Smoothed Particle Hydrodynamics: Turbulence and MHD

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Turbulence in the Interstellar Medium

- highly supersonic, Mach numbers ~ 5-20

- isothermal to good approximation

- unknown driving mechanism, but “large scale”

- super-Alfvenic - magnetic fields mildly important

- statistics of turbulence may determine distribution of stellar masses (IMF) (Padoan & Nordlund 2002)

Larson (1981)
A simple approach is to study isothermal turbulence in periodic box, driven artificially in fourier space at “large scales”

- previous disagreement between SPH and grid codes (Padoan et al., 2007; Ballesteros-Peredes et al., 2006)

- but based on very low resolution SPH simulations (~$58^3$ particles)
Smoothed Particle Hydrodynamics

\[ \rho(\mathbf{r}) = \sum_{j=1}^{N} m_j W(|\mathbf{r} - \mathbf{r}_j|, h) \]
SPH (PHANTOM) vs. Grid (FLASH)
SPH vs. Grid
Max density
Power spectra

- Kinetic energy goes like $k^{-2}$ - “Burgulence”
A new universality?

- Kritsuk et al. (2007) suggest $\rho^{1/3} v$ should scale like Kolmogorov ($k^{-5/3}$)

![Graphs showing the scaling behavior of different simulations.](image)

- Some support for this, however not much inertial range even at $512^3$
Grid (FLASH)

t=0.05  grid, $128^3$

t=0.1

t=0.15
Tracer particles, with SPH density calculation

$t=0.05$ grid tracers, $128^3$

$t=0.1$

$t=0.15$ grid tracers, $512^3$
SPH (PHANTOM)

- $t=0.05$, SPH, $128^3$
- $t=0.1$, SPH, $256^3$
- $t=0.15$, SPH, $512^3$
PDFs

![Graph showing PDFs with different configurations: SPH, 128^3, SPH, 256^3, SPH, 512^3, grid, 128^3, grid, 256^3, grid, 512^3. The x-axis represents rho ranging from 10^-5 to 1000, and the y-axis represents PDF (ln rho) ranging from 10^-11 to 0.1.](image)
PDFs with tracer particles - I

![PDF plot with legend]

- SPH, $128^3$
- SPH, $256^3$
- SPH, $512^3$
- grid, $128^3$
- grid, $256^3$
- grid, $512^3$
- tracers, $128^3$
- tracers, $256^3$
Tracer particles tend to get “stuck” at high densities (follow the mass, but don’t feel any differential forces below the grid scale)
MHD
Smoothed Particle Magnetohydrodynamics

Four main issues:

- numerical instability related to $B(\text{div} B)$ term in conservative MHD force (particles attract unstoppably) (Phillips & Monaghan 1985)
  
  Morris (1996), Borve et al. (2001), Price & Monaghan (2004a)

- formulation of dissipative terms associated with MHD shocks
  
  Price & Monaghan (2004a)

- incorporating variable smoothing length self-consistently
  
  Price & Monaghan (2004b)

- maintenance of the $\text{div} B = 0$ constraint
  
  Price & Monaghan (2005), using divergence cleaning schemes
Euler Potentials / “Clebsch variables”

\[ \mathbf{B} = \nabla \alpha \times \nabla \beta \]
Advantage

\[
\frac{d\alpha}{dt} = 0; \quad \frac{d\beta}{dt} = 0
\]

Induction equation
Disadvantage

\[
\frac{d \alpha}{dt} = 0; \quad \frac{d \beta}{dt} = 0
\]

- mapping from initial->final particle distribution
- field cannot wind more than once around
- difficult to incorporate non-ideal MHD terms
The Vector Potential \( \mathbf{B} = \nabla \times \mathbf{A} \)

\[
\frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times \mathbf{B} - \eta \mathbf{J} + \nabla \phi,
\]

\[
\frac{d\mathbf{A}}{dt} = \mathbf{v} \times \nabla \times \mathbf{A} + (\mathbf{v} \cdot \nabla) \mathbf{A} + \mathbf{v} \times \mathbf{B}_{\text{ext}} - \eta \mathbf{J} + \nabla \phi.
\]

