SIMULATING TURBULENCE
How (not) to do it
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NUMERICAL METHODS

Grid

SPH

Padoan et al. (2007), commenting on Ballesteros-Paredes et al. (2006):

“The complete absence of an inertial range with a reasonable slope, or with a reasonable dependence of the slope on the Mach number, makes their SPH simulations totally inadequate for testing the turbulent fragmentation model...”

Fig. 8.—Power spectra compensated for the slope of the Stagger code HD run, $\beta = 1.9$. The TVD and SPH power spectra are the same as in Fig. 2 of Ballesteros-Paredes et al. (2006) for the Mach numbers 3 and 6.
Price & Federrath (2010): Comparison of driven, Mach 10 turbulence
Kinetic energy spectra (time averaged)

Burgers-like $k^{-2}$ spectrum in the kinetic energy for Mach 10 hydro

Price & Federrath (2010)
Density-weighted energy spectra \( (\rho^{1/3}v) \)

Confirms Kritsuk et al. (2007) suggestion of Kolmogorov-like \( k^{-5/3} \) spectrum in this variable

Price & Federrath (2010)
Density resolution

max density in SPH at $128^3$ similar to max grid density at $512^3$

Price & Federrath (2010)
Density PDFs:
Density variance - Mach number relation in solenoidally-driven turbulence


\[ \sigma_s^2 = \ln(1 + b^2 M^2), \quad b = 0.5 \]

\[ \sigma_s^2 = \ln(1 + b^2 M^2), \quad b = 0.33 \]

Std. dev in log rho

Lemaster & Stone best fit
Trying to measure the (linear) density variance

If log normal, expect

\[ \sigma^2_{\rho} = e^{\sigma^2_s} - 1 \]

\[ s \equiv \log(\rho/\bar{\rho}) \]
Comparison to observations

Brunt (2010) (based on new method for inferring 3D variance from 2D observations)
(see Brunt, Federrath and Price 2010)

need COMPRESSIVE DRIVING or GRAVITY

3. Application to the Taurus Molecular Cloud

We now apply the method to the $^{13}$CO J=1–0 spectral line imaging observations of the Taurus molecular cloud. Figure 2 shows the $^{13}$CO emission integrated over the velocity range [0, 12] km s$^{-1}$ in the Taurus molecular cloud. Following the procedure described in Section 3, we first estimate the variance in the normalized projected field. We assume that the $^{13}$CO integrated intensity, $I$, is linearly proportional to the column density, $N$: the advantages and disadvantages of this assumption are discussed below. Taking $I \propto N$, we calculate $\sigma_{3N}/\sigma_{1N} = \sigma_{3I}/\sigma_{1I}$, where $I_1$ is the mean intensity. The observed variance in the field, $\sigma_{3I} + \sigma_{3N}$, is the sum of the signal variance, $\sigma_{3I}$, and the noise variance, $\sigma_{3N}$. We measure $\sigma_{3I} + \sigma_{3N} = 3.92$ and $\sigma_{3I} = 3.40$ units are all in km s$^{-1}$. With a measured $I_1 = 2.17$ km s$^{-1}$, we then find $\sigma_{3N}/\sigma_{1N} = 3.36$. The power spectrum of the integrated intensity field is now calculated. We use a square field of size 3159 pixels x 3159 pixels in which the map is embedded, and compute the power spectrum using a Fast Fourier Transform. Application of tapers to smoothly roll-off the field edges had an insignificant effect on the results.

**Fig. 1.** Integrated intensity map of the $^{13}$CO J=1–0 line over the velocity range [0, 12] km s$^{-1}$ in the Taurus molecular cloud.
Kainulainen et al. (2009):

PDFs are shown in Fig. 1.

The dispersions of the fitted log-normal functions are shown in Table 1.

The cumulative forms of the PDFs shown in Figs. 2.

While supersonic turbulence is expected to develop a density dispersion in logarithmic units. The fits are shown in Figs.

PDF close to a log-normal distribution, prominent deviations from that shape are predicted in strongly self-gravitating systems (e.g. Federrath et al. 2008a). Recent observational studies have indeed indicated that the column density PDF close to a log-normal distribution, prominent deviations from that shape are predicted in strongly self-gravitating systems.

