Forcing a sparse minor

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Abstract

This paper addresses the following question for a given graph H: what is the minimum number $f(H)$ such that every graph with average degree at least $f(H)$ contains H as a minor? Due to connections with Hadwiger's Conjecture, this question has been studied in depth when H is a complete graph. Kostochka and Thomason independently proved that $f(K_t) = ct\sqrt{\ln t}$. More generally, Myers and Thomason determined $f(H)$ when H has a super-linear number of edges. We focus on the case when H has a linear number of edges. Our main result, which complements the result of Myers and Thomason, states that if H has t vertices and average degree d at least some absolute constant, then $f(H) \leqslant 3.895\sqrt{\ln d}\,t$. Furthermore, motivated by the case when H has small average degree, we prove that if H has t vertices and q edges, then $f(H) \leqslant t + 6.291q$ (where the coefficient of 1 in the t term is best possible).

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1 Introduction

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. This paper studies average degree conditions that force an H-minor. In particular, it focuses on the infimum of all numbers d such that every graph with average degree at least d contains H as a minor, which we denote by $f(H)$. We are interested in determining bounds on $f(H)$ that are a function of the number of edges and vertices of H.

We distinguish two types of graphs H (or to be more precise, families of graphs H). We consider H to be 'dense' if $|E(H)| \geqslant |V(H)|^{1+\tau}$ for some constant $\tau > 0$. On the other

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hand, we consider H to be 'sparse' if $|E(H)| \leqslant c|V(H)|$ for some constant c (independent of $|V(H)|$). This paper focuses on $f(H)$ for graphs H that are not dense, and especially those that are sparse.

Previous work in this field concerns dense H. Indeed, largely motivated by Hadwiger's Conjecture, $f(H)$ was first studied for $H = K_t$, the complete graph on t vertices. Dirac [7] proved that for $t \leqslant 5$, every n-vertex K_t-minor-free graph has at most $(t-2)n - \binom{t-1}{2}$ edges, and this bound is tight. Mader [18] extended this result for $t \leqslant 7$. It follows that $f(K_t) = 2t - 4$ for $t \leqslant 7$. For $t \geqslant 8$ there are K_t-minor-free graphs with more than $(t-2)n - \binom{t-1}{2}$ edges. However, results of Jørgensen [9] and Song and Thomas [23] respectively imply that $f(K_8) = 12$ and $f(K_9) = 14$. Thus $f(K_t) = 2t-4$ for $t \leqslant 9$. Song [24] proved that $f(K_{10}) \leqslant 22$ and $f(K_{11}) \leqslant 26$, and conjectured that both these bounds can be improved.

The first upper bound on $f(K_t)$ for general t was due to Mader [17], who proved that $f(K_t) \leqslant 2^{t-2}$. Mader [18] later proved that $f(K_t) \in \mathcal{O}(t \ln t)$. Kostochka [11, 12] and de la Vega [5] (based on the work of Bollobás et al. [2]) independently proved the lower bound, $f(K_t) \in \Omega(t\sqrt{\ln t})$. A matching upper bound of $f(K_t) \in \mathcal{O}(t\sqrt{\ln t})$ was independently proved by Kostochka [11, 12] and Thomason [26]. Later, Thomason [27] determined the asymptotic constant:

$$f(K_t) = (\alpha + o(1))t\sqrt{\ln t}, \tag{1}$$

where $\alpha = 0.628\dots$ is an explicit constant, and $o(1)$ denotes a term tending to 0 as $t \to \infty$. Myers [20] characterised the extremal K_t-minor-free graphs as unions of pseudo-random graphs.

Myers and Thomason [22] generalised (1) for dense graphs H as follows. They introduced a graph parameter γ with the property that if $t = |V(H)|$ then

$$f(H) = (\alpha\,\gamma(H) + o(1))t\sqrt{\ln t}, \tag{2}$$

where $\gamma(H) \leqslant 1$ and $o(1)$ denotes a term (slowly) tending to 0 as $t \to \infty$. Note that when H is sparse, the $o(1)$ term might dominate $\gamma(H)$, in which case this result says little about $f(H)$, as discussed by Myers and Thomason [22, Section 7]. For example, (2) does not determine f for specially structured graphs such as unbalanced complete bipartite graphs; see Section 1.2 below.

Moreover, Myers and Thomason [22] proved that if H has $t^{1+\tau}$ edges, for some constant $\tau > 0$, then $\gamma(H) \leqslant \sqrt{\tau}$ (with equality for almost all H and for all regular H). That is, if H has average degree $d = 2t^{\tau}$, then

$$f(H) \leqslant \alpha\sqrt{\ln d}\, t + o(t\sqrt{\ln t}). \tag{3}$$

Since the $o(1)$ term tends to 0 slowly, this bound also says little when H is sparse.

1.1 Non-Dense Graphs H

With respect to typical non-dense graphs, we prove the following theorem in the same direction as (3) except it does apply when G is not dense.

Theorem 1. *There is an absolute constant d_0, such that for every graph H with t vertices and average degree $d \geqslant d_0$,*

$$f(H) \leqslant 3.895\sqrt{\ln d}\, t.$$

The lower bounds on $f(H)$ due to Myers and Thomason [22] apply even when H is sparse. It follows that Theorem 1 is tight up to a constant factor for numerous graphs H. In particular, if H is sufficiently large, and is random, or regular, or even if the maximum and minimum degrees are close, then Theorem 1 is tight[1]. Theorem 1 is proved in Section 5.