Use Gauge that gives Galilean invariance: \( \phi = \mathbf{v} \cdot \mathbf{A} \)

\[
\frac{d\mathbf{A}}{dt} = -\mathbf{A} \times (\nabla \times \mathbf{v}) - (\mathbf{A} \cdot \nabla) \mathbf{v} + \mathbf{v} \times \mathbf{B}_{\text{ext}} - \eta \mathbf{J}.
\]

Also correct low speed \((v << c)\) and magnetically dominated \((E < cB)\) limit for electromagnetism (de Montigny & Rousseaux 2007, Am. J. Phys 75, 984)
SPMHD with a vector potential

\[
L_{sph} = \sum_b m_b \left[ \frac{1}{2} v_b^2 - u_b(\rho_b, s_b) - \frac{1}{2\mu_0} \frac{B_b^2}{\rho_b} \right].
\]

take \( \delta L = 0 \),

Choose:

\[
B_a = (\nabla \times A)_a + B_{ext} = \frac{1}{\Omega_a \rho_a} \sum_b m_b (A_a - A_b) \times \nabla_a W_{ab}(h_a) + B_{ext},
\]

\[
\frac{dA_a^i}{dt} = \frac{A_a^i}{\Omega_a \rho_a} \sum_b m_b (v_a^i - v_b^i) \frac{\partial W_{ab}(h_a)}{\partial x_a^i} + \epsilon_{ijk} v_a^j B_{ext,a}^k,
\]
Perturbations upon perturbations...

\[
\delta(\rho_b B_b) = \frac{1}{\Omega_b} \sum_c m_c (A_b - A_c) \times [(\delta x_b - \delta x_c) \cdot \nabla] \nabla_b W_{bc}(h_b) \\
+ \frac{1}{\Omega_b} \sum_c m_c (\delta A_b - \delta A_c) \times \nabla_b W_{bc}(h_b) + B_{ext} \delta \rho_b \\
+ \left[ H_b + \frac{B_{b, int} \zeta_b}{\Omega_b} \right] \delta \rho_b + \frac{B_{b, int} \rho_b}{\Omega_b} \frac{\partial h_b}{\partial \rho_b} \sum_c m_c [(\delta x_b - \delta x_c) \cdot \nabla] \frac{\partial W_{bc}(h_b)}{\partial h_b},
\]

\[
\delta A^b_k = A^b_m \frac{1}{\Omega_b \rho_b} \sum_d m_d (\delta x^m_b - \delta x^m_d) \frac{\partial W_{bd}(h_b)}{\partial x^k_b} + \epsilon_{k mn} \delta x^m_b B^n_{ext, b}.
\]
\[ \int \left\{ -m_a \frac{dv^i_a}{dt} - \sum_b \frac{m_b}{\Omega_b} \left[ \frac{P_b}{\rho_b^2} - \frac{3}{2\mu_0} \left( \frac{B_b}{\rho_b} \right)^2 + \frac{\xi_b}{\rho_b^2} \right] \sum_c m_c \frac{\partial W_{bc}(h_b)}{\partial x^i_b} (\delta_{ba} - \delta_{ca}) \right. \]

\[ \left. - \frac{\epsilon_{jkl}}{\mu_0} \sum_b \frac{m_b}{\Omega_b} \frac{B^j_b}{\rho_b^2} \sum_c m_c (A^b_k - A^c_k) \frac{\partial^2 W_{bc}(h_b)}{\partial x^i_b \partial x^l_b} (\delta_{ba} - \delta_{ca}) \right. \]

\[ \left. - \frac{1}{\mu_0} \sum_b \frac{m_b}{\Omega_b} \frac{B^j_b B^{j\int b}}{\rho_b} \frac{\partial h_b}{\partial \rho_b} \sum_c m_c (\delta_{ba} - \delta_{ca}) \frac{\partial^2 W_{bc}(h_b)}{\partial x^i_b \partial h_b} \right. \]