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PDF close to a log-normal distribution, prominent deviations from that shape are predicted in strongly self-gravitating systems.
WHAT ABOUT LOW MACH NUMBER (ICM/IGM) TURBULENCE?
In contrast, our moving mesh technique does yield power-law scaling laws... consistent with expectations...

We argue that large errors in SPH's gradient estimated and associated subsonic velocity noise are ultimately responsible...

This casts doubt about the reliability of SPH for simulations of cosmic structure formation...

...we find that the widely employed standard formulation of SPH quite badly fails in the subsonic regime...

Instead of building up a Kolmogorov-like turbulent cascade, large-scale eddies are quickly damped close to the driving scale...

ABSTRACT

Highly supersonic, compressible turbulence is thought to modify the thermodynamic structure of gas in virialized dark matter halos and affect small-scale mixing processes in the gas. However, the properties of astrophysical turbulence have been restricted to the supersonic regime, where we focus on comparing the accuracy of our new moving-mesh technique AREPO with previous results, we show that the widely employed standard formulation of SPH quite badly fails in the subsonic regime. Instead of building up a Kolmogorov-like turbulent cascade, large-scale eddies are quickly damped close to the driving scale and decay into small-scale velocity noise. In contrast, our moving-mesh technique does yield power-law scaling laws for the power spectra of velocity, vorticity and density, consistent with expectations for turbulence. We argue that large errors in SPH's gradient estimated and associated subsonic velocity noise are ultimately responsible for producing non-physical results in the subsonic regime. This casts doubt about the reliability of SPH simulations of cosmic structure formation, especially if turbulence in clusters of galaxies is indeed significant. In contrast, SPH's performance is much better for supersonic turbulence, which can be adequately captured with SPH. When comparing our simulations of turbulence, our moving-mesh approach shows good agreement with the non-spherical results, although with somewhat better resolving power at the cost of reduced advection errors and the automatic adaptivity of the mesh.

Key words: hydrodynamics, shock waves, turbulence, methods & numerical particle hydrodynamics and moving-mesh simulations

1 INTRODUCTION

Astrophysical gas dynamics in the interstellar and intergalactic medium is typically characterized by very high Reynolds numbers, thanks to the comparatively low gas densities encountered in these environments, which imply a very low physical viscosity for the involved gas. We may hence expect that turbulent cascades over large dynamic ranges are rather prevalent, provided effective driving processes exist. Such turbulence can then be an important feature of gas dynamics, for example providing an additional effective pressure contribution, or leading to enhanced mixing of chemical elements in the gas.

In fact, it is believed that turbulence in the interstellar medium (ISM) plays a key role in the process of star formation, affecting processes such as the lifetime of molecular clouds, and the overall efficiency of star formation (e.g. Klessen et al. 2000; Mac Low & Klessen 2004). Here the turbulence is highly supersonic, and presumably driven primarily by supernova explosions. In addition, the strong radiative cool-
Figure 3. Visual comparison of the turbulent velocity field (top row), the density field (middle row) and the enstrophy $|\nabla \times \mathbf{v}|^2$ (bottom row) in quasi-stationary turbulence with $\mathcal{M} \sim 0.3$, simulated with different numerical techniques. Shown are thin slices through the middle of the periodic simulation box. From left to right, we show our moving grid result, an equivalent calculation on a static mesh, and an SPH calculation, as labeled.
What’s going on?

Figure 5. Convergence study for the velocity power spectrum of $\mathcal{M} \sim 0.3$ subsonic turbulence. The panel on top shows results for AREPO, from a resolution of $64^3$ to $512^3$ cells. The panel on the bottom gives the corresponding results for SPH. However, even at a high resolution as high $512^3$ particles, no extended inertial range of turbulence can be identified in SPH. The thin grey lines show the power-law expected for Kolmogorov’s theory.
We argue that the origin of this noise lies in errors of SPH’s gradient estimate. Numerous studies have pointed out that the standard approach followed in SPH for constructing derivatives of smoothed fluid quantities involves rather large error terms, especially for the comparatively irregular particle distributions in multi-dimensional simulations. The problem lies in a lack of consistency of the ordinary density estimates (which do not conserve volume, i.e. the sum of $m_i/\rho_i$ is not guaranteed to add up to the total volume) and in a low order of the gradient estimate itself (e.g. Quinlan et al. 2006; Gaburov & Nitadori 2011; Amicarelli et al. 2011). In practice, this means that there can be pressure forces on particles even though all individual pressure values of the particles are equal,
A clue:

