Forcing a sparse minor

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Abstract

This paper addresses the following question for a given graph $H$: what is the minimum number $f(H)$ such that every graph with average degree at least $f(H)$ contains $H$ as a minor? Due to connections with Hadwiger’s Conjecture, this question has been studied in depth when $H$ is a complete graph. Kostochka and Thomason independently proved that $f(K_t) = ct\sqrt{\ln t}$. More generally, Myers and Thomason determined $f(H)$ when $H$ has a super-linear number of edges. We focus on the case when $H$ has a linear number of edges. Our main result, which complements the result of Myers and Thomason, states that if $H$ has $t$ vertices and average degree $d$ at least some absolute constant, then $f(H) \leq 3.895\sqrt{\ln d} t$. Furthermore, motivated by the case when $H$ has small average degree, we prove that if $H$ has $t$ vertices and $q$ edges, then $f(H) \leq t + 6.291q$ (where the coefficient of 1 in the $t$ term is best possible).

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1 Introduction

A graph $H$ is a minor of a graph $G$ if a graph isomorphic to $H$ can be obtained from a subgraph of $G$ by contracting edges. This paper studies average degree conditions that force an $H$-minor. In particular, it focuses on the infimum of all numbers $d$ such that every graph with average degree at least $d$ contains $H$ as a minor, which we denote by $f(H)$. We are interested in determining bounds on $f(H)$ that are a function of the number of edges and vertices of $H$.

We distinguish two types of graphs $H$ (or to be more precise, families of graphs $H$). We consider $H$ to be ‘dense’ if $|E(H)| \geq |V(H)|^{1+\tau}$ for some constant $\tau > 0$. On the other

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hand, we consider $H$ to be ‘sparse’ if $|E(H)| \leq c|V(H)|$ for some constant $c$ (independent of $|V(H)|$). This paper focuses on $f(H)$ for graphs $H$ that are not dense, and especially those that are sparse.

Previous work in this field concerns dense $H$. Indeed, largely motivated by Hadwiger’s Conjecture, $f(H)$ was first studied for $H = K_t$, the complete graph on $t$ vertices. Diirac [7] proved that for $t \leq 5$, every $n$-vertex $K_t$-minor-free graph has at most $(t - 2)n - \binom{t-1}{2}$ edges, and this bound is tight. Mader [18] extended this result for $t \leq 7$. It follows that $f(K_t) = 2t - 4$ for $t \leq 7$. For $t \geq 8$ there are $K_t$-minor-free graphs with more than $(t - 2)n - \binom{t-1}{2}$ edges. However, results of Jørgensen [9] and Song and Thomas [23] respectively imply that $f(K_8) = 12$ and $f(K_9) = 14$. Thus $f(K_t) = 2t - 4$ for $t \leq 9$. Song [24] proved that $f(K_{10}) \leq 22$ and $f(K_{11}) \leq 26$, and conjectured that both these bounds can be improved.

The first upper bound on $f(K_t)$ for general $t$ was due to Mader [17], who proved that $f(K_t) \leq 2t^2 - 2$. Mader [18] later proved that $f(K_t) \in \mathcal{O}(t \ln t)$. Kostochka [11, 12] and de la Vega [5] (based on the work of Bollobás et al. [2]) independently proved the lower bound, $f(K_t) \in \Omega(t \sqrt{\ln t})$. A matching upper bound of $f(K_t) \in \mathcal{O}(t \sqrt{\ln t})$ was independently proved by Kostochka [11, 12] and Thomason [26]. Later, Thomason [27] determined the asymptotic constant:

$$f(K_t) = (\alpha + o(1))t \sqrt{\ln t},$$

where $\alpha = 0.628\ldots$ is an explicit constant, and $o(1)$ denotes a term tending to 0 as $t \to \infty$. Myers [20] characterised the extremal $K_t$-minor-free graphs as unions of pseudo-random graphs.

Myers and Thomason [22] generalised (1) for dense graphs $H$ as follows. They introduced a graph parameter $\gamma$ with the property that if $t = |V(H)|$ then

$$f(H) = (\alpha \gamma(H) + o(1))t \sqrt{\ln t},$$

where $\gamma(H) \leq 1$ and $o(1)$ denotes a term (slowly) tending to 0 as $t \to \infty$. Note that when $H$ is sparse, the $o(1)$ term might dominate $\gamma(H)$, in which case this result says little about $f(H)$, as discussed by Myers and Thomason [22, Section 7]. For example, (2) does not determine $f$ for specially structured graphs such as unbalanced complete bipartite graphs; see Section 1.2 below.

Moreover, Myers and Thomason [22] proved that if $H$ has $t^{1+\tau}$ edges, for some constant $\tau > 0$, then $\gamma(H) \leq \sqrt{\tau}$ (with equality for almost all $H$ and for all regular $H$). That is, if $H$ has average degree $d = 2t^\tau$, then

$$f(H) \leq \alpha \sqrt{\ln d} t + o(t \sqrt{\ln t}).$$

(3)

Since the $o(1)$ term tends to 0 slowly, this bound also says little when $H$ is sparse.
1.1 Non-Dense Graphs $H$

With respect to typical non-dense graphs, we prove the following theorem in the same direction as (3) except it does apply when $G$ is not dense.

**Theorem 1.** There is an absolute constant $d_0$, such that for every graph $H$ with $t$ vertices and average degree $d \geq d_0$,

\[
f(H) \leq 3.895\sqrt{\ln d} t.
\]

The lower bounds on $f(H)$ due to Myers and Thomason [22] apply even when $H$ is sparse. It follows that Theorem 1 is tight up to a constant factor for numerous graphs $H$. In particular, if $H$ is sufficiently large, and is random, or regular, or even if the maximum and minimum degrees are close, then Theorem 1 is tight\(^1\). Theorem 1 is proved in Section 5.

When $d$ is very small, Theorem 1 is not applicable. Thus, motivated by the case of graphs $H$ with small average degree, we investigate linear bounds of the form

\[
f(H) \leq \alpha |V(H)| + \beta |E(H)|
\]

for explicit constants $\alpha$ and $\beta$. A first question in this regard is the smallest possible values for $\alpha$ and $\beta$. We can push $\beta$ as close to 0 as we like. Indeed, Theorem 1 immediately implies that for every $\beta > 0$ there is a constant $c = c(\beta)$ such that $f(H) \leq ct + \beta q$ for every graph $H$ with $t$ vertices and $q$ edges. On the other hand, $\alpha \geq 1$ in any such bound, since $K_{t-1}$ has average degree $t - 2$ but does not contain the graph with $t$ vertices and no edges as a minor. At this extremity we prove the following (in Section 3):

**Theorem 2.** For every graph $H$ with $t$ vertices and $q$ edges,

\[
f(H) \leq t + 6.291q.
\]

Note that $\beta \geq \frac{1}{3}$ in any bound of the form $f(H) \leq t + \beta q$ (since in Section 6 we observe that if $H$ consists of $k \geq 1$ disjoint triangles, then $f(H) = 4k - 2 = t + \frac{3}{2} - 2$).

Also, note that a linear bound of the form $f(H) \leq \alpha t + \beta q$ can also be concluded from a theorem of Fox and Sudakov [8, Theorem 5.1] in conjunction with an old lemma of Mader (our Lemma 4).

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\(^1\)For example, consider a $d$-regular graph $H$ on $t$ vertices. In the notation of Myers and Thomason [22], $\tau(H) = \log_t \left( \frac{\sqrt{2}}{2} \right)$. Their Theorem 4.8 and Corollary 4.9 imply that $\gamma(H) \to \sqrt{\tau(H)}$ as $t \to \infty$. Let $n := \lceil \gamma(H) t \sqrt{\ln t} \rceil \to \sqrt{\ln dt}$. Myers and Thomason [22, Theorem 2.3] prove that $H$ is a minor of a random graph $G(n, \frac{1}{2})$ with probability tending to 0 as $t \to \infty$. Thus some graph with average degree $\frac{n}{2}$ contains no $H$-minor. Thus $f(H) \geq c \sqrt{\ln dt}$. 

1.2 Specially Structured Graphs

Attention in the literature has also been focused on specially structured graphs, as we now discuss. We propose some open problems in this regard in our concluding section.

Let $K_{s,t}$ be the complete bipartite graph with $s \leq t$. First consider when $s$ is small. Chudnovsky et al. [3] proved that the maximum number of edges in a $K_{2,t}$-minor-free graph is at most $\frac{1}{2}(t+1)(n-1)$, which is tight for infinitely many values of $n$. This implies that $f(K_{2,t}) = t + 1$. Myers [21] had earlier proved the same result for sufficiently large $t$.