\[ \left. - \frac{1}{\mu_0} \sum_b \frac{m_b}{\Omega_b} \frac{B^j_b}{\rho_b^2} \left[ 2\delta^l_i B_{\text{ext}}^j - \delta^j_i B_{\text{ext}}^l \right] \sum_c m_c \frac{\partial W_{bc}(h_b)}{\partial x^l_b} (\delta_{ba} - \delta_{ca}) \right. \]

\[ \left. - \frac{\epsilon_{jkl}}{\mu_0} \sum_b \frac{m_b}{\Omega_b} \frac{B^j_b}{\rho_b^2} \sum_c m_c \frac{A^b_i}{\Omega_b \rho_b} \left[ \sum_d m_d \frac{\partial W_{bd}(h_b)}{\partial x^k_b} (\delta_{ba} - \delta_{da}) \right] \frac{\partial W_{bc}(h_b)}{\partial x^l_b} \right. \]

\[ \left. + \frac{\epsilon_{jkl}}{\mu_0} \sum_b \frac{m_b}{\Omega_b} \frac{B^j_b}{\rho_b^2} \sum_c m_c \frac{A^c_i}{\Omega_c \rho_c} \left[ \sum_d m_d \frac{\partial W_{cd}(h_c)}{\partial x^k_c} (\delta_{ca} - \delta_{da}) \right] \frac{\partial W_{bc}(h_b)}{\partial x^l_b} \right\} \delta x^i_a dt = 0, \]
Equations of motion

\[
\frac{dv_a^i}{dt} = - \sum_b m_b \left[ \frac{P_a - \frac{3}{2\mu_0} B_a^2}{\rho_a \Omega_a} \frac{\partial W_{ab}(h_a)}{\partial x_a^i} + \frac{P_b - \frac{3}{2\mu_0} B_b^2}{\rho_b \Omega_b} \frac{\partial W_{ab}(h_b)}{\partial x_a^i} \right] \quad \text{isotropic term}
\]

\[
- \frac{\epsilon_{jkl}}{\mu_0} \sum_b m_b (A_k^a - A_k^b) \left[ \frac{B_j^a}{\Omega_a \rho_a} \frac{\partial^2 W_{ab}(h_a)}{\partial x_a^i \partial x_a^l} + \frac{B_j^b}{\Omega_b \rho_b} \frac{\partial^2 W_{ab}(h_b)}{\partial x_a^i \partial x_a^l} \right] \quad \text{2D term}
\]

\[
- \frac{1}{\mu_0} \sum_b m_b \left[ \frac{B_j^a B_{int,a}^j}{\Omega_a \rho_a} \frac{\partial h_a}{\partial \rho_a} \frac{\partial^2 W_{ab}(h_a)}{\partial x_a^i \partial h_a} + \frac{B_j^b B_{int,b}^j}{\Omega_b \rho_b} \frac{\partial h_b}{\partial \rho_b} \frac{\partial^2 W_{ab}(h_b)}{\partial x_a^i \partial h_b} \right] \quad \text{2D } \nabla h \text{ term}
\]

\[
- \frac{1}{\mu_0} \left[ 2\delta_i^l B_j^e - \delta_i^j B_e^\text{ext} \right] \sum_b m_b \left[ \frac{B_j^a}{\Omega_a \rho_a} \frac{\partial W_{ab}(h_a)}{\partial x_a^l} + \frac{B_j^b}{\Omega_b \rho_b} \frac{\partial W_{ab}(h_b)}{\partial x_a^l} \right] \quad \text{2.5D/}B_{\text{ext}} \text{ term}
\]

\[
- \sum_b m_b \left[ \frac{A_i^a}{\Omega_a \rho_a^2} J_a^k \frac{\partial W_{ab}(h_a)}{\partial x_a^k} + \frac{A_i^b}{\Omega_b \rho_b^2} J_b^k \frac{\partial W_{ab}(h_b)}{\partial x_a^k} \right], \quad \text{3D term}
\]
Equations of motion (simplified)