To suppress the artificial viscosity in regions of strong shear, Balsara (1995) proposed a simple viscosity limiter in the form of an additional multiplicative factor \((f_i + f_j)/2\) for the viscous tensor, defined as

\[
 f_i = \frac{\mid \nabla \cdot \mathbf{v} \mid_i}{\mid \nabla \cdot \mathbf{v} \mid_i + \mid \nabla \times \mathbf{v} \mid_i}. \tag{3}
\]

This limiter is often used in cosmological SPH simulations and also available in the GADGET code. In our default simulations, we have refrained from enabling it, but we have also run comparison simulations where it is used, as discussed in our results section.
Figure 6. Dissipation power spectra for AREPO and SPH runs at different resolutions compared to the corresponding shape of the velocity power spectrum at 256 resolution. For the mesh code, the dissipation actually peaks on scales where the power spectrum starts to deviate from Kolmogorov’s self-similar scaling. In contrast, SPH shows very strong dissipation already on larger scales, preventing the buildup of a turbulent cascade. In addition, the dissipation is also strong on small scales, close to the resolution limit, where the small-scale noise developing in SPH is constantly damped away.

In the top panel of Figure 2, we compare the velocity power spectra of these two simulations with the S8 simulation. Note that at the resolution of 128^3 employed for these tests, the S0 run with 80 neighbors is expected to have effectively the same mass and spatial resolution as the S8 simulation with our default choice of -smoothing neighbors. Interestingly, the power spectra look indeed very similar on large scales, i.e., there is no noticeable improvement due to the higher number of smoothing neighbors at a fixed mass resolution. Only the small-scale noise is reduced when the number of neighbors is increased.

We now turn to the artificial viscosity parameterization, which is another area where one may hope that simple changes could lead to significant improvements in the results obtained for turbulence. In particular, the problematic damping of the injected turbulence energy already on large scales suggests that a reduction of the viscosity may help. A lower viscosity seems also warranted because in our subsonic regime shocks are not really expected, suggesting that artificial viscosity may perhaps not be needed at all or only at a minimal level. We have hence first repeated our default simulations by enabling the so-called Balsara reduction factor for the Figure 7. Dependence of SPH turbulence results on numerical nuisance parameters. The panel on top gives results for the velocity power spectrum when the number of SPH smoothing neighbours is increased, from our default of 64 to 180, and finally to 512. Formally, the later run with 128^3 particles has the same mass and spatial resolution as our S1 run with 64^3 particles, hence the latter is included as a dashed line. The bottom panel illustrates the effect of changing the SPH viscosity parameterization. For lower $\alpha$, the velocity power on large scales goes up, but the shape of the power spectrum does not improve. Note however that this also increases the

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BUT WHAT IS THE REYNOLDS NUMBER?

\[ R_e \equiv \frac{VL}{\nu} \]

Stokes (1851), Reynolds (1883)
DISSIPATION IN SPH

There is none (it is a Hamiltonian system) ...except what you explicitly add.

AV terms give:

\[ \nu \approx \frac{1}{10} \alpha v_{\text{sig}} h; \quad \zeta \approx \frac{1}{6} \alpha v_{\text{sig}} h \]

Monaghan & Lattanzio (1985): \( \alpha = 1 \)

Morris & Monaghan (1997): \( \alpha (x, t) \in [0.1, 1] \)
Reynolds numbers in SPH

\[ R_e = \frac{10}{\alpha} M \frac{L}{h}, \]

Price & Federrath (2010), Mach 10:

c.f. Elmegreen & Scalo (2004): 
\( R_e \sim 10^5 - 10^7 \) in ISM
REYNOLDS NUMBERS IN SPH

\[ \mathcal{R}_e = \frac{10}{\alpha} \mathcal{M} \frac{L}{h}, \]

Linear dependence on Mach number

\[ \mathcal{R}_e = 2.4n^{1/3} \left( \frac{\mathcal{M}}{0.3} \right) \left( \frac{\alpha}{1.0} \right)^{-1} \left( \frac{N_{\text{ngb}}}{64} \right)^{-1/3}, \]