Kostochka and Prince [14] proved that for $t \geq 6300$ and $n \geq t + 3$, every $n$-vertex graph $G$ with more than $\frac{1}{2}(t+3)(n-2)+1$ edges has a $K_{3,t}$ minor, and this bound is tight. Thus $f(K_{3,t}) = 2t + 6$ for $t \geq 6300$.

Now consider general complete bipartite graphs. Myers [21] conjectured that for every integer $s$ there exists a positive constant $c$ such that $f(K_{s,t}) \leq ct$ for every integer $t$. Kühn and Osthus [16] and Kostochka and Prince [13] independently proved certain strengthenings of this conjecture. Let $K^*_{s,t}$ denote the graph obtained from $K_{s,t}$ by adding all edges between the $s$ vertices of degree $t$. Of course, $f(K_{s,t}) \leq f(K^*_{s,t})$. Kühn and Osthus [16] proved that $f(K^*_{s,t}) \leq (1+\epsilon)t$ for all $t \geq t(\epsilon)$ and $s \leq \epsilon^6 t / \log t$. Kostochka and Prince [13] proved that $t + 3s - 5\sqrt{s} \leq f(K_{s,t}) \leq f(K^*_{s,t}) \leq t + 3s$ for $t > (180s \log_2 s)^{1+6s \log_2 s}$. Kostochka and Prince [15] refined their method to conclude a similar upper bound of $f(K^*_{s,t}) \leq t + 8s \log_2 s$ under a more reasonable assumption about $t$, namely that $t / \log_2 t \geq 1000s$. This result is best possible in the sense that the 1000 and 8 cannot be simultaneously reduced to $1/18$, say. Again, considering $K_{s,t}$ rather than $K^*_{s,t}$ does not significantly affect the bounds.

See [28–30] for various results concerning average or minimum degree conditions that force several copies of a given graph as a minor or subdivision.

2 A Minor with Large Minimum Degree

The standard approach to find an $H$-minor in a graph $G$ of high average degree involves first finding a minor $G'$ of $G$ with high minimum degree and few vertices. Then it is shown that $H$ is a minor of $G'$ and hence of $G$. This approach was introduced by Mader [18]. This section presents a variety of results proving that such $G'$ exist. Many of these results can be found in reference [18], but since this paper is written in German, we include all the proofs. A graph $G$ is minor-minimal with respect to some set $\mathcal{G}$ of graphs if $G \in \mathcal{G}$ and every proper minor of $G$ is not in $\mathcal{G}$.

**Lemma 3.** Let $G$ be a minor-minimal graph with average degree at least $d$. Then every edge of $G$ is in at least $\left\lfloor \frac{d}{2} \right\rfloor$ triangles, and every vertex has degree at least $\left\lfloor \frac{d}{2} \right\rfloor + 1$.

**Proof.** Say $G$ has $n$ vertices and $m$ edges, where $2m \geq dn$. Suppose on the contrary that
G contains an edge e in \( t < \left\lfloor \frac{d}{2} \right\rfloor \) triangles. Then \( G/e \) has \( n - 1 \) vertices and \( m - 1 - t \) edges. Thus \( G/e \) has average degree \( \frac{2(m-1-t)}{n-1} > \frac{2m}{n} \geq d \). Hence G is not minor-minimal with average degree at least \( d \). This contradiction proves that every edge of \( G \) is in at least \( \left\lfloor \frac{d}{2} \right\rfloor \) triangles. Thus every vertex has degree at least \( \left\lfloor \frac{d}{2} \right\rfloor + 1 \). \( \square \)

**Lemma 4.** Every graph with average degree \( d \geq 1 \) contains a minor with at most \( \left\lfloor \frac{d^2+1}{d+1} \right\rfloor \) vertices and minimum degree at least \( \left\lfloor \frac{d}{2} \right\rfloor \), as well as a minor with at most \( \left\lfloor \frac{d^2+1}{d+1} \right\rfloor + 1 \) vertices and minimum degree at least \( \left\lfloor \frac{d}{2} \right\rfloor + 1 \).

**Proof.** It suffices to prove the result for minor-minimal graphs \( G \) with average degree at least \( d \). By Lemma 3, each edge of \( G \) is in at least \( \left\lfloor \frac{d}{2} \right\rfloor \) triangles. Say \( G \) has \( n \) vertices and \( m \) edges. Thus \( m \geq \left\lceil \frac{dn}{2} \right\rceil \). If \( m > \left\lceil \frac{dn}{2} \right\rceil \) then deleting any one edge maintains the average degree condition, thus contradicting the minimality of \( G \). Hence \( m = \left\lceil \frac{dn}{2} \right\rceil \), and \( G \) has average degree \( \frac{2m}{n} = \frac{2}{n} \left\lceil \frac{dn}{2} \right\rceil < \frac{2}{n} \left( \frac{dn}{2} + 1 \right) = d + \frac{2}{n} \leq \frac{d^2 + d + 2}{d+1} \) since \( n \geq d + 1 \). Thus \( G \) has a vertex \( v \) with degree at most \( \left\lfloor \frac{d^2 + d + 2}{d+1} - 1 \right\rfloor = \left\lfloor \frac{d^2 + 1}{d+1} \right\rfloor \). Hence the subgraph of \( G \) induced by the neighbours of \( v \) has at most \( \left\lfloor \frac{d^2 + 1}{d+1} \right\rfloor \) vertices and minimum degree at least \( \left\lfloor \frac{d}{2} \right\rfloor \). Moreover, the subgraph of \( G \) induced by the closed neighbourhood of \( v \) has at most \( \left\lfloor \frac{d^2 + 1}{d+1} \right\rfloor + 1 \) vertices and minimum degree at least \( \left\lfloor \frac{d}{2} \right\rfloor + 1 \). \( \square \)

Mader [18] introduced the following key definition. For an integer \( k \geq 1 \), let \( X_k \) be the set of graphs \( G \) with \( |V(G)| \geq k \) and \( |E(G)| \geq k|V(G)| - \binom{k+1}{2} \).

**Lemma 5.** Let \( G \) be a minor-minimal graph in \( X_k \). Then \( |E(G)| = k|V(G)| - \binom{k+1}{2} \), and either \( G \) is isomorphic to \( K_k \) or the neighbourhood of each vertex in \( G \) induces a subgraph with minimum degree at least \( k \).

**Proof.** Say \( G \) has \( n \) vertices and \( m \) edges. Then \( m = k|V(G)| - \binom{k+1}{2} \), otherwise delete an edge. If \( n = k \) then \( m = k^2 - \binom{k+1}{2} = \binom{k}{2} \), implying that \( G \) is isomorphic to \( K_k \), as desired. Now assume that \( n \geq k + 1 \). Let \( vw \) be an edge of \( G \). Say \( vw \) is in \( t \) triangles. Then \( G/vw \) has \( n - 1 \geq k \) vertices and \( m - t - 1 \) edges. Since \( G \) is minor-minimal, \( G/vw \) is not in \( X_k \). Thus

\[
k n - \binom{k+1}{2} - t - 1 = m - t - 1 = |E(G/vw)| \leq k(n - 1) - \binom{k+1}{2} - 1,
\]

implying \( t \geq k \). That is, each edge is in at least \( k \) triangles. Therefore, the neighbourhood of each vertex induces a subgraph with minimum degree at least \( k \). \( \square \)

The following lemma is proved by mimicking a proof by Mader [18] for the special case of \( c_1 = 4 \) and \( c_2 = \frac{3}{2} \).

**Lemma 6.** Fix constants \( c_1 > 2 \) and \( c_2 > 1 \). For each integer \( k \geq 1 \), every graph \( G \) with average degree at least \( 4k \) has a minor with:
(1) at most \( (\frac{c}{2} + 1)k \) vertices and minimum degree at least \( 2k \), or
(2) at most \( 2k + 1 \) vertices and minimum degree at least \( (1 + \frac{1}{c_1})k \), or
(3) at most \( c_2k \) vertices and minimum degree at least \( k \), or
(4) at most \( (4 - \frac{c}{2})k \) vertices and minimum degree at least \( c_2k \), or
(5) \( k \) vertices and minimum degree \( k - 1 \) (that is, \( K_k \)).

Proof. Since \( G \) has average degree at least \( 4k \), \( G \in X_{2k} \). Let \( G' \) be a minor-minimal minor of \( G \) in \( X_{2k} \). Thus \( |E(G')| = 2k |V(G')| - \left( \frac{2k+1}{2} \right) \). Hence \( G' \) has average degree less than \( 4k \). Let \( v \) be a vertex in \( G' \) with degree less than \( 4k \). If \( G' \) is isomorphic to \( K_{2k} \) then outcome (5) holds. Otherwise, by Lemma 5, \( G_0 := G'[N(v)] \) has minimum degree at least \( 2k \). If \( G_0 \) has at most \( (\frac{c}{2} + 1)k \) vertices then it satisfies outcome (1) and we are done. Now assume that \( G_0 \) has at least \( (\frac{c}{2} + 1)k \) vertices. Since \( G_0 \) has minimum degree at least \( 2k \), \( |E(G_0)| \geq k |V(G_0)| \). Delete edges from \( G_0 \) until \( |E(G_0)| = k |V(G_0)| \).