When d is very small, Theorem 1 is not applicable. Thus, motivated by the case of graphs H with small average degree, we investigate linear bounds of the form

$$f(H) \leqslant \alpha|V(H)| + \beta|E(H)|$$

for explicit constants α and β. A first question in this regard is the smallest possible values for α and β. We can push β as close to 0 as we like. Indeed, Theorem 1 immediately implies that for every $\beta > 0$ there is a constant $c = c(\beta)$ such that $f(H) \leqslant ct + \beta q$ for every graph H with t vertices and q edges, On the other hand, $\alpha \geqslant 1$ in any such bound, since K_{t-1} has average degree $t-2$ but does not contain the graph with t vertices and no edges as a minor. At this extremity we prove the following (in Section 3):

Theorem 2. *For every graph H with t vertices and q edges,*

$$f(H) \leqslant t + 6.291q.$$

Note that $\beta \geqslant \frac{1}{3}$ in any bound of the form $f(H) \leqslant t + \beta q$ (since in Section 6 we observe that if H consists of $k \geqslant 1$ disjoint triangles, then $f(H) = 4k - 2 = t + \frac{q}{3} - 2$).

Also, note that a linear bound of the form $f(H) \leqslant \alpha t + \beta q$ can also be concluded from a theorem of Fox and Sudakov [8, Theorem 5.1] in conjunction with an old lemma of Mader (our Lemma 4).

[1] For example, consider a d-regular graph H on t vertices. In the notation of Myers and Thomason [22], $\tau(H) = \log_t(\frac{d}{2})$. Their Theorem 4.8 and Corollary 4.9 imply that $\gamma(H) \to \sqrt{\tau(H)}$ as $t \to \infty$. Let $n := \lfloor \gamma(H) t \sqrt{\ln t} \rfloor \to \sqrt{\ln d}t$. Myers and Thomason [22, Theorem 2.3] prove that H is a minor of a random graph $G(n, \frac{1}{2})$ with probability tending to 0 as $t \to \infty$. Thus some graph with average degree $\frac{n}{2}$ contains no H-minor. Thus $f(H) \geqslant c\sqrt{\ln d}t$.

1.2 Specially Structured Graphs

Attention in the literature has also been focused on specially structured graphs, as we now discuss. We propose some open problems in this regard in our concluding section.

Let $K_{s,t}$ be the complete bipartite graph with $s \leqslant t$. First consider when s is small. Chudnovsky et al. [3] proved that the maximum number of edges in a $K_{2,t}$-minor-free graph is at most $\frac{1}{2}(t+1)(n-1)$, which is tight for infinitely many values of n. This implies that $f(K_{2,t}) = t+1$. Myers [21] had earlier proved the same result for sufficiently large t. Kostochka and Prince [14] proved that for $t \geqslant 6300$ and $n \geqslant t+3$, every n-vertex graph G with more than $\frac{1}{2}(t+3)(n-2)+1$ edges has a $K_{3,t}$ minor, and this bound is tight. Thus $f(K_{3,t}) = 2t+6$ for $t \geqslant 6300$.

Now consider general complete bipartite graphs. Myers [21] conjectured that for every integer s there exists a positive constant c such that $f(K_{s,t}) \leqslant ct$ for every integer t. Kühn and Osthus [16] and Kostochka and Prince [13] independently proved certain strengthenings of this conjecture. Let $K^*_{s,t}$ denote the graph obtained from $K_{s,t}$ by adding all edges between the s vertices of degree t. Of course, $f(K_{s,t}) \leqslant f(K^*_{s,t})$. Kühn and Osthus [16] proved that $f(K^*_{s,t}) \leqslant (1+\epsilon)t$ for all $t \geqslant t(\epsilon)$ and $s \leqslant \epsilon^6 t/\log t$. Kostochka and Prince [13] proved that $t+3s-5\sqrt{s} \leqslant f(K_{s,t}) \leqslant f(K^*_{s,t}) \leqslant t+3s$ for $t > (180 s \log_2 s)^{1+6s\log_2 s}$. Kostochka and Prince [15] refined their method to conclude a similar upper bound of $f(K^*_{s,t}) \leqslant t+8s\log_2 s$ under a more reasonable assumption about t, namely that $t/\log_2 t \geqslant 1000s$. This result is best possible in the sense that the 1000 and 8 cannot be simultaneously reduced to $1/18$, say. Again, considering $K_{s,t}$ rather than $K^*_{s,t}$ does not significantly affect the bounds.

See [28–30] for various results concerning average or minimum degree conditions that force several copies of a given graph as a minor or subdivision.

2 A Minor with Large Minimum Degree

The standard approach to find an H-minor in a graph G of high average degree involves first finding a minor G' of G with high minimum degree and few vertices. Then it is shown that H is a minor of G' and hence of G. This approach was introduced by Mader [18]. This section presents a variety of results proving that such G' exist. Many of these results can be found in reference [18], but since this paper is written in German, we include all the proofs. A graph G is *minor-minimal* with respect to some set $\mathcal{G}$ of graphs if $G \in \mathcal{G}$ and every proper minor of G is not in $\mathcal{G}$.