\[
\frac{dv_a^i}{dt} = \sum_b m_b \left[ \left( \frac{S_a^{ij}}{\rho_a^2 \Omega_a} + \frac{(A_{ab} \times B_a)^i}{\mu_0 \rho_a^2 \Omega_a} \frac{\partial}{\partial x_a^i} + \psi_a \delta_j^i \frac{\partial}{\partial h_a} \right) \frac{\partial W_{ab}(h_a)}{\partial x_a^j} + \left( \frac{S_b^{ij}}{\rho_b^2 \Omega_b} + \frac{(A_{ab} \times B_b)^i}{\mu_0 \rho_b^2 \Omega_b} \frac{\partial}{\partial x_a^i} + \psi_b \delta_j^i \frac{\partial}{\partial h_b} \right) \frac{\partial W_{ab}(h_b)}{\partial x_a^j} \right]
\]

\[
S^{ij} \equiv -P \delta^{ij} + \frac{1}{\mu_0} \left[ B^i B^{j}_{\text{ext}} + \delta^{ij} \left( \frac{3}{2} B^2 - 2 \mathbf{B} \cdot \mathbf{B}_{\text{ext}} - \xi \right) \right] - A^i J^j,
\]

- conserves energy, momentum and entropy exactly and simultaneously
Does it work?
With some hacks...

4.2 Brio-Wu problem
• bad using $\eta_{\text{nonlin}}$
• much better with quintic (but NOT with symmetric $J$).
• NOT stabilised by constant stress subtraction.
• it is the formulation of $J$ in $dA/dt$ that matters.

4.3 1.5D shock tubes
Brio-Wu: stabilised by $-B(\text{div } B)$
mshk3: works OK with $-B(\text{div } B)$
mshk7: works OK with $-B(\text{div } B)$

4.4 Circularly polarized Alfvén wave
The circularly polarised Alfvén wave is an exact, non-linear solution of the MHD equations. It is particularly useful as a test problem as it allows one to compute the evolution of a non-linear wave of arbitrary amplitude indefinitely, since the wave does not compress the gas and therefore does not steepen into a shock. The parameters for the test problem used here are identical to those described by Tóth (2000) for Eulerian codes and the setup for SPH is identical to that described in Paper III for the standard SPMHD scheme except that here we set up the magnetic field in terms of the vector potential.

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We can do OK

\[\rho, p, y, B_y, n\]
Circularly polarised Alfven wave, field loop advection (2D)

First 25 crossings

1000 crossings

ZERO dissipation until you add some
What about all that divergence-free wonder?

- 2D test problem: Orszag-Tang Vortex

\[
\begin{align*}
[\nu_x, \nu_y] &= \nu_0 [-\sin(2\pi y), \sin(2\pi x)] \\
[B_x, B_y] &= B_0 [-\sin(2\pi y), \sin(4\pi x)]
\end{align*}
\]
2D Orszag-Tang Vortex: Energy conservation

The graph illustrates the total energy over time for two different formulations:

- **Hacked-together formulation**: represented by the solid black line.
- **Beautiful, consistent energy-conserving formulation**: represented by the dashed red line.

The energy appears to decrease with time, with the hacked-together formulation showing a more pronounced decrease compared to the energy-conserving formulation.
Field lines
Conclusions

On turbulence:

• We find good agreement between SPH and grid codes on the statistics of supersonic turbulence

• SPH does a good job of simulating highly compressible turbulence by placing resolution in high density regions.

• tracer particles have the possibility of dramatically improving the density resolution in grid-based simulations at little extra cost. A hybrid scheme?

On SPMHD:

• vector potential is not a viable approach for MHD in SPH. Numerical instabilities are MUCH WORSE than in the standard approach.

• better to look at generalised versions of the Euler potentials