Define \( k' := \lceil \frac{c}{2}k \rceil \). We now define a sequence of graphs \( G_0, G_1, \ldots, G_{k'} \) that satisfy \( |V(G_i)| = |V(G_0)| - i \) and

\[
|E(G_i)| - \frac{i k}{c_1} \leq |E(G_i)| \leq k |V(G_i)|.
\]

Given \( G_i \) where \( 0 \leq i \leq k' - 1 \), construct \( G_{i+1} \) as follows. First suppose that each edge \( e \) in \( G_i \) is in at least \( (1 + \frac{1}{c_1})k - 1 \) triangles. Since \( |E(G_i)| \leq k |V(G_i)| \), some vertex \( v \) in \( G_i \) has degree at most \( 2k \). Thus the closed neighbourhood of \( v \) induces a subgraph with at most \( 2k + 1 \) vertices and minimum degree at least \( (1 + \frac{1}{c_1})k \), which satisfies outcome (2). Now assume some edge \( e \) in \( G_i \) is in at most \( (1 + \frac{1}{c_1})k - 1 \) triangles. Let \( G_{i+1} \) be obtained from \( G_i \) by contracting \( e \). Thus \( |V(G_{i+1})| = |V(G_i)| - 1 = |V(G_0)| - i - 1 \) and

\[
|E(G_{i+1})| \geq |E(G_i)| - (1 + \frac{1}{c_1})k \geq k |V(G_i)| - \frac{i k}{c_1} - (1 + \frac{1}{c_1})k = k |V(G_{i+1})| - \frac{(i + 1)k}{c_1}.
\]

If \( G_{i+1} \) has more than \( k |V(G_{i+1})| \) edges, then delete edges until \( |E(G_{i+1})| = k |V(G_{i+1})| \). Thus \( G_{i+1} \) satisfies the stated properties.

Consider the final graph \( F := G_{k'} \). It satisfies

\[
|V(F)| = |V(G_0)| - k' \geq \left( \frac{c_1}{2} + 1 \right)k - k' \geq k \quad \text{and}
\]

\[
|E(F)| \geq k |V(F)| - \frac{k'}{c_1} k \geq k |V(F)| - \frac{k+1}{2}.
\]

Thus \( F \in X_k \). Let \( F' \) be a minor-minimal minor of \( F \) in \( X_k \). Thus \( |V(F')| \leq |V(F)| = |V(G_0)| - k' \leq (4 - \frac{c}{2})k \). If \( F' \) is isomorphic to \( K_k \) then outcome (5) holds. Otherwise, by Lemma 5, the neighbourhood of each vertex in \( F' \) induces a subgraph with minimum degree at least \( k \). If \( F' \) has a vertex of degree at most \( c_2k \) then \( F'[N(v)] \) has at most \( c_2k \) vertices and has minimum degree at least \( k \), which satisfies outcome (3). Otherwise \( F' \) has minimum degree at least \( c_2k \) and at most \( (4 - \frac{c}{2})k \) vertices, which satisfies outcome (4). \( \square \)
It is natural to maximise the ratio between the minimum degree and the number of vertices in our minor. The next lemma does that:

**Lemma 7.** For every integer \( k \geq 1 \), every graph with average degree at least \( 4k \) contains a complete graph \( K_k \) as a minor or contains a minor with \( n \) vertices and minimum degree \( \delta \), where \( \delta \geq 0.6518n \) and \( 2\delta - n \geq 0.4659k \) and \( k \leq \delta < n \leq 4k \).

**Proof.** Apply Lemma 6 with \( c_1 = 3.2929 \) and \( c_2 = 1.5341 \).

Maximising the difference between twice the minimum degree and the number of vertices in our minor will be useful below. The next lemma does that.

**Lemma 8.** For every integer \( k \geq 1 \), every graph with average degree at least \( 4k \) contains a complete graph \( K_k \) as a minor or contains a minor with \( n \) vertices and minimum degree \( \delta \), where \( \delta \geq 0.6273n \) and \( 2\delta - n \geq 0.5773k \) and \( k \leq \delta < n \leq 4k \).

**Proof.** Apply Lemma 6 with \( c_1 = 3.4641 \) and \( c_2 = 1.4227 \).

### 3 Deterministic Linear Bounds

This section establishes a number of linear bounds on \( f(H) \). All the proofs are deterministic. The following well known lemma will be useful. We include the proof for completeness.

**Lemma 9.** Every graph \( G \) with minimum degree at least \( \ell - 1 \) contains every tree on \( \ell \geq 2 \) vertices as a subgraph.

**Proof.** We proceed by induction on \( \ell \) (with \( G \) fixed). The base case with \( \ell = 1 \) is trivial. Assume that \( \ell \geq 2 \). Let \( T \) be a tree on \( \ell \) vertices. Let \( v \) be a leaf of \( T \) adjacent to \( w \). By induction, \( G \) contains a subgraph \( X \) isomorphic to \( T - v \). Let \( w' \) be the image of \( w \) in \( X \). Since \( \deg_G(w') \geq \ell - 1 > |V(X - w')| = \ell - 2 \), there is a neighbour \( v' \) of \( w' \) in \( G - X \). Mapping \( v \) to \( v' \) gives a subgraph of \( G \) isomorphic to \( T \).

A graph \( G \) is 2-degenerate if every non-empty subgraph of \( G \) has a vertex of degree at most 2.

**Lemma 10.** Let \( G \) be a graph with \( n \geq 1 \) vertices and minimum degree \( \delta \), with \( 2\delta - n \geq t - 2 \). Then \( G \) contains every 2-degenerate graph on \( t \geq 1 \) vertices as a subgraph.

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2The optimised constants used in the proof of Lemma 7 (and elsewhere in the paper) were computed using AMPL and the MINOS 5.5 nonlinear equation solver.
Lemma 11. Every graph $G$ with average degree at least $6.929t$ contains every 2-degenerate graph $H$ on $t \geq 1$ vertices as a minor.

Proof. If $t \leq 4$ then $G$ contains $K_4$ and thus $H$ as a minor (since $6.929t > 4 = f(K_4)$). Now assume that $t \geq 5$. By assumption, $G$ has average degree at least $4k$, where $k := \lceil (t - 2)/0.5773 \rceil$. If $G$ contains a $K_k$ minor, then $G$ contains $H$ as a minor (since $t \geq 5$ implies $k \geq t$). Otherwise, by Lemma 8, $G$ contains a minor $G'$ with $n$ vertices and minimum degree $\delta$ where $k \leq \delta \leq n \leq 4k$ and $2\delta - n \geq 0.5773k \geq t - 2$. By Lemma 10, $G'$ contains $H$ as a subgraph. Thus $G$ contains $H$ as a minor.

We obtain the following straightforward linear bound for forcing an $H$-minor. The 1-subdivision of a graph $H$ is the graph obtained from $H$ by subdividing each edge of $H$ exactly once. A $(\leq 1)$-subdivision of $H$ is a graph obtained from $H$ by subdividing each edge of $H$ at most once.

Proposition 12. Let $H$ be a graph with $t$ vertices and $q$ edges. Then every graph $G$ with average degree at least $6.929(t + q)$ contains $H$ as a minor.

Proof. If $H'$ is the 1-subdivision of $H$, then $H'$ has $t + q$ vertices and is 2-degenerate. By Lemma 11, $G$ contains $H'$ and thus $H$ as a minor.

This result is improved in Theorem 15 below. First we need the following easy generalisation of Lemma 10. A subgraph $H'$ of a graph $H$ is spanning if $V(H') = V(H)$.

Lemma 13. Let $H$ be a graph with $t \geq 1$ vertices and $q$ edges. Assume that $H$ contains a 2-degenerate spanning subgraph $H'$ with $q'$ edges. Let $G$ be a graph with $n \geq 1$ vertices
and minimum degree $\delta$, with $2\delta - n \geq q - q' + t - 2$. Then $G$ contains a $(\leq 1)$-subdivision of $H$ as a subgraph.

**Proof.** Let $H''$ be the graph obtained from $H$ by subdividing each edge not in $H'$ once. Thus $H''$ is 2-degenerate, and has $t + q - q'$ vertices. By Lemma 10, $G$ contains $H''$ as a subgraph, which is a $(\leq 1)$-subdivision of $H$. \hfill $\square$

**Lemma 14.** Let $H$ be a graph with $t \geq 1$ vertices and $q$ edges. Assume that $H$ contains a 2-degenerate spanning subgraph $H'$ with $q'$ edges. Let $G$ be a graph with $n \geq 1$ vertices and average degree at least $6.929(q - q' + t)$. Then $G$ contains $H$ as a minor.