Lemma 3. *Let G be a minor-minimal graph with average degree at least d. Then every edge of G is in at least $\lfloor \frac{d}{2} \rfloor$ triangles, and every vertex has degree at least $\lfloor \frac{d}{2} \rfloor + 1$.*

Proof. Say G has n vertices and m edges, where $2m \geqslant dn$. Suppose on the contrary that

G contains an edge e in $t < \lfloor \frac{d}{2} \rfloor$ triangles. Then G/e has $n-1$ vertices and $m-1-t$ edges. Thus G/e has average degree $\frac{2(m-1-t)}{n-1} > \frac{2m}{n} \geqslant d$. Hence G is not minor-minimal with average degree at least d. This contradiction proves that every edge of G is in at least $\lfloor \frac{d}{2} \rfloor$ triangles. Thus every vertex has degree at least $\lfloor \frac{d}{2} \rfloor + 1$. □

Lemma 4. *Every graph with average degree $d \geqslant 1$ contains a minor with at most $\lceil \frac{d^2+1}{d+1} \rceil$ vertices and minimum degree at least $\lfloor \frac{d}{2} \rfloor$, as well as a minor with at most $\lceil \frac{d^2+1}{d+1} \rceil + 1$ vertices and minimum degree at least $\lfloor \frac{d}{2} \rfloor + 1$.*

Proof. It suffices to prove the result for minor-minimal graphs G with average degree at least d. By Lemma 3, each edge of G is in at least $\lfloor \frac{d}{2} \rfloor$ triangles. Say G has n vertices and m edges. Thus $m \geqslant \lceil \frac{dn}{2} \rceil$. If $m > \lceil \frac{dn}{2} \rceil$ then deleting any one edge maintains the average degree condition, thus contradicting the minimality of G. Hence $m = \lceil \frac{dn}{2} \rceil$, and G has average degree $\frac{2m}{n} = \frac{2}{n} \lceil \frac{dn}{2} \rceil < \frac{2}{n}(\frac{dn}{2} + 1) = d + \frac{2}{n} \leqslant \frac{d^2+d+2}{d+1}$ since $n \geqslant d+1$. Thus G has a vertex v with degree at most $\lceil \frac{d^2+d+2}{d+1} - 1 \rceil = \lceil \frac{d^2+1}{d+1} \rceil$. Hence the subgraph of G induced by the neighbours of v has at most $\lceil \frac{d^2+1}{d+1} \rceil$ vertices and minimum degree at least $\lfloor \frac{d}{2} \rfloor$. Moreover, the subgraph of G induced by the closed neighbourhood of v has at most $\lceil \frac{d^2+1}{d+1} \rceil + 1$ vertices and minimum degree at least $\lfloor \frac{d}{2} \rfloor + 1$. □

Mader [18] introduced the following key definition. For an integer $k \geqslant 1$, let X_k be the set of graphs G with $|V(G)| \geqslant k$ and $|E(G)| \geqslant k|V(G)| - \binom{k+1}{2}$.

Lemma 5. *Let G be a minor-minimal graph in X_k. Then $|E(G)| = k|V(G)| - \binom{k+1}{2}$, and either G is isomorphic to K_k or the neighbourhood of each vertex in G induces a subgraph with minimum degree at least k.*

Proof. Say G has n vertices and m edges. Then $m = k|V(G)| - \binom{k+1}{2}$, otherwise delete an edge. If $n = k$ then $m = k^2 - \binom{k+1}{2} = \binom{k}{2}$, implying that G is isomorphic to K_k, as desired. Now assume that $n \geqslant k+1$. Let vw be an edge of G. Say vw is in t triangles. Then G/vw has $n-1 \geqslant k$ vertices and $m-t-1$ edges. Since G is minor-minimal, G/vw is not in X_k. Thus

$$kn - \binom{k+1}{2} - t - 1 = m - t - 1 = |E(G/vw)| \leqslant k(n-1) - \binom{k+1}{2} - 1,$$

implying $t \geqslant k$. That is, each edge is in at least k triangles. Therefore, the neighbourhood of each vertex induces a subgraph with minimum degree at least k. □

The following lemma is proved by mimicking a proof by Mader [18] for the special case of $c_1 = 4$ and $c_2 = \frac{3}{2}$.

Lemma 6. *Fix constants $c_1 > 2$ and $c_2 > 1$. For each integer $k \geqslant 1$, every graph G with average degree at least $4k$ has a minor with:*

(1) *at most* $(\frac{c_1}{2}+1)k$ *vertices and minimum degree at least* $2k$*, or*
(2) *at most* $2k+1$ *vertices and minimum degree at least* $(1+\frac{1}{c_1})k$*, or*
(3) *at most* c_2k *vertices and minimum degree at least* k*, or*
(4) *at most* $(4-\frac{c_1}{2})k$ *vertices and minimum degree at least* c_2k*, or*
(5) k *vertices and minimum degree* $k-1$ *(that is,* K_k*).*

Proof. Since G has average degree at least $4k$, $G \in X_{2k}$. Let G' be a minor-minimal minor of G in X_{2k}. Thus $|E(G')| = 2k|V(G')| - \binom{2k+1}{2}$. Hence G' has average degree less than $4k$. Let v be a vertex in G' with degree less than $4k$. If G' is isomorphic to K_{2k} then outcome (5) holds. Otherwise, by Lemma 5, $G_0 := G'[N(v)]$ has minimum degree at least $2k$. If G_0 has at most $(\frac{c_1}{2}+1)k$ vertices then it satisfies outcome (1) and we are done. Now assume that G_0 has at least $(\frac{c_1}{2}+1)k$ vertices. Since G_0 has minimum degree at least $2k$, $|E(G_0)| \geqslant k|V(G_0)|$. Delete edges from G_0 until $|E(G_0)| = k|V(G_0)|$.

