net effect is a very much reduced star formation rate / efficiency per $t_{ff}$

Dolag & Stasyszyn (2009):
SPH+MHD makes it’s way into GADGET

application to galaxy clusters, magnetic field evolution and dynamics in spiral galaxies (Kotarba et al. 2009, Stasyszyn et al. 2010)
Limitations of the Euler potentials approach

\[ B = \nabla \alpha \times \nabla \beta \]
\[ \frac{d\alpha}{dt} = 0 \]
\[ \frac{d\beta}{dt} = 0 \]

- advection of magnetic fields: no change in topology \((A \cdot B = 0)\)
- does not follow wind-up of magnetic fields
- difficult to model resistive effects -- reconnection processes not treated correctly

Axel Brandenburg (at KITP 2007): “Why don’t you just use the vector potential?”

is \( B = \nabla \times A \) a better approach?

\[ \frac{\partial A}{\partial t} = v \times B + \nabla \phi \quad \Rightarrow \quad \frac{dA}{dt} = -A_i \nabla v^i \]
2.4.4 Equations of motion

Putting the perturbations (31) and (33) [the second term of which has been expanded into (38) and (39)] into (13) we have

$$\int \left\{ -m_i \frac{d^2 \mathbf{x}_i}{dt^2} - \sum_m m_i \mathbf{F}_i^{m} \mathbf{x}_i - \frac{3}{2} \mathbf{v}_i \left( \mathbf{v}_i \right)^2 + \frac{3}{2} \sum_m \mathbf{m}^i \mathbf{D}^m_i \right\} = \sum_m m_i \mathbf{F}_i^{m} \mathbf{x}_i$$

Price 2010 (paper IV): 25 pages* of pain later, we had derived the ultimate vector potential formulation in SPH...

...it was beautiful, derived elegantly from a Lagrangian variational principle, the method was exactly conservative, novel, the divergence was constrained...

$$\delta \int L \, dt = 0$$

where $$L^g$$ are the (potentially many) integrals of the variational principle of least action is satisfied by the equations of motion in the form

$$\frac{d}{dt} \left( \frac{\mathbf{p}_i}{\mu_i} \right) = - \sum_m \mathbf{F}_i^{m} \mathbf{x}_i - \frac{3}{2} \mathbf{v}_i \left( \mathbf{v}_i \right)^2 + \frac{3}{2} \sum_m \mathbf{m}^i \mathbf{D}^m_i$$

and therefore quite reasonable — but only for a finite time

$$\text{...could this be the ultimate method for MHD in SPH?}$$

where the current $$\mathbf{j}^\alpha$$ is defined according to

$$\int \mathbf{j}^\alpha \, dt = 0$$

1st problem: same old numerical instability (in 2D and 3D)

...and did it work?

2nd problem: unconstrained growth of non-physical components of $$\mathbf{A}$$ in 3D problems
“Axel, the answer is no.”

(for an interesting reason)

...namely that the magnetic Galilean limit of Maxwell’s equations requires enforcement of div $A = 0$, similar to the original constraint on the B field.

(Price 2010, note submitted to J. Comp. Phys.)

Current directions on SPH+MHD

- generalised Euler potentials method
  
  $B = \nabla \alpha_1 \times \nabla \beta_1 + \nabla \alpha_2 \times \nabla \beta_2 + \nabla \alpha_3 \times \nabla \beta_3$

  $B = \nabla \alpha_1 \times \nabla X_0 + \nabla \alpha_2 \times \nabla Y_0 + \nabla \alpha_3 \times \nabla Z_0$

  Allows remapping procedure (reconnection “by hand”):

  $\alpha_1^* = \alpha_1 \nabla \beta_1$
  $\alpha_2^* = \alpha_2 \nabla \beta_2$

  $[\beta_1^*, \beta_2^*, \beta_3^*] = [X_i, Y_i, Z_i]$

- exact implementation of projection method for div $B$

  $B^* = B - \nabla \phi$
  $\nabla^2 \phi = \nabla \cdot B$

  was tried by PM05, but with only approximate solution. With exact method might be better

- two fluid implementation (ions/neutrals)

completely independent of the ideal MHD implementation

why haven’t we finished all this yet?
Padoan et al. (2007):

“Numerical simulations can ... account for ... turbulence in ... star formation only if they can generate an inertial range of turbulence, which requires both low numerical diffusivity and large numerical resolution. Furthermore... the magnetic field cannot be neglected”

“SPH simulations of large scale star formation to date fail in all three fronts: numerical diffusivity, numerical resolution, and presence of magnetic fields. This should cast serious doubts on the value of comparing predictions based on SPH simulations with observational data (see also Agertz et al. 2006).”

Comparison of Mach 10, hydro turbulence

SPH=PHANTOM  grid=FLASH
Kinetic energy spectra

Burgers-like $k^2$ spectrum in the kinetic energy for Mach 10 hydro

Price & Federrath (2010)
Density-weighted energy spectra \( (\rho^{1/3}v) \)

Confirms Kritsuk et al. (2007) suggestion of Kolmogorov-like \( k^{-5/3} \) spectrum in this variable

Price & Federrath (2010)

Density resolution

max density in SPH at \( 128^3 \) similar to max grid density at \( 512^3 \)

Price & Federrath (2010)
**Probability Distribution Functions**

![Graph showing log PDF of \(\log(\rho/\rho_0)\) vs. \(\log(\rho/\rho_0)\).](image)

- \(\rho_0\) is the reference density.
- \(\rho\) is the density.
- The log PDF distribution is shown for different grid configurations.
- Price & Federrath (2010)

**Density variance -- Mach number relation**

![Graph showing density variance \(\sigma^2\) vs. Mach number.](image)

- \(\sigma^2 = \ln(1 + b^2 M^2)\), \(b = 0.5\)
- \(\sigma^2 = \ln(1 + b^2 M^2)\), \(b = 0.33\)
- Lemaster & Stone best fit
- Price, Federrath & Brunt (2010, in prep)
Trying to measure the (linear) density variance

If log normal, expect

\[ \sigma^2 / \rho = e^{\sigma^2 s'} - 1 \]

\[ s' \equiv \log(\rho / \bar{\rho}) \]

Comparison to observations

need COMPRESSIVE DRIVING or GRAVITY

Brunt (2010)
(based on new method for inferring 3D variance from 2D observations)
(see Brunt, Federrath and Price 2010)

Price, Federrath & Brunt (2010, in prep)
Conclusions

• being a television presenter is easier than getting MHD in SPH to work

• MHD in SPH would work if people stopped making unsubstantiated swipes* at SPH

• Magnetic fields can significantly change star formation even at supercritical field strengths, so we need MHD in SPH

• SPH and grid codes agree very well on the statistics of turbulence when the resolutions are comparable: nparts = ncells to get similar spectra, but SPH much better at resolving dense structures.

• The standard-deviation– Mach number relation in supersonic turbulence seems robust up to Mach 20, but observed density variances are much higher than can be produced with solenoidally-driven turbulence alone

*defined as any paper where the criticism is based purely on a citation to Agertz et al. (2006)