**Proof.** If $t \leq 4$ then $G$ contains $K_4$ and thus $H$ as a minor (since $6.929(q - q' + t) > 4 = f(K_4)$). Now assume that $t \geq 5$. By assumption, $G$ has average degree at least $4k$, where $k := [(q - q' + t - 2)/0.5773]$. If $G$ contains a $K_k$ minor, then $G$ contains $H$ as a minor (since $t \geq 5$ implies $k \geq t$). Otherwise, by Lemma 8, $G$ contains a minor $G'$ with $n$ vertices and minimum degree $\delta$ where $k \leq \delta \leq n \leq 4k$ and $2\delta - n \geq 0.5773k \geq q - q' + t - 2$. By Lemma 13, $G'$ contains $H$ as a subgraph. Thus $G$ contains $H$ as a minor. \hfill $\square$

**Theorem 15.** For every graph $H$ with $i$ isolated vertices and $q$ edges, every graph $G$ with average degree at least $i + 6.929q$ contains $H$ as a minor.

**Proof.** Let $c := 6.929$. Let $t := |V(H)|$. First note that $q \geq \frac{t-i}{2}$. We proceed by induction on $|V(H)| + |V(G)|$. The result is trivial if $|V(H)| \leq 1$. Now assume that $|V(H)| \geq 2$. Let $G$ be a graph with $n$ vertices, $m$ edges, and average degree $\frac{2m}{n} \geq i + cq$. We may assume that $G$ is minor-minimal with average degree at least $i + cq$. By Lemma 3, $G$ has minimum degree at least $\lceil \frac{i+cq}{2} \rceil + 1 \geq q$.

First suppose that $H$ contains an isolated vertex $v$. Let $w$ be a vertex of minimum degree in $G$. Thus $\deg(w) \leq \frac{2m}{n}$. Hence the average degree of $G - w$ is

$$\frac{2(m - \deg(w))}{n - 1} \geq \frac{2m - \frac{2m}{n} - (n - 1)}{n - 1} = \frac{2m}{n} - 1 \geq (t - 1) + cq.$$

By induction, $G - w$ contains $H - v$ as a minor. Thus $G$ contains $H$ as a minor (with $v$ embedded at $w$). Now assume that $i = 0$.

Now suppose that some component $T$ of $H$ is a tree. Let $\ell := |V(T)|$. Since $H$ has no isolated vertex, $\ell \geq 2$. Also, $q = |E(H)| \geq |E(T)| = \ell - 1$ and $G$ has minimum degree at least $\ell - 1$. By Lemma 9, there is a subgraph $T'$ of $G$ isomorphic to $T$. Let $G' := G - V(T')$. Note that $|E(G')| > m - \ell n$. By assumption, (a) $2m \geq cq n$. Since $c \geq 4$ and $\ell \geq 2$, we have $c(\ell - 1) \geq 2\ell$, implying (b) $-2\ell n \geq -c(\ell - 1)n$. Also $q \geq \ell - 1$, implying (c) $0 \geq -c\ell q + c\ell(\ell - 1)$. Adding (a), (b) and (c) gives

$$2m - 2\ell n \geq cq n - c\ell q - c(\ell - 1)n + c\ell(\ell - 1) = c(q - (\ell - 1))(n - \ell).$$
Hence the average degree of $G'$ is
\[\frac{2|E(G')|}{|V(G')|} \geq \frac{2(m - \ell n)}{n - 1} \geq c(q - (\ell - 1)).\]

$H - V(T)$ has no isolated vertices and $q - (\ell - 1)$ edges. By induction, $G'$ contains $H - V(T)$ as a minor. Hence $G$ contains $H$ as a minor, with $T$ mapped to $T'$. Now assume that no component of $H$ is a tree: Thus $q \geq t$.

Let $H_1, \ldots, H_k$ be the components of $H$. Each $H_i$ contains a spanning subgraph $H'_i$ consisting of a tree plus one edge. Let $H' := H'_1 \cup \cdots \cup H'_k$. Thus $|E(H'_i)| = |V(H_i)|$ and $|E(H')| = |V(H)| = t$. Observe that $H'$ is 2-degenerate. By Lemma 14 with $q' = t$, $G$ contains $H$ as a minor.

Note that the entire proof of Theorem 15 is deterministic and leads to an algorithm for finding an $H$-minor in $G$ that has time complexity polynomial in both $|V(H)|$ and $|V(G)|$.

4 Probabilistic Linear Bounds

This section applies the probabilistic method to improve the linear bounds in Theorem 15.

**Lemma 16.** Let $H$ be a graph with $t$ vertices and $q$ edges. Let $G$ be a graph with $n \geq t$ vertices and average degree at least $d$. Then there is a spanning subgraph $R$ of $H$ with at least $dq \frac{n}{n-1}$ edges, such that $R$ is isomorphic to a subgraph of $G$.

**Proof.** Say $G$ has $m$ edges. Then $m \geq \frac{1}{2}dn$. Let $f$ be a random injection $f$ from $V(H)$ to $V(G)$. Then by the linearity of expectation,
\[
\mathbb{E}(\{|vw \in E(H) : f(v)f(w) \in E(G)\}|) = \sum_{vw \in E(H)} \mathbb{P}(f(v)f(w) \in E(G)) \\
= \sum_{vw \in E(H)} \frac{m}{\binom{n}{2}} \\
\geq \frac{dq}{n-1}.
\]

Thus there exists an injection $f$ from $V(H)$ to $V(G)$ such that $|\{vw \in E(H) : f(v)f(w) \in E(G)\}| \geq \frac{dq}{n-1}$. Then the spanning subgraph $R$ of $H$ with $E(R) := \{vw \in E(H) : f(v)f(w) \in E(G)\}$ satisfies the claim.

**Lemma 17.** Let $H$ be a graph with $t$ vertices and $q$ edges. Let $G$ be a graph with at most $n$ vertices and minimum degree at least $\delta$, such that
\[2\delta + 4 + \frac{\delta q}{n-1} \geq n + t + q.\]
Then $G$ contains a $(\leq 1)$-subdivision of $H$ as a subgraph.

**Proof.** By Lemma 16, there is a spanning subgraph $R$ of $H$ with at least $\frac{q_5}{n-1}$ edges, such that $R$ is isomorphic to a subgraph of $G$. For each vertex $v$ of $H$, let $v'$ be the corresponding vertex of $G$ (defined by this isomorphism). Observe that the number of edges $vw$ of $H$ such that $v'w'$ is not an edge of $G$ is at most $q(1 - \frac{\delta}{n-1})$. For each such edge we choose a common neighbour of $v'$ and $w'$ and route $vw$ by a path in $G$ with one internal vertex. Consider each edge $vw$ of $H$ such that $v'w'$ is not an edge of $G$ in turn. Both $v'$ and $w'$ have degree at least $\delta$ and they are not adjacent. Thus $v'$ and $w'$ have at least $2\delta - (n - 2)$ common neighbours. Since $2\delta - (n - 2) \geq (t - 2) + q(1 - \frac{\delta}{n-1})$, there is a common neighbour $x$ of $v'$ and $w'$ that is not already used by a vertex in $V(H) \setminus \{v, w\}$ or by a division vertex already assigned. Hence we may route $vw$ by the path $v'xw'$ in $G$. \[\square\]

Now we combine Lemma 6 and Lemma 17.

**Lemma 18.** Let $c_1 > 2$ and $c_2 > 1$. Define $a_1 := \frac{c_1}{2} + 1$, $b_1 := 2$, $a_2 := 2$, $b_2 := 1 + \frac{1}{c_1}$, $a_3 := c_2$, $b_3 := 1$, $a_4 := 4 - \frac{c_2}{2}$ and $b_4 := c_2$. Assume that for $1 \leq i \leq 4$,

$$0 < 2b_i - a_i \leq 3 \text{ and } b_i < a_i.$$

Let $\alpha \geq 4$ and $\beta$ be numbers such that for $1 \leq i \leq 4$,

$$\alpha \geq \frac{4}{2b_i - a_i} \text{ and } \beta \geq \frac{4(a_i - b_i)}{a_i(2b_i - a_i)}.$$

Then, for every graph $H$ with $t$ vertices and $q$ edges,

$$f(H) \leq \alpha t + \beta q.$$

**Proof.** We are given a $t$-vertex $q$-edge graph $H$ and a graph $G$ with average degree at least $\alpha t + \beta q \geq 4k$, where $k := \lfloor \frac{1}{4}(\alpha t + \beta q) \rfloor$. Since $c_1 > 2$ and $c_2 > 1$, Lemma 6 is applicable to $G$. If case (5) occurs in Lemma 6, then $K_k$ is a minor of $G$, which implies that $H$ is a minor of $G$ (since $\alpha > 4$ implies $t \leq k$). Now assume that case (i) occurs in Lemma 6 for some $i \in \{1, 2, 3, 4\}$. Let $a := a_i$ and $b := b_i$. Thus $G$ contains a minor $G'$ with $n \leq ak + 1$ vertices and minimum degree $\delta \geq bk$. By the assumptions,

$$(2b - a)k + 3 \geq (2b - a)(k + 1) > \frac{(2b - a)(\alpha t + \beta q)}{4} \geq t + \left(\frac{a - b}{a}\right)q = t + \left(1 - \frac{b}{a}\right)q.$$

Thus

$$2bk + 4 + \frac{b}{a}q \geq t + q + ak + 1.$$

Since $n \leq ak + 1$ and $\delta \geq bk$, we have $\frac{b}{a} \leq \frac{\delta}{n-1}$. Thus

$$2\delta + 4 + \frac{\delta q}{n-1} \geq t + q + n.$$
By Lemma 17, $G'$ contains a $(\leq 1)$-subdivision of $H$ as a subgraph. Hence $G$ contains $H$ as a minor. 