Define $k' := \lfloor \frac{c_1}{2}k \rfloor$. We now define a sequence of graphs $G_0, G_1, \ldots, G_{k'}$ that satisfy $|V(G_i)| = |V(G_0)| - i$ and

$$k|V(G_i)| - \frac{ik}{c_1} \leqslant |E(G_i)| \leqslant k|V(G_i)|.$$

Given G_i where $0 \leqslant i \leqslant k'-1$, construct G_{i+1} as follows. First suppose that each edge e in G_i is in at least $(1+\frac{1}{c_1})k-1$ triangles. Since $|E(G_i)| \leqslant k|V(G_i)|$, some vertex v in G_i has degree at most $2k$. Thus the closed neighbourhood of v induces a subgraph with at most $2k+1$ vertices and minimum degree at least $(1+\frac{1}{c_1})k$, which satisfies outcome (2). Now assume some edge e in G_i is in at most $(1+\frac{1}{c_1})k-1$ triangles. Let G_{i+1} be obtained from G_i by contracting e. Thus $|V(G_{i+1})| = |V(G_i)| - 1 = |V(G_0)| - i - 1$ and

$$|E(G_{i+1})| \geqslant |E(G_i)| - (1+\frac{1}{c_1})k \geqslant k|V(G_i)| - \frac{ik}{c_1} - (1+\frac{1}{c_1})k = k|V(G_{i+1})| - \frac{(i+1)k}{c_1}.$$

If G_{i+1} has more than $k|V(G_{i+1})|$ edges, then delete edges until $|E(G_{i+1})| = k|V(G_{i+1})|$. Thus G_{i+1} satisfies the stated properties.

Consider the final graph $F := G_{k'}$. It satisfies

$$|V(F)| = |V(G_0)| - k' \geqslant (\frac{c_1}{2}+1)k - k' \geqslant k \text{ and}$$
$$|E(F)| \geqslant k|V(F)| - \frac{k'}{c_1}k \geqslant k|V(F)| - \tbinom{k+1}{2}.$$

Thus $F \in X_k$. Let F' be a minor-minimal minor of F in X_k. Thus $|V(F')| \leqslant |V(F)| = |V(G_0)| - k' \leqslant (4-\frac{c_1}{2})k$. If F' is isomorphic to K_k then outcome (5) holds. Otherwise, by Lemma 5, the neighbourhood of each vertex in F' induces a subgraph with minimum degree at least k. If F' has a vertex of degree at most c_2k then $F'[N(v)]$ has at most c_2k vertices and has minimum degree at least k, which satisfies outcome (3). Otherwise F' has minimum degree at least c_2k and at most $(4-\frac{c_1}{2})k$ vertices, which satisfies outcome (4). ∎

It is natural to maximise the ratio between the minimum degree and the number of vertices in our minor. The next lemma does that[2]:

Lemma 7. *For every integer $k \geqslant 1$, every graph with average degree at least $4k$ contains a complete graph K_k as a minor or contains a minor with n vertices and minimum degree δ, where $\delta \geqslant 0.6518n$ and $2\delta - n \geqslant 0.4659k$ and $k \leqslant \delta < n \leqslant 4k$.*

Proof. Apply Lemma 6 with $c_1 = 3.2929$ and $c_2 = 1.5341$. □

Maximising the difference between twice the minimum degree and the number of vertices in our minor will be useful below. The next lemma does that.

Lemma 8. *For every integer $k \geqslant 1$, every graph with average degree at least $4k$ contains a complete graph K_k as a minor or contains a minor with n vertices and minimum degree δ, where $\delta \geqslant 0.6273n$ and $2\delta - n \geqslant 0.5773k$ and $k \leqslant \delta < n \leqslant 4k$.*

Proof. Apply Lemma 6 with $c_1 = 3.4641$ and $c_2 = 1.4227$. □

3 Deterministic Linear Bounds

This section establishes a number of linear bounds on $f(H)$. All the proofs are deterministic. The following well known lemma will be useful. We include the proof for completeness.

Lemma 9. *Every graph G with minimum degree at least $\ell - 1$ contains every tree on $\ell \geqslant 2$ vertices as a subgraph.*

Proof. We proceed by induction on ℓ (with G fixed). The base case with $\ell = 1$ is trivial. Assume that $\ell \geqslant 2$. Let T be a tree on ℓ vertices. Let v be a leaf of T adjacent to w. By induction, G contains a subgraph X isomorphic to $T - v$. Let w' be the image of w in X. Since $\deg_G(w') \geqslant \ell - 1 > |V(X - w')| = \ell - 2$, there is a neighbour v' of w' in $G - X$. Mapping v to v' gives a subgraph of G isomorphic to T. □

A graph G is *2-degenerate* if every non-empty subgraph of G has a vertex of degree at most 2.

Lemma 10. *Let G be a graph with $n \geqslant 1$ vertices and minimum degree δ, with $2\delta - n \geqslant t - 2$. Then G contains every 2-degenerate graph on $t \geqslant 1$ vertices as a subgraph.*

[2] The optimised constants used in the proof of Lemma 7 (and elsewhere in the paper) were computed using AMPL and the MINOS 5.5 nonlinear equation solver.

Proof. We proceed by induction on $t \geqslant 1$ (with G fixed). The result is trivial for $t = 1$. Let H be a 2-degenerate graph on t vertices.

First suppose that there is a degree-1 vertex v in H adjacent to x. By induction, $H - v$ is a subgraph of G. Let x' be the image of x in G. Since $2\delta - n \geqslant t - 2$ and $n > \delta$, we have $\deg_G(x') \geqslant \delta > t - 2$. Thus some neighbour of x' is not used by the $t-2$ vertices in $H - x - v$. Embed v at this neighbour, to obtain H as a subgraph of G.