In BS calculations:

<table>
<thead>
<tr>
<th>( n )</th>
<th>64^3</th>
<th>128^3</th>
<th>256^3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{R}_e )</td>
<td>154</td>
<td>307</td>
<td>614</td>
</tr>
</tbody>
</table>
Using standard (15yo) viscosity switches:

see also Dolag et al. (2005) and Valdarnini (2011) on importance of viscosity switches for SPH simulations of ICM/IGM turbulence
Resolving high Reynolds numbers in smoothed particle hydrodynamics simulations of subsonic turbulence

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ABSTRACT

Accounting for the Reynolds number is critical in numerical simulations of turbulence, particularly for subsonic flow. For smoothed particle hydrodynamics (SPH) with constant artificial viscosity coefficient $\alpha$, it is shown that the effective Reynolds number in the absence of explicit physical viscosity terms scales linearly with the Mach number – compared to mesh schemes, where the effective Reynolds number is largely independent of the flow velocity. As a result, SPH simulations with $\alpha = 1$ will have low Reynolds numbers in the subsonic regime compared to mesh codes, which may be insufficient to resolve turbulent flow. This explains the failure of Bauer & Springel to find agreement between the moving-mesh code $\textsc{arepo}$ and the $\textsc{gadget}$ SPH code on simulations of driven, subsonic ($v \sim 0.3c_s$) turbulence appropriate to the intergalactic/intracluster medium, where it was alleged that SPH is somehow fundamentally incapable of producing a Kolmogorov-like turbulent cascade. We show that turbulent flow with a Kolmogorov spectrum can be easily recovered by employing standard methods for reducing $\alpha$ away from shocks.

Key words: hydrodynamics – turbulence – methods: numerical – galaxies: clusters: intracluster medium – intergalactic medium.

1 INTRODUCTION

Turbulence in astrophysics is of key importance for the interstellar medium (ISM), intracluster medium (ICM) and intergalactic medium (IGM). Compressible, hydrodynamic turbulence is characterized by two dimensionless parameters, the Mach number $\mathcal{M} \equiv V/c_s$ and the Reynolds number (Stokes 1851; Reynolds 1883)

$$R_e \equiv \frac{V L}{v}, \tag{1}$$

where $V$ is the flow velocity, $L$ is a typical length-scale, $v$ is the viscosity of the fluid and $c_s$ is the sound speed. Physically, these parameters estimate the relative importance of each of the terms in the Navier–Stokes equations – the Mach number specifies the ratio of the inertial term, $(\mathbf{v} \cdot \nabla) \mathbf{v}$, to the pressure term, $\nabla P/\rho$, while the Reynolds number specifies the ratio of the inertial term to the viscous dissipation term, $v \nabla^2 \mathbf{v}$. Mathematically, these two parameters – along with the boundary conditions and driving – entirely characterize the flow.

Given the importance of turbulence in theoretical models, it is crucial that agreement can be found between codes used for simulations of the ISM and ICM/IGM. Several comparison projects have been published recently comparing simulations of both decaying (Kitsionas et al. 2009) and driven (Price & Federrath 2010a) supersonic turbulence relevant to molecular clouds. However, fewer calculations appropriate to the ICM or IGM have been performed. In a recent preprint, Bauer & Springel (2011) have set out to extend the high Mach number comparisons to the mildly compressible, driven, subsonic turbulence thought to be appropriate to the ICM and IGM. In this case, the motions are comparable to or smaller than the sound speed, turbulent motions are dissipated by viscosity, and the flow is mainly characterized by the Reynolds number, similar to turbulence in the laboratory. In particular, it is well known from laboratory studies that the transition from laminar flow to fully developed turbulence only occurs once a critical Reynolds number is reached – for example, for Poiseuille flow (water flowing in a pipe) this is observed for $R_e \gtrsim 2000$ (e.g. Reynolds 1895).