Optimising $\beta$ in Lemma 18 gives:

**Proposition 19.** For every graph $H$ with $t$ vertices and $q$ edges,

$$f(H) \leq 7.477t + 2.375q.$$ 

**Proof.** Apply Lemma 18 with $\alpha = 7.477$ and $\beta = 2.375$ and $c_1 = 3.375$ and $c_2 = 1.465$. \hfill \Box

We now prove the bound introduced in Section 1.

**Proof of Theorem 2.** We proceed by induction on $t$ with the following hypothesis: Every graph $G$ with average degree at least $t + cq$ contains every graph $H$ on $t$ vertices and $q$ edges as a minor, where $c := 6.291$. The result is trivial if $t \leq 1$. Now assume that $t \geq 2$. Let $G$ be a graph with $n$ vertices, $m$ edges, and average degree $\frac{2m}{n} \geq t + cq$. We may assume that $G$ is minor-minimal with average degree at least $t + cq$. By Lemma 3, $G$ has minimum degree at least $\lfloor \frac{t + cq}{2} \rfloor + 1 \geq q$.

Case 1. $H$ contains an isolated vertex $v$: Let $w$ be a vertex of minimum degree in $G$. Thus 

$$\deg(w) \leq \frac{2m}{n}. \text{ Hence the average degree of } G - w \text{ is}$$

$$\frac{2(m - \deg(w))}{n - 1} \geq \frac{2m - \frac{2m}{n} - (n - 1)}{n - 1} = \frac{2m}{n} - 1 \geq (t - 1) + cq.$$ 

By induction, $G - w$ contains $H - v$ as a minor. Thus $G$ contains $H$ as a minor (with $v$ embedded at $w$). Now assume that $H$ has no isolated vertex.

Case 2. Some component $T$ of $H$ is a tree: Let $\ell := |V(T)|$. Note that $t = |V(H)| \geq |V(T)| = \ell \geq 2$ and $q = |E(H)| \geq |E(T)| = \ell - 1$, implying that $G$ has minimum degree at least $\ell - 1$. By Lemma 9, there is a subgraph $T'$ of $G$ isomorphic to $T$. Let $G' := G - V(T')$. Note that $|E(G')| > m - \ell n$. By assumption, (a) $2m \geq (t + cq)n$. Since $c \geq 2$ and $\ell \geq 2$, we have $c(\ell - 1) \geq \ell$, implying (b) $-2\ell n \geq -(\ell + c\ell - c)n$. Since $q \geq \ell - 1$, we have (c) $0 \geq -c\ell(q - \ell + 1)$. Since $t \geq 1$, we have (d) $0 \geq \ell - \ell t$. Adding (a), (b), (c) and (d) gives

$$2m - 2\ell n \geq (t + cq)n - (\ell + c\ell - c)n - c\ell(q - \ell + 1) + \ell - \ell t$$

$$= n((t - \ell) + c(q - \ell + 1)) - \ell((t - \ell) + c(q - \ell + 1))$$

$$= (\ell - \ell(t - \ell) + c(q - \ell + 1))(n - \ell).$$

Hence the average degree of $G'$ is

$$\frac{2|E(G')|}{|V(G')|} \geq \frac{2(m - \ell n)}{n - \ell} \geq (t - \ell) + c(q - (\ell - 1)).$$
Since \( H - V(T) \) has \( t - \ell \) vertices and \( q - (\ell - 1) \) edges, by induction, \( G' \) contains \( H - V(T) \) as a minor. Hence \( G \) contains \( H \) as a minor, with \( T \) mapped to \( T' \). Now assume that no component of \( H \) is a tree. Thus \( q \geq t \).

Case 3. \( t + cq \geq \alpha t + \beta q \), where \( \alpha := 6.9687 \) and \( \beta := 2.484 \). Then \( G \) has average degree at least \( \alpha t + \beta q \), and thus contains \( H \) as a minor by Lemma 18 with \( c_1 = 3.484 \) and \( c_2 = 1.426 \).

Case 4: Now assume that \( (\alpha - 1)t \geq (c - \beta)q \). Let \( H_1, \ldots, H_k \) be the components of \( H \). Each \( H_i \) contains a spanning subgraph \( H'_i \) consisting of a tree plus one edge. Let \( H' := H'_1 \cup \cdots \cup H'_k \). Thus \( |E(H'_i)| = |V(H_i)| \) and \( |E(H')} = |V(H)| = t \). Observe that \( Q \) is 2-degenerate.

Define \( k := \lceil \frac{1}{2}(t + c(q - 2)) \rceil \). Thus \( G \) has average degree at least \( t + cq > 4k \). By Lemma 8, \( G \) contains a complete graph \( K_k \) as a minor, or \( G \) contains a minor \( G' \) with \( n' \) vertices and minimum degree \( \delta \), where \( 2\delta - n' \geq \sigma k \) and \( \sigma = 0.5773 \). In the first case, \( H \) is a subgraph of \( K_k \) (since \( k \geq q \geq t \)), implying \( H \) is a minor of \( G \). In the second case,

\[
2\delta - n' \geq \sigma k \geq \frac{\sigma}{4}(t + c(q - 2)) \geq \frac{\sigma(c - \beta)}{4(\alpha - 1)} q + \frac{\sigma c}{4}(q - 2) \geq q - 2,
\]

where the final inequality follows by considering the actual numerical values. Thus, by Lemma 13 with \( q' = t \), \( G' \) contains \( H \) as a minor. Therefore \( G \) contains \( H \) as a minor. \( \square \)

The bound in Theorem 2 is stronger than the bound in Theorem 15 when \( q \geq 1.567(t - i) \) (which is roughly when the non-isolated vertices in \( H \) have average degree at least 3).

## 5 General Result

The following lemma is at the heart of the proof of our main result (Theorem 1).

**Lemma 20.** For all \( \lambda \in (\frac{1}{2}, 1) \) and \( \epsilon \in (0, \lambda) \) there exists \( d_0 \) such that for every graph \( H \) with \( t \) vertices and average degree \( d \geq d_0 \), every graph \( G \) with \( n \geq (1 + \epsilon)\lceil \sqrt{\log_6 d} \rceil t \) vertices and minimum degree at least \( \lambda n \) contains \( H \) as a minor, where \( b = (1 - \lambda + \epsilon)^{-1} \).

We first sketch the proof. Say \( V(H) = \{1, 2, \ldots, t\} \). Our goal is to exhibit disjoint subsets \( X_1, \ldots, X_t \) of \( V(G) \) such that:

(a) \( G[X_i] \) is connected for \( 1 \leq i \leq t \), and

(b) for each edge \( ij \) of \( H \) there is an edge of \( G \) between \( X_i \) and \( X_j \).

We choose the \( X_i \) in three stages. In the first two stages, we choose disjoint sets \( S_1, \ldots, S_t \) and \( T_1, \ldots, T_t \) randomly, with the \( S_i \) non-empty, such that:
(i) every pair of vertices of \( G \) have many common neighbours not in \( S_1 \cup \cdots \cup S_t \cup T_1 \cup \cdots \cup T_t \),

(ii) for a small number of edges \( ij \in E(H) \), there is no edge between \( S_i \cup T_i \) and \( S_j \cup T_j \), and

(iii) the total number (summed over all \( i \)) of components in \( G[S_i \cup T_i] \) is small.

Having done so, it is straightforward to greedily chooses disjoint sets \( U_1, \ldots, U_t \), where \( |U_i| \) equals the number of components of \( G[S_i \cup T_i] \) minus 1, plus the number of edges \( ij \) of \( H \) with \( j > i \) such that there is no edge of \( G \) between \( S_i \cup T_i \) and \( S_j \cup T_j \), so that (a) and (b) hold for \( X_i = S_i \cup T_i \cup U_i \).