Now assume that H has minimum degree 2. Since H is 2-degenerate, there is a degree-2 vertex v in H adjacent to x and y. By induction, $H-v$ is a subgraph of G. Let x' and y' be the images of x and y in G. Say x' and y' have c common neighbours. Thus x' has at least $\delta-c-1$ neighbours that are not y' and not adjacent to y'. Similarly, y' has at least $\delta-c-1$ neighbours that are not x' and not adjacent to x'. Thus $n \geqslant 2+c+2(\delta-c-1) = 2\delta - c$, implying $c \geqslant 2\delta - n \geqslant t-2$. At most $t-3$ of the common neighbours of x' and y' are used by $H - v$. So embed v at one of the remaining common neighbours of x' and y'. And H is a subgraph of G. □

Lemma 11. *Every graph G with average degree at least $6.929t$ contains every 2-degenerate graph H on $t \geqslant 1$ vertices as a minor.*

Proof. If $t \leqslant 4$ then G contains K_4 and thus H as a minor (since $6.929t > 4 = f(K_4)$). Now assume that $t \geqslant 5$. By assumption, G has average degree at least $4k$, where $k := \lceil (t-2)/0.5773 \rceil$. If G contains a K_k minor, then G contains H as a minor (since $t \geqslant 5$ implies $k \geqslant t$). Otherwise, by Lemma 8, G contains a minor G' with n vertices and minimum degree δ where $k \leqslant \delta \leqslant n \leqslant 4k$ and $2\delta - n \geqslant 0.5773k \geqslant t-2$. By Lemma 10, G' contains H as a subgraph. Thus G contains H as a minor. □

We obtain the following straightforward linear bound for forcing an H-minor. The *1-subdivision* of a graph H is the graph obtained from H by subdividing each edge of H exactly once. A $(\leqslant 1)$-*subdivision* of H is a graph obtained from H by subdividing each edge of H at most once.

Proposition 12. *Let H be a graph with t vertices and q edges. Then every graph G with average degree at least $6.929(t+q)$ contains H as a minor.*

Proof. If H' is the 1-subdivision of H, then H' has $t+q$ vertices and is 2-degenerate. By Lemma 11, G contains H' and thus H as a minor. □

This result is improved in Theorem 15 below. First we need the following easy generalisation of Lemma 10. A subgraph H' of a graph H is *spanning* if $V(H') = V(H)$.

Lemma 13. *Let H be a graph with $t \geqslant 1$ vertices and q edges. Assume that H contains a 2-degenerate spanning subgraph H' with q' edges. Let G be a graph with $n \geqslant 1$ vertices* *and minimum degree δ, with $2\delta - n \geqslant q - q' + t - 2$. Then G contains a $(\leqslant 1)$-subdivision of H as a subgraph.*

Proof. Let H'' be the graph obtained from H by subdividing each edge not in H' once. Thus H'' is 2-degenerate, and has $t + q - q'$ vertices. By Lemma 10, G contains H'' as a subgraph, which is a $(\leqslant 1)$-subdivision of H. □

Lemma 14. *Let H be a graph with $t \geqslant 1$ vertices and q edges. Assume that H contains a 2-degenerate spanning subgraph H' with q' edges. Let G be a graph with $n \geqslant 1$ vertices and average degree at least $6.929(q - q' + t)$. Then G contains H as a minor.*

Proof. If $t \leqslant 4$ then G contains K_4 and thus H as a minor (since $6.929(q - q' + t) > 4 = f(K_4)$). Now assume that $t \geqslant 5$. By assumption, G has average degree at least $4k$, where $k := \lceil (q - q' + t - 2)/0.5773 \rceil$. If G contains a K_k minor, then G contains H as a minor (since $t \geqslant 5$ implies $k \geqslant t$). Otherwise, by Lemma 8, G contains a minor G' with n vertices and minimum degree δ where $k \leqslant \delta \leqslant n \leqslant 4k$ and $2\delta - n \geqslant 0.5773k \geqslant q - q' + t - 2$. By Lemma 13, G' contains H as a subgraph. Thus G contains H as a minor. □

Theorem 15. *For every graph H with i isolated vertices and q edges, every graph G with average degree at least $i + 6.929q$ contains H as a minor.*

Proof. Let $c := 6.929$. Let $t := |V(H)|$. First note that $q \geqslant \frac{t-i}{2}$. We proceed by induction on $|V(H)| + |V(G)|$. The result is trivial if $|V(H)| \leqslant 1$. Now assume that $|V(H)| \geqslant 2$. Let G be a graph with n vertices, m edges, and average degree $\frac{2m}{n} \geqslant i + cq$. We may assume that G is minor-minimal with average degree at least $i + cq$. By Lemma 3, G has minimum degree at least $\lfloor \frac{i+cq}{2} \rfloor + 1 \geqslant q$.

First suppose that H contains an isolated vertex v. Let w be a vertex of minimum degree in G. Thus $\deg(w) \leqslant \frac{2m}{n}$. Hence the average degree of $G - w$ is

$$\frac{2(m - \deg(w))}{n-1} \geqslant \frac{2m - \frac{2m}{n} - (n-1)}{n-1} = \frac{2m}{n} - 1 \geqslant (t-1) + cq.$$

By induction, $G - w$ contains $H - v$ as a minor. Thus G contains H as a minor (with v embedded at w). Now assume that $i = 0$.