Since at low Mach number the Reynolds number controls not only the transition to turbulence, but also the character of such turbulence (e.g. the extent of the inertial range), it is critical to specify, or at least estimate, the Reynolds number employed in numerical simulations of turbulence in order to compare with the physical Reynolds numbers in the problems of interest. For the ISM, the physical Reynolds numbers are high [e.g. Elmegreen & Scalo (2004) estimate $R_e \sim 10^5$–$10^7$ for the cold ISM] so the approach adopted has been to fix the Mach number and try to reach high numerical Reynolds numbers by minimizing numerical dissipation away from shocks. Estimates for $R_e$ in the ICM/IGM are more difficult. Brunetti & Lazarian (2007) estimate $R_e \sim 52$, but

Figure 3. Visual comparison of the turbulent turbulence with $\mathcal{M} \sim 0.3$, simulated wit to right, we show our moving grid result,
Reynolds numbers are achievable in SPH. Dehnen et al. switch it may be expected that significantly higher to simulate linear waves for over 3 periods with essentially no numerical noise. In particular, they show that they are able to achieve similar results with SPH an improved viscosity switch is adequate for very low Mach number incompressible calculations. It is necessary to capture the physical dissipation that occurs due to viscosity parameters, such as the zero viscosity was we have already discussed in Sec. 7. This implies that ultimately it is quite incorrect to try to simulate purely hydrodynamic ICM turbulence at high Reynolds number without taking into account more detailed physics, such as magnetic fields.

The switch proposed recently by Cullen et al. as well as reducing the viscosity to very low values where it is not necessary to achieve similar results with SPH an improved viscosity switch is needed. The switch could substantially improve SPH simulations of turbulence at high Reynolds number without taking into account numerical code. However, this may also imply that ultimately it is quite incorrect to try to simulate purely hydrodynamic ICM turbulence at high Reynolds number without taking into account the Reynolds number employed in the calculations, requiring more detailed physics, such as magnetic fields.

The state of the art expected to control not only whether the flow is turbulent but also the character of such turbulence. In particular, differences in the Reynolds number in numerical turbulence simulations, particularly in galaxy clusters, are shown in Fig. 5, showing the time-averaged spectrum from 4.4−10^5 to 4.4−10^6 of the sound speed and τ/3 such that α=0 and τ/3, respectively, as indicated. The shaded regions show the 4/3 dependence of the dissipation scale. The turnover is also consistent with the expected slope at large scales. Though the spectrum is clear that already this is a dramatic improvement on the SPH turbulence calculations using the Morris & Monaghan viscosity switch employed in this Letter we estimate that a power-law inertial range in the power spectrum strongly depends on the Reynolds number employed in the calculations, requiring more detailed physics, such as magnetic fields.

Indeed, both the Price viscosity switch employed in their preprint, using either a Balsara switch and also a run with the interpolation procedure as demonstrated in Fig. 9 of Bauer & Springel, argue that “large errors in SPH’s” gravity apply in the presence of magnetic fields. Reaching α=0 or the Balsara switch calculations show a power-law spectrum using only standard SPH gradient calculations. This explains the failure to produce a turbulent Kolmogorov-like turbulent cascade in their SPH calculations. Fig. 5 demonstrates that this argument is incorrect, since we are able to achieve the same results using only standard SPH gradients. However, we find that the appearance of the turnover “turns over” at relatively small k−5/3 slope is evident at large scales. For the cold ISM, the Reynolds numbers they achieve are not presently achievable with any numerical code. However, this may also imply that ultimately it is quite incorrect to try to simulate purely hydrodynamic ICM turbulence at high Reynolds number without taking into account more detailed physics, such as magnetic fields.

Figure 2. Time-averaged k−5/3-compensated power spectra from subsonic SPH turbulence calculations using the Morris & Monaghan (1997) viscosity switch at a resolution of 64^3, 128^3 and 256^3 particles, as indicated, for which the corresponding Reynolds numbers are ~ 1500, 3000 and 6000, respectively. The shaded regions show the 1σ errors from the time-averaging. At the highest Reynolds numbers a Kolmogorov-like k−5/3 slope is evident at large scales.
Also, much better viscosity switches now available (e.g. Cullen & Dehnen 2010)
CONCLUSIONS

• Don’t believe everything you read on astro-ph

• SPH gives comparable results to grid methods for turbulence studies, but more efficient only if one is interested in the density field / gravity is involved

• Know your Reynolds number - it defines the flow!

• Viscosity switches are the key to high Reynolds numbers in SPH at low Mach number - also easier to achieve high Re at high Mach number