It remains to choose the \( S_i \) and \( T_i \) so that (i), (ii), and (iii) are satisfied. In the first stage we randomly choose disjoint sets \( S_1, \ldots, S_t \) each with \( \ell = \lceil \sqrt{\log_b d} \rceil \) vertices. In the second stage, we randomly choose the \( T_i \) and show that (i), (ii) and (iii) hold with positive probability. Some of the \( T_i \) are empty, the rest of which have \( 2\ell^2 \) vertices. \( T_i \) is non-empty precisely if the size of the neighbourhood of \( S_i \) is below a certain threshold. We need to add the \( T_i \) to such \( S_i \) in the second phase to ensure that (ii) holds. In the first phase, we focus on bounding the number of \( i \) for which the neighbourhood of \( S_i \) is small. This allows us to bound the number of vertices used in the second phase, which helps in proving (i). In the following proof, no effort is made to minimise \( d_0 \).

**Proof of Lemma 20.** Note that in \( G \), every pair of vertices have at least \( (2\lambda - 1)n \) common neighbours (and \( 2\lambda - 1 > 0 \)). Let \( \ell := \lceil \sqrt{\log_b d} \rceil \). Define

\[
\nu := \left( \frac{1 - \lambda}{1 - 2\lambda + \epsilon} \right) ^ \ell \quad \text{and} \quad \mu := (1.5692)^\ell (1 - \lambda)^{5\ell^2/6}.
\]

Observe that \( 0 < \nu, \mu < 1 \) (since \( \lambda > \frac{1}{2} \)), and \( \nu \) and \( \mu \) tend to 0 exponentially as \( \ell \to \infty \). Now define

\[
\theta := 5(\nu + \mu)(\ell + \ell^2) + 5\nu\ell^2 + 8.
\]

Elementary calculus shows that \( \theta \) is bounded by a function of \( \epsilon \) and \( \lambda \) independent of \( \ell \). Thus, taking \( d_0 \) at least some function of \( \epsilon \) and \( \lambda \), since \( d \geq d_0 \), we may assume that \( d, \ell, t \) and \( n \) are at least functions of \( \epsilon, \lambda \) and \( \theta \). In particular, we assume:

\[
\epsilon(1 - \epsilon)(2\lambda - 1)\ell \geq 2\theta \quad (4)
\]

\[
\exp\left( \frac{\epsilon^2(2\lambda - 1)^2n}{8} \right) \geq 10 \binom{n}{2} \quad (5)
\]

\[
\exp\left( \frac{\epsilon^4n}{2(1 + \epsilon)^2} \right) \geq 10n. \quad (6)
\]
For a set $S$ of vertices in $G$, a vertex $v$ of $G$ is a non-neighbour of $S$ if $v$ is not in $S$ and $v$ is not adjacent to a vertex in $S$.

Say $V(H) = \{1, 2, \ldots, t\}$. Let $S_1, \ldots, S_t, T_1, \ldots, T_t$ be pairwise disjoint subsets of $V(G)$. Say $S_i$ is bad if $S_i$ has at least $(n - \ell)(1 - \lambda + \epsilon)^\ell$ non-neighbours, otherwise $S_i$ is good. Say $S_i$ is disjointed if $G[S_i]$ has a connected component with at most $\ell^2$ vertices. An edge $ij \in E(H)$ is problematic if $S_i$ or $S_j$ is good (or both), but there is no edge in $G$ between $S_i$ and $S_j$. An edge $ij \in E(H)$ is nasty if $S_i$ and $S_j$ are both bad and there is no edge in $G$ between $S_i \cup T_i$ and $S_j \cup T_j$.

Below we prove the following two claims.

**Claim 1.** There exists subsets $S_1, \ldots, S_t$ of $V(G)$ satisfying the following properties:

- $(P0)$ $S_1, \ldots, S_t$ are pairwise disjoint, and $|S_i| = \ell$ for $1 \leq i \leq t$.
- $(P1)$ At most $5\nu t$ of the $S_i$ are bad.
- $(P2)$ At most $5\mu t$ of the $S_i$ are disjointed.
- $(P3)$ At most $2t$ edges of $H$ are problematic.
- $(P4)$ For all vertices $v, w \in V(G)$,
  \[ |(N(v) \cap N(w)) \setminus (S_1 \cup \cdots \cup S_t)| \geq \left(1 - \frac{\ell t}{n} - \frac{\epsilon}{2}\right) |N(v) \cap N(w)|. \]
- $(P5)$ For each vertex $v \in V(G)$,
  \[ |N(v) \setminus (S_1 \cup \cdots \cup S_t)| \geq \left(1 - \frac{\ell t}{n} - \frac{\epsilon^2}{\lambda(1+\epsilon)}\right) |N(v)|. \]

**Claim 2.** Given subsets $S_1, \ldots, S_t$ of $V(G)$ that satisfy $(P0)$, $(P1)$, $(P2)$, $(P3)$, $(P4)$ and $(P5)$, there exist subsets $T_1, \ldots, T_t$ of $V(G)$ satisfying the following properties:

- $(Q0)$ $S_1, \ldots, S_t, T_1, \ldots, T_t$ are pairwise disjoint, and for $1 \leq i \leq t$,
  \[ |T_i| = \begin{cases} \ell^2 & \text{if } S_i \text{ is bad}, \\ 0 & \text{if } S_i \text{ is good}. \end{cases} \]
- $(Q1)$ At most $\frac{t}{2}$ edges of $H$ are nasty.

Before proving these claims we show that they imply the lemma. By $(P0)$, \[ |S_1 \cup \cdots \cup S_t| = \ell t, \] and by $(P1)$ and $(Q0)$, \[ |T_1 \cup \cdots \cup T_t| \leq 5\nu t \cdot \ell^2. \] Mark each vertex in $\bigcup_i S_i \cup T_i$ as used.

For $i = 1, 2, \ldots, t$, choose a set $U_i$ of less than $r_i$ vertices in $G$ as follows, where $r_i$ is the number of components of $G[S_i \cup T_i]$. Note that if $S_i$ is good and not disjointed,
then $|S_i \cup T_i| = \ell$ and each component of $G[S_i \cup T_i]$ has more than $\ell^2$ vertices, implying $r_i \leq 5$. Otherwise (if $S_i$ is bad or disjointed) all we need is that $r_i \leq |S_i| + |T_i| \leq \ell + \ell^2$. For $1 \leq j \leq r_i$, let $x_j$ be an arbitrary vertex in the $j$-th component of $G[S_i \cup T_i]$. For $j = 1, \ldots, r_i - 1$, choose an unused common neighbour $z$ of $x_j$ and $x_{j+1}$, add $z$ to $U_i$, and mark $z$ as used.

To prove that such a vertex $z$ exists, we first estimate $|\bigcup_i U_i|$. By (P1) and (P2), at most $5(\nu + \mu)t$ of the $S_i$ are bad or disjointed. Each of these contribute at most $\ell + \ell^2$ vertices to $\bigcup_i U_i$. For each $S_i$ that is good and not disjointed, at most 5 vertices are added to $\bigcup_i U_i$. In total, by (8),

$$\left| \bigcup_i U_i \right| \leq 5(\nu + \mu)(\ell + \ell^2)t + 5t \quad \text{and} \quad (9)$$

$$\left| \bigcup_i T_i \cup U_i \right| \leq 5(\nu + \mu)(\ell + \ell^2)t + 5t + 5\nu\ell^2t = (\theta - 3)t. \quad (10)$$

By (P4) and (4) and (10), and since $n \geq (1 + \epsilon)\ell t$,

$$|N(x_j) \cap N(x_{j+1})| \setminus \bigcup_i (S_i \cup T_i \cup U_i)| \geq \left(1 - \frac{\ell t}{n} - \frac{\epsilon}{2} \right) (2\lambda - 1)n - (\theta - 3)t$$

$$\geq \left(1 - \frac{1}{1 + \epsilon} - \frac{\epsilon}{2} \right) (2\lambda - 1)(1 + \epsilon)\ell t - (\theta - 3)t$$

$$\geq \frac{\epsilon}{2} (1 - \epsilon)(2\lambda - 1)\ell t - (\theta - 3)t$$

$$\geq 3t > 0.$$

The used vertices are precisely $\bigcup_i (S_i \cup T_i \cup U_i)$. Thus the above inequality says that there is an unused common neighbour $z$ of $x_j$ and $x_{j+1}$, as claimed. By construction, each subgraph $G[S_i \cup T_i \cup U_i]$ is connected.