Now suppose that some component T of H is a tree. Let $\ell := |V(T)|$. Since H has no isolated vertex, $\ell \geqslant 2$. Also, $q = |E(H)| \geqslant |E(T)| = \ell - 1$ and G has minimum degree at least $\ell - 1$. By Lemma 9, there is a subgraph T' of G isomorphic to T. Let $G' := G - V(T')$. Note that $|E(G')| > m - \ell n$. By assumption, (a) $2m \geqslant cqn$. Since $c \geqslant 4$ and $\ell \geqslant 2$, we have $c(\ell - 1) \geqslant 2\ell$, implying (b) $-2\ell n \geqslant -c(\ell - 1)n$. Also $q \geqslant \ell - 1$, implying (c) $0 \geqslant -c\ell q + c\ell(\ell - 1)$. Adding (a), (b) and (c) gives

$$2m - 2\ell n \geqslant cqn - c\ell q - c(\ell - 1)n + c\ell(\ell - 1) = c(q - (\ell - 1))(n - \ell).$$

Hence the average degree of G' is

$$\frac{2|E(G')|}{|V(G')|} \geqslant \frac{2(m-\ell n)}{n-\ell} \geqslant c(q-(\ell-1)).$$

$H-V(T)$ has no isolated vertices and $q-(\ell-1)$ edges. By induction, G' contains $H-V(T)$ as a minor. Hence G contains H as a minor, with T mapped to T'. Now assume that no component of H is a tree: Thus $q \geqslant t$.

Let $H_1, \dots, H_k$ be the components of H. Each H_i contains a spanning subgraph H_i' consisting of a tree plus one edge. Let $H' := H_1' \cup \dots \cup H_k'$. Thus $|E(H_i')| = |V(H_i)|$ and $|E(H')| = |V(H)| = t$. Observe that H' is 2-degenerate. By Lemma 14 with $q' = t$, G contains H as a minor. □

Note that the entire proof of Theorem 15 is deterministic and leads to an algorithm for finding an H-minor in G that has time complexity polynomial in both $|V(H)|$ and $|V(G)|$.

4 Probabilistic Linear Bounds

This section applies the probabilistic method to improve the linear bounds in Theorem 15.

Lemma 16. *Let H be a graph with t vertices and q edges. Let G be a graph with $n \geqslant t$ vertices and average degree at least d. Then there is a spanning subgraph R of H with at least $\frac{dq}{n-1}$ edges, such that R is isomorphic to a subgraph of G.*

Proof. Say G has m edges. Then $m \geqslant \frac{1}{2}dn$. Let f be a random injection f from $V(H)$ to $V(G)$. Then by the linearity of expectation,

$$\begin{aligned}\mathbb{E}(|\{vw \in E(H) : f(v)f(w) \in E(G)\}|) &= \sum_{vw \in E(H)} \mathbb{P}(f(v)f(w) \in E(G)) \\ &= \sum_{vw \in E(H)} \frac{m}{\binom{n}{2}} \\ &\geqslant \frac{dq}{n-1}.\end{aligned}$$

Thus there exists an injection f from $V(H)$ to $V(G)$ such that $|\{vw \in E(H) : f(v)f(w) \in E(G)\}| \geqslant \frac{dq}{n-1}$. Then the spanning subgraph R of H with $E(R) := \{vw \in E(H) : f(v)f(w) \in E(G)\}$ satisfies the claim. □

Lemma 17. *Let H be a graph with t vertices and q edges. Let G be a graph with at most n vertices and minimum degree at least δ, such that*

$$2\delta + 4 + \frac{\delta q}{n-1} \geqslant n + t + q.$$

Then G contains a $(\leqslant 1)$-subdivision of H as a subgraph.

Proof. By Lemma 16, there is a spanning subgraph R of H with at least $\frac{q\delta}{n-1}$ edges, such that R is isomorphic to a subgraph of G. For each vertex v of H, let v' be the corresponding vertex of G (defined by this isomorphism). Observe that the number of edges vw of H such that $v'w'$ is not an edge of G is at most $q(1-\frac{\delta}{n-1})$. For each such edge we choose a common neighbour of v' and w' and route vw by a path in G with one internal vertex. Consider each edge vw of H such that $v'w'$ is not an edge of G in turn. Both v' and w' have degree at least δ and they are not adjacent. Thus v' and w' have at least $2\delta-(n-2)$ common neighbours. Since $2\delta-(n-2)\geqslant(t-2)+q(1-\frac{\delta}{n-1})$, there is a common neighbour x of v' and w' that is not already used by a vertex in $V(H)\setminus\{v,w\}$ or by a division vertex already assigned. Hence we may route vw by the path $v'xw'$ in G. $\square$

Now we combine Lemma 6 and Lemma 17.

Lemma 18. *Let $c_1>2$ and $c_2>1$. Define $a_1:=\frac{c_1}{2}+1$, $b_1:=2$, $a_2:=2$, $b_2:=1+\frac{1}{c_1}$, $a_3:=c_2$, $b_3:=1$, $a_4:=4-\frac{c_1}{2}$ and $b_4:=c_2$. Assume that for $1\leqslant i\leqslant 4$,*

$$0<2b_i-a_i\leqslant 3 \text{ and } b_i<a_i.$$

Let $\alpha\geqslant 4$ and β be numbers such that for $1\leqslant i\leqslant 4$,

$$\alpha\geqslant\frac{4}{2b_i-a_i} \text{ and } \beta\geqslant\frac{4(a_i-b_i)}{a_i(2b_i-a_i)}.$$

Then, for every graph H with t vertices and q edges,

$$f(H)\leqslant\alpha t+\beta q.$$

Proof. We are given a t-vertex q-edge graph H and a graph G with average degree at least $\alpha t+\beta q\geqslant 4k$, where $k:=\lfloor\frac{1}{4}(\alpha t+\beta q)\rfloor$. Since $c_1>2$ and $c_2>1$, Lemma 6 is applicable to G. If case (5) occurs in Lemma 6, then K_k is a minor of G, which implies that H is a minor of G (since $\alpha\geqslant 4$ implies $t\leqslant k$). Now assume that case (i) occurs in Lemma 6 for some $i\in\{1,2,3,4\}$. Let $a:=a_i$ and $b:=b_i$. Thus G contains a minor G' with $n\leqslant ak+1$ vertices and minimum degree $\delta\geqslant bk$. By the assumptions,

$$(2b-a)k+3\;\geqslant\;(2b-a)(k+1)\;>\;\frac{(2b-a)(\alpha t+\beta q)}{4}\;\geqslant\;t+\left(\frac{a-b}{a}\right)q\;=\;t+\left(1-\frac{b}{a}\right)q.$$

