A SIMPLE PROOF OF THE FÁRY-WAGNER THEOREM

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The purpose of this note is to give a simple proof of the following fundamental result independently due to Fáry [1] and Wagner [2]. A plane graph is a simple graph embedded in the plane without edge crossings. Combinatorially speaking, there is a circular ordering of the edges incident to each vertex, and a nominated outerface.

**Theorem.** Every plane graph has a drawing in which every edge is straight.

**Proof.** A triangulation is a plane graph in which every face is bounded by three edges. Edges can be added to a plane graph to obtain a plane triangulation. Thus it suffices to prove the theorem for plane triangulations $G$. We proceed by induction on $|V(G)|$. The base case with $|V(G)| = 3$ is trivial. Now suppose that $|V(G)| \geq 4$. A separating triangle of $G$ is a 3-cycle that contains a vertex in its interior and in its exterior. If $G$ has no separating triangles, then let $vw$ be any edge of $G$. Otherwise, let $vw$ be an edge incident to a vertex that is in the interior of an innermost separating triangle of $G$.

Now $vw$ is on the boundary of two faces, say $vwp$ and $vwq$. Since $vw$ is not in a separating triangle, $p$ and $q$ are the only common neighbours of $v$ and $w$. Let $(vp, vw, vq, vx_1, vx_2, \ldots, vx_k)$ and $(wq, vw, wp, wy_1, wy_2, \ldots, wy_\ell)$ be the clockwise ordering of the edges incident to $v$ and $w$ respectively.

Let $G'$ be the plane triangulation obtained from $G$ by contracting the edge $vw$ into a single vertex $s$. Replace the pairs of parallel edges $\{vp, wp\}$ and $\{vq, wq\}$ in $G$ by edges $sp$ and $sq$ in $G'$. The clockwise ordering of the edges of $G'$ incident to $s$ is $(sp, sy_1, sy_2, \ldots, sy_\epsilon, sq, sx_1, sx_2, \ldots, sx_\kappa)$. By induction, $G'$ has a drawing in which every edge is straight (and the circular ordering of the edges incident to $s$ are preserved). For all $\epsilon > 0$, let $C_\epsilon(s)$

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In fact, for every vertex $v$ there is an edge incident to $v$ whose endpoints have at most two common neighbours. This is because the neighbourhood of $v$ has no $K_4$-minor (it is even outerplanar), and every graph with no $K_4$-minor has a vertex of degree at most two.
denote the circle of radius $\epsilon$ centred at $s$. For each neighbour $t$ of $s$ in $G'$, let $R_\epsilon(t)$ denote the region consisting of the union of all open segments between $t$ and a point in $C_\epsilon(s)$. There is an $\epsilon > 0$ such that all neighbours $t$ of $s$ are in the exterior of $C_\epsilon(s)$ and the only edges of $G'$ that intersect $R_\epsilon(t)$ are incident to $s$.

There is a line $L$ through $s$ with $p$ on one side of $L$ and $q$ on the other side, as otherwise the edges $sp$ and $sq$ would overlap. Now $sp$ and $sq$ break $C_\epsilon(s)$ into two arcs, one that intersects the edges $\{sx_i : 1 \leq i \leq k\}$, and one that intersects the edges $\{sy_j : 1 \leq j \leq \ell\}$. The set $L \cap C_\epsilon(s)$ consists of two points. Position $v$ and $w$ at these two points, with $v$ on the side of $C_\epsilon(s)$ that intersects the edges $\{sx_i : 1 \leq i \leq k\}$, and with $w$ on the other side. Delete $s$ and its incident edges. Draw the edges of $G$ incident to $v$ or $w$ straight. Thus $vw$ is contained in $L$. Since $p$ and $q$ are on different sides of $L$, the edges incident to $v$ or $w$ do not cross. By the choice of $\epsilon$, edges incident to $v$ or $w$ do not cross other edges of $G$. Thus we obtain the desired drawing of $G$. \[\square\]

References