Suppose that there is no edge in $G$ between $S_i \cup T_i \cup U_i$ and $S_j \cup T_j \cup U_j$ for some edge $ij \in E(H)$. If $S_i$ or $S_j$ is good, then $ij$ is problematic, otherwise $ij$ is nasty. Thus, by (P3) and (Q1) there are at most $3t$ such edges. Choose an unused common neighbour $z$ of some vertex in $S_i \cup T_i \cup U_i$ and some vertex in $S_j \cup T_j \cup U_j$, add $z$ to $U_i$, and mark $z$ as used. This step increases $|\bigcup_i U_i|$ by at most $3t$, implying that $|\bigcup_i T_i \cup U_i| \leq \theta t$ by (10). By the argument above, such a vertex $z$ exists. Now $S_1, \ldots, S_t, T_1, \ldots, T_t, U_1, \ldots, U_t$ are pairwise disjoint, $G[S_i \cup T_i \cup U_i]$ is connected for each $i$, and for each edge $ij \in E(H)$, there is an edge in $G$ between $S_i \cup T_i \cup U_i$ and $S_j \cup T_j \cup U_j$. Thus $G$ contains $H$ as a minor (by contracting each set $S_i \cup T_i \cup U_i$). It remains to prove Claims 1 and 2.

**Proof of Claim 1.** Choose $S_1, \ldots, S_t \subseteq V(G)$ satisfying (P0) uniformly at random. Since $n > \ell t = |S_1 \cup \cdots \cup S_t|$, such subsets exist. We now bound the probability that each of (P1), (P2), (P3), (P4) and (P5) fail.

(P1): Consider a subset $S_i$ and a vertex $v \in G - S_i$. Since $v$ has degree at least $\lambda n$ in $G$, and since $S_i$ is chosen at random in $V(G)$, for each vertex $x \in S_i$, the probability that $v$ is
Thus the expected number of bad $S_i$ is more than $5\nu t$. Since (P1) fails if the number of bad $S_i$ is more than $5\nu t$, Markov’s inequality implies that (P1) fails with probability less than $\frac{1}{5}$.

(P2): Consider a disjointed set $S_i$. The number of subsets of $S_i$ with at most $\frac{\ell}{6}$ vertices is

$$\sum_{j=0}^{\lfloor \ell/6 \rfloor} \binom{\ell}{j} \leq 2^{h(1/6)\ell} < (1.5692)^\ell,$$

where $h(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ is the binary entropy function. Let $v$ be a vertex in a component of $G[S_i]$ with at most $\frac{\ell}{6}$ vertices. Thus $v$ is not adjacent to the at least $\frac{5}{6}\ell$ vertices in the other components of $G[S_i]$. These other vertices were chosen randomly. Thus the probability that $S_i$ is disjointed is less than $(1.5692)^\ell(1-\lambda)^{5\ell/6} = \mu$, and the expected number of disjointed $S_i$ is at most $\mu t$. Since (P2) fails if the number of disjointed $S_i$ is more than $5\mu t$, Markov’s inequality implies that (P2) fails with probability less than $\frac{1}{5}$.

(P3): Consider a problematic edge $ij$ in $H$, where $S_i$ is good. Thus $S_i$ has at most $(n-\ell)(1-\lambda+\epsilon)^\ell$ non-neighbours. Since there is no edge between $S_i$ and $S_j$, every vertex in $S_j$ is one of these at most $(n-\ell)(1-\lambda+\epsilon)^\ell$ non-neighbours of $S_i$. Since $S_j$ is chosen randomly out of the $n-\ell$ vertices in $G - S_i$, the probability that each of the $\ell$ vertices in $S_j$ is a non-neighbour of $S_i$ is at most $(1-\lambda+\epsilon)^\ell \leq \frac{1}{2}$. (This is the key inequality in the whole proof.) Thus the probability that $ij \in E(H)$ is problematic is at most $\frac{1}{2}$. By the linearity of expectation, the expected number of problematic edges (out of a total of $\frac{\ell}{2}$) is at most $\frac{5}{2}$. Since (P3) fails if the number of problematic edges is more than $\frac{5}{2}$, Markov’s inequality implies that the probability that (P3) fails is less than $\frac{1}{5}$.

(P4): Consider a pair of distinct vertices $v, w \in V(G)$. Let $X$ be the random variable $|\{N(v) \cap N(w)\} \setminus (S_1 \cup \cdots \cup S_t)|$. Since $S_1 \cup \cdots \cup S_t$ consists of $\ell t$ vertices chosen randomly from the $n$ vertices in $G$, we have $\mathbb{E}(X) = (1 - \frac{\ell t}{n})|N(v) \cap N(w)|$. If (P4) fails for $v, w$ then $X - \mathbb{E}(X) < -\frac{\epsilon}{2}|N(v) \cap N(w)|$. Hence

$$\mathbb{P}((\text{P4} \text{ fails for } v, w) \leq \mathbb{P}(|X - \mathbb{E}(X)| \geq \frac{\epsilon}{2}|N(v) \cap N(w)|).$$

The selection of $S_1, \ldots, S_t$ may be considered as $\ell t$ trials, each choosing a random vertex from the vertices not already chosen. Changing the outcome of any one trial changes $\mathbb{E}(X)$ by at most 1. Thus by Azuma’s inequality\(^3\) with $x = \frac{\epsilon}{2}|N(v) \cap N(w)|$,

$$\mathbb{P}((\text{P4} \text{ fails for } v, w) \leq \mathbb{P}(|X - \mathbb{E}(X)| > x) \leq 2 \exp \left( \frac{-(\frac{\epsilon}{2}|N(v) \cap N(w)|)^2}{2\ell t} \right).$$

\(^3\)Azuma’s inequality \cite{azuma1967} says that if $X$ is a random variable determined by $n$ trials $R_1, \ldots, R_n$, such that
Since \(|N(v) \cap N(w)| \geq (2\lambda - 1)n\) and \(n > \ell t\) and by (5),
\[
\mathbb{P}(\text{(P4) fails for } v, w) \leq 2 \exp\left(\frac{-\epsilon^2(2\lambda - 1)^2n^2}{8\ell t}\right) < 2 \exp\left(\frac{-\epsilon^2(2\lambda - 1)^2n}{8}\right) \leq \left(\frac{n}{2}\right)^{-1}.
\]
By the union bound, the probability that (P4) fails (for some pair of distinct vertices in \(G\)) is less than \(\frac{1}{5}\).

(P5): Consider a vertex \(v \in V(G)\). Let \(X\) be the random variable \(|N(v) \setminus (S_1 \cup \cdots \cup S_t)|\). Since \(S_1 \cup \cdots \cup S_t\) consists of \(\ell t\) vertices chosen randomly from the \(n\) vertices in \(G\), we have \(\mathbb{E}(X) = (1 - \frac{\ell t}{n})|N(v)|\). If (P5) fails for \(v\) then \(X - \mathbb{E}(X) < -\frac{\epsilon^2}{\lambda(1+\epsilon)}|N(v)|\). Hence
\[
\mathbb{P}(\text{(P5) fails for } v) \leq \mathbb{P}(|X - \mathbb{E}(X)| \geq \frac{\epsilon^2}{\lambda(1+\epsilon)}|N(v)|).
\]
As before, Azuma’s inequality is applicable with \(x = \frac{\epsilon^2}{\lambda(1+\epsilon)}|N(v)|\), giving
\[
\mathbb{P}(\text{(P5) fails for } v) \leq 2 \exp\left( - \left( \frac{\epsilon^2|N(v)|}{\lambda(1+\epsilon)} \right)^2 / 2\ell t \right).
\]
Since \(|N(v)| \geq \lambda n\) and \(n > \ell t\), and by (6),
\[
\mathbb{P}(\text{(P5) fails for } v) \leq 2 \exp\left( \frac{-\epsilon^4n^2}{2(1+\epsilon)^2\ell t} \right) < 2 \exp\left( \frac{-\epsilon^4n}{2(1+\epsilon)^2} \right) \leq (5n)^{-1}.
\]
By the union bound, the probability that (P5) fails (for some vertex in \(G\)) is less than \(\frac{1}{5}\).