Thus

$$2bk+4+\frac{b}{a}q\;\geqslant\;t+q+ak+1.$$

Since $n\leqslant ak+1$ and $\delta\geqslant bk$, we have $\frac{b}{a}\leqslant\frac{\delta}{n-1}$. Thus

$$2\delta+4+\frac{\delta q}{n-1}\;\geqslant\;t+q+n.$$

By Lemma 17, G' contains a ($\leqslant 1$)-subdivision of H as a subgraph. Hence G contains H as a minor. $\square$

Optimising β in Lemma 18 gives:

Proposition 19. *For every graph H with t vertices and q edges,*

$$f(H) \leqslant 7.477t + 2.375q.$$

Proof. Apply Lemma 18 with $\alpha = 7.477$ and $\beta = 2.375$ and $c_1 = 3.375$ and $c_2 = 1.465$. $\square$

We now prove the bound introduced in Section 1.

Proof of Theorem 2. We proceed by induction on t with the following hypothesis: Every graph G with average degree at least $t + cq$ contains every graph H on t vertices and q edges as a minor, where $c := 6.291$. The result is trivial if $t \leqslant 1$. Now assume that $t \geqslant 2$. Let G be a graph with n vertices, m edges, and average degree $\frac{2m}{n} \geqslant t + cq$. We may assume that G is minor-minimal with average degree at least $t + cq$. By Lemma 3, G has minimum degree at least $\lfloor \frac{t+cq}{2} \rfloor + 1 \geqslant q$.

Case 1. H contains an isolated vertex v: Let w be a vertex of minimum degree in G. Thus $\deg(w) \leqslant \frac{2m}{n}$. Hence the average degree of $G - w$ is

$$\frac{2(m - \deg(w))}{n-1} \geqslant \frac{2m - \frac{2m}{n} - (n-1)}{n-1} = \frac{2m}{n} - 1 \geqslant (t-1) + cq.$$

By induction, $G - w$ contains $H - v$ as a minor. Thus G contains H as a minor (with v embedded at w). Now assume that H has no isolated vertex.

Case 2. Some component T of H is a tree: Let $\ell := |V(T)|$. Note that $t = |V(H)| \geqslant |V(T)| = \ell \geqslant 2$ and $q = |E(H)| \geqslant |E(T)| = \ell - 1$, implying that G has minimum degree at least $\ell - 1$. By Lemma 9, there is a subgraph T' of G isomorphic to T. Let $G' := G - V(T')$. Note that $|E(G')| > m - \ell n$. By assumption, (a) $2m \geqslant (t + cq)n$. Since $c \geqslant 2$ and $\ell \geqslant 2$, we have $c(\ell - 1) \geqslant \ell$, implying (b) $-2\ell n \geqslant -(\ell + c\ell - c)n$. Since $q \geqslant \ell - 1$, we have (c) $0 \geqslant -c\ell(q - \ell + 1)$. Since $t \geqslant 1$, we have (d) $0 \geqslant \ell - \ell t$. Adding (a), (b), (c) and (d) gives

$$\begin{aligned} 2m - 2\ell n &\geqslant (t + cq)n - (\ell + c\ell - c)n - c\ell(q - \ell + 1) + \ell - \ell t \\ &= n\big((t - \ell) + c(q - \ell + 1)\big) - \ell\big((t - \ell) + c(q - \ell + 1)\big) \\ &= \big((t - \ell) + c(q - \ell + 1)\big)(n - \ell). \end{aligned}$$

Hence the average degree of G' is

$$\frac{2|E(G')|}{|V(G')|} \geqslant \frac{2(m - \ell n)}{n - \ell} \geqslant (t - \ell) + c(q - (\ell - 1)).$$

Since $H-V(T)$ has $t-\ell$ vertices and $q-(\ell-1)$ edges, by induction, G' contains $H-V(T)$ as a minor. Hence G contains H as a minor, with T mapped to T'. Now assume that no component of H is a tree. Thus $q \geqslant t$.

Case 3. $t + cq \geqslant \alpha t + \beta q$, where $\alpha := 6.9687$ and $\beta := 2.484$: Then G has average degree at least $\alpha t + \beta q$, and thus contains H as a minor by Lemma 18 with $c_1 = 3.484$ and $c_2 = 1.426$.

Case 4: Now assume that $(\alpha - 1)t \geqslant (c - \beta)q$. Let $H_1, \dots, H_k$ be the components of H. Each H_i contains a spanning subgraph H'_i consisting of a tree plus one edge. Let $H' := H'_1 \cup \dots \cup H'_k$. Thus $|E(H'_i)| = |V(H_i)|$ and $|E(H')| = |V(H)| = t$. Observe that Q is 2-degenerate.

Define $k := \lceil \frac{1}{4}(t + c(q-2)) \rceil$. Thus G has average degree at least $t + cq > 4k$. By Lemma 8, G contains a complete graph K_k as a minor, or G contains a minor G' with n' vertices and minimum degree δ, where $2\delta - n' \geqslant \sigma k$ and $\sigma = 0.5773$. In the first case, H is a subgraph of K_k (since $k \geqslant q \geqslant t$), implying H is a minor of G. In the second case,

$$2\delta - n' \geqslant \sigma k \geqslant \frac{\sigma}{4}(t + c(q-2)) \geqslant \frac{\sigma(c-\beta)}{4(\alpha-1)}q + \frac{\sigma c}{4}(q-2) \geqslant q - 2,$$

where the final inequality follows by considering the actual numerical values. Thus, by Lemma 13 with $q' = t$, G' contains H as a minor. Therefore G contains H as a minor. $\square$

The bound in Theorem 2 is stronger than the bound in Theorem 15 when $q \geqslant 1.567(t - i)$ (which is roughly when the non-isolated vertices in H have average degree at least 3).