We have shown that each of (P1)–(P5) fail with probability less than \(\frac{1}{5}\). By the union bound, the probability that at least one of (P1)–(P5) fails is less than 1. Thus the probability that none of (P1)–(P5) fails is greater than 0. Thus there exists \(S_1, \ldots, S_t\) such that all of (P1)–(P5) hold. \(\square\)

**Proof of Claim 2.** Let \(W := V(G) \setminus (S_1 \cup \cdots \cup S_t)\). By (P5), since \(G\) has minimum degree at least \(\lambda n\), and since \(n \geq (1 + \epsilon)\ell t\), the subgraph \(G[W]\) has minimum degree at least
\[
\left(1 - \frac{\ell t}{n} - \frac{\epsilon^2}{\lambda(1+\epsilon)}\right)\lambda n = \lambda(n - \ell t) - \frac{\epsilon^2n}{1+\epsilon} \geq (\lambda - \epsilon)(n - \ell t) = (\lambda - \epsilon)|W|.
\]
Choose \(T_1, \ldots, T_t \subseteq W\) satisfying (Q0) uniformly at random. Such subsets exist, since by (8) and (4) (which implies that \(\epsilon \geq 5\nu\ell\)),
\[
|W| = n - \ell t \geq \epsilon \ell t \geq 5\nu\ell^2t \geq |T_1 \cup \cdots \cup T_t|.
\]
for each \(i\), and any two possible sequences of outcomes \(r_1, \ldots, r_i\) and \(r_1, \ldots, r_{i-1}, r'_i\),
\[
|\mathbb{E}(X | R_1 = r_1, \ldots, R_i = r_i) - \mathbb{E}(X | R_1 = r_1, \ldots, R_{i-1} = r_{i-1}, R_i = r'_i)| \leq c_i,
\]
then \(\mathbb{P}(|X - \mathbb{E}(X)| > x) \leq 2 \exp(-x^2/(2\sum c_i^2))\). In all our applications, \(c_i = 1\).
If \( ij \in E(H) \) is a nasty edge, and \( v \) is any vertex in \( T_j \), then \( v \) is adjacent to no vertex in \( T_i \). Since \( v \) and \( T_i \) were chosen randomly in \( W \), and \( v \) is adjacent to at least \( (\lambda - \epsilon)|W| \) vertices in \( W \), the probability that \( v \) is adjacent to no vertex in \( T_i \) is at most

\[
(1 - (\lambda - \epsilon))^{|T_i|} = b^{-\epsilon^2} \leq b^{-\log_b d} = d^{-1}.
\]

Thus the probability of an edge in \( H \) being nasty is at most \( d^{-1} \). Hence the expected number of nasty edges (out of a total of \( d t \)) is at most \( t^2 \). With positive probability the number of nasty edges is at most \( t^2 \). Hence there exists \( T_1, \ldots, T_t \) such that (Q1) holds. \( \square \)

This completes the proof of Lemma 20. \( \square \)

**Proof of Theorem 1.** Define \( \epsilon := 0.00001 \) and \( \lambda := 0.6518 \) and \( b := (1 - \lambda + \epsilon)^{-1} > 2.8718 \). Let \( H \) be a graph with \( t \) vertices and average degree \( d \geq d_0 \), where \( d_0 \) is sufficiently large compared to \( \epsilon \) and \( \lambda \) (and thus an absolute constant). Let \( G \) be a graph with average degree at least \( 3.895 \sqrt{\ln d} t \). Define \( k := \left\lceil (1 + \epsilon)\left\lceil \sqrt{\log_b d} \right\rceil t \right\rceil \). Now

\[
3.895 \sqrt{\ln b} > 3.895 \sqrt{\ln 2.8718} > 4(1 + \epsilon).
\]

Let \( \eta := 3.895 - 4(1 + \epsilon)/\sqrt{\ln b} \), which is positive. Thus \( 3.895 - \eta = 4(1 + \epsilon)/\sqrt{\ln b} \) and

\[
(3.895 - \eta)\sqrt{\ln d} t = 4(1 + \epsilon)\sqrt{\ln d} t/\sqrt{\ln b} = 4(1 + \epsilon) \sqrt{\log_b d} \sqrt{t}.
\]

For sufficiently large \( d_0 \) and \( d \geq d_0 \), we have \( \eta \sqrt{\ln d} t \geq 4(1 + \epsilon)t + 4 \). Adding these two inequalities gives

\[
3.895 \sqrt{\ln d} t \geq 4(1 + \epsilon) \sqrt{\log_b d} t + 4(1 + \epsilon)t + 4 \geq 4[(1 + \epsilon)\left\lceil \sqrt{\log_b d} \right\rceil t] = 4k.
\]

Thus \( G \) has average degree at least \( 4k \). By Lemma 7, either \( G \) contains \( K_k \) as a minor or \( G \) contains a minor \( G' \) with \( n > k \) vertices and minimum degree at least \( \lambda n \). In the first case, \( G \) contains \( H \) as a minor (since \( k \geq t \) for sufficiently large \( d_0 \) and \( d \geq d_0 \)). In the second case, by Lemma 20, there exists \( d_0 \) depending only on \( \epsilon \) and \( \lambda \), such that \( G' \), and thus \( G \), contains \( H \) as a minor (assuming \( d \geq d_0 \)). \( \square \)

### 6 Open Problems

We conclude with a number of open problems that focus on \( f(H) \) for various well-structured (non-random) graphs \( H \).

- Let \( H \) consist of \( k \geq 1 \) disjoint triangles. Corradi and Hajnal [4] proved that every graph of minimum degree at least \( 2k \) and order at least \( 3k \) contains \( k \) disjoint cycles, and thus contains \( H \) as a minor. Let \( G \) be a graph with average degree at least
$4k-2$ for some positive integer $k$. By Lemma 3, $G$ has a minor with minimum degree at least $2k$ and average degree at least $4k-2$ (implying the number of vertices is at least $4k - 1 \geq 3k$). By the above result of Corradi and Hajnal [4], $G$ contains $H$ as a minor, and $f(H) \leq 4k-2$. (The same conclusion also follows from a result of Justesen [10].) In fact, $f(H) = 4k-2$ since if $G$ is the complete bipartite graph $K_{2k-1,n}$ with $n \gg k$, then the average degree of $G$ tends to $4k - 2$ as $n \to \infty$, but $G$ contains no $H$-minor since each cycle includes at least two vertices on each side. We conjecture the following generalisation: Every graph with average degree at least $\frac{4}{3}t - 2$ contains every 2-regular graph on $t$ vertices as a minor.

- Fix integers $d \ll s \ll t$. Let $H_0$ be a $d$-regular graph on $t$ vertices. Myers and Thomason [22] prove that $f(H_0) \geq c\sqrt{\log dt}$. Let $H$ be the graph obtained from $H_0$ by adding $s$ dominant vertices. Thus $H$ has average degree about $2s$. Hence $c_1\sqrt{\log dt} \leq f(H_0) \leq f(H) \leq c_2\sqrt{\log st}$ by Theorem 1. Where $f(H)$ lies between $c\sqrt{\log dt}$ and $c\sqrt{\log st}$ is an interesting open problem.

- What is the least function $g$ such that every graph with average degree at least $g(k) \cdot t$ contains every graph with $t$ vertices and treewidth at most $k$ as a minor? Note that “graph with $t$ vertices and treewidth at most $k$” can be replaced by “$k$-tree on $t$ vertices” in the above. Since every such $k$-tree has less than $kt$ edges, Proposition 19 and Theorem 1 respectively imply that $g(k) \leq 7.477 + 2.375k$ and $g(k) \in O(\sqrt{\log k})$. Since every 2-tree is 2-degenerate, $g(2) \leq 6.929$ by Lemma 11.

- What is the minimum constant $c$ such that every graph with average degree at least $ct^2$ contains the $t \times t$ grid as a minor? Since the $t \times t$ grid is 2-degenerate, $c \leq 6.929$ by Lemma 11.

- What is the least constant $c$ such that every graph with average degree at least $ct$ contains every planar graph with $t$ vertices as a minor? Since such a planar graph has less than $3t$ edges, Proposition 19 implies that $c \leq 14.602$.

- What is the least function $g$ such that every graph with average degree at least $g(k) \cdot t$ contains every $K_k$-minor-free graph with $t$ vertices as a minor? Since every $K_k$-minor-free graph has average degree $O(k^{\sqrt{\log k}})$, Theorem 1 implies that $g(k) \in O(\sqrt{\log k})$.

- Every graph with average degree at least $10t^2$ contains a subdivision of $K_t$ as a subgraph. A proof of this result is given by Diestel [6] based on results on highly connected subgraphs by Mader [19] and on linkages by Thomas and Wollan [25]. This method immediately generalises to prove that for every graph $H$ with $t$ vertices and $q$ edges, every graph with average degree at least $4t + 20q$ contains a subdivision of $H$ as a subgraph. Determining the best constants in such a result is an interesting line of research. Note that there is a linear lower bound for a graph $H$ with $t$ vertices and $q$ edges, such that every set of at least $\frac{t}{2}$ vertices induces a subgraph with at
least $\epsilon q$ edges, for some $\epsilon > 0$. Say $K_{n,n}$ contains a subdivision of $H$. At least $\frac{t}{2}$ original vertices of $H$ are on one side of $K_{n,n}$. Thus at least $\epsilon q$ edges have a division vertex on the other side of $K_{n,n}$, implying $n \geq \epsilon q$. Hence, average degree at least $\epsilon q$ is needed to force a subdivision of $H$.

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