5 General Result

The following lemma is at the heart of the proof of our main result (Theorem 1).

Lemma 20. *For all $\lambda \in (\frac{1}{2}, 1)$ and $\epsilon \in (0, \lambda)$ there exists d_0 such that for every graph H with t vertices and average degree $d \geqslant d_0$, every graph G with $n \geqslant (1+\epsilon)\lceil \sqrt{\log_b d} \rceil t$ vertices and minimum degree at least λn contains H as a minor, where $b = (1 - \lambda + \epsilon)^{-1}$.*

We first sketch the proof. Say $V(H) = \{1, 2, \dots, t\}$. Our goal is to exhibit disjoint subsets $X_1, \dots, X_t$ of $V(G)$ such that:

(a) $G[X_i]$ is connected for $1 \leqslant i \leqslant t$, and

(b) for each edge ij of H there is an edge of G between X_i and X_j.

We choose the X_i in three stages. In the first two stages, we choose disjoint sets $S_1, \dots, S_t$ and $T_1, \dots, T_t$ randomly, with the S_i non-empty, such that:

(i) every pair of vertices of G have many common neighbours not in $S_1 \cup \dots \cup S_t \cup T_1 \cup \dots \cup T_t$,

(ii) for a small number of edges $ij \in E(H)$, there is no edge between $S_i \cup T_i$ and $S_j \cup T_j$, and

(iii) the total number (summed over all i) of components in $G[S_i \cup T_i]$ is small.

Having done so, it is straightforward to greedily chooses disjoint sets $U_1, \dots, U_t$, where $|U_i|$ equals the number of components of $G[S_i \cup T_i]$ minus 1, plus the number of edges ij of H with $j > i$ such that there is no edge of G between $S_i \cup T_i$ and $S_j \cup T_j$, so that (a) and (b) hold for $X_i = S_i \cup T_i \cup U_i$.

It remains to choose the S_i and T_i so that (i), (ii), and (iii) are satisfied. In the first stage we randomly choose disjoint sets $S_1, \dots, S_t$ each with $\ell = \lceil \sqrt{\log_b d} \rceil$ vertices. In the second stage, we randomly choose the T_i and show that (i), (ii) and (iii) hold with positive probability. Some of the T_i are empty, the rest of which have $2\ell^2$ vertices. T_i is non-empty precisely if the size of the neighbourhood of S_i is below a certain threshold. We need to add the T_i to such S_i in the second phase to ensure that (ii) holds. In the first phase, we focus on bounding the number of i for which the neighbourhood of S_i is small. This allows us to bound the number of vertices used in the second phase, which helps in proving (i). In the following proof, no effort is made to minimise d_0.

Proof of Lemma 20. Note that in G, every pair of vertices have at least $(2\lambda - 1)n$ common neighbours (and $2\lambda - 1 > 0$). Note that $b > 1$. Let $\ell := \lceil \sqrt{\log_b d} \rceil$. Define

$$\nu := \left(\frac{1-\lambda}{1-\lambda+\epsilon} \right)^{\ell} \quad \text{and} \quad \mu := (1.5692)^{\ell}(1-\lambda)^{5\ell/6}.$$

Observe that $0 < \nu, \mu < 1$ (since $\lambda > \frac{1}{2}$), and ν and μ tend to 0 exponentially as $\ell \to \infty$. Now define

$$\theta := 5(\nu + \mu)(\ell + \ell^2) + 5\nu\ell^2 + 8.$$

Elementary calculus shows that θ is bounded by a function of ϵ and λ independent of ℓ. Thus, taking d_0 at least some function of ϵ and λ, since $d \geqslant d_0$, we may assume that d, ℓ, t and n are at least functions of ϵ, λ and θ. In particular, we assume:

$$\epsilon(1-\epsilon)(2\lambda-1)\ell \geqslant 2\theta \tag{4}$$

$$\exp\left(\frac{\epsilon^2(2\lambda-1)^2 n}{8} \right) \geqslant 10\tbinom{n}{2} \tag{5}$$

$$\exp\left(\frac{\epsilon^4 n}{2(1+\epsilon)^2} \right) \geqslant 10n. \tag{6}$$

For a set S of vertices in G, a vertex v of G is a *non-neighbour* of S if v is not in S and v is not adjacent to a vertex in S.

Say $V(H) = \{1, 2, \dots, t\}$. Let $S_1, \dots, S_t, T_1, \dots, T_t$ be pairwise disjoint subsets of $V(G)$. Say S_i is *bad* if S_i has at least $(n - \ell)(1 - \lambda + \epsilon)^\ell$ non-neighbours, otherwise S_i is *good*. Say S_i is *disjointed* if $G[S_i]$ has a connected component with at most $\frac{\ell}{6}$ vertices. An edge $ij \in E(H)$ is *problematic* if S_i or S_j is good (or both), but there is no edge in G between S_i and S_j. An edge $ij \in E(H)$ is *nasty* if S_i and S_j are both bad and there is no edge in G between $S_i \cup T_i$ and $S_j \cup T_j$. Below we prove the following two claims.

Claim 1. *There exists subsets $S_1, \dots, S_t$ of $V(G)$ satisfying the following properties:*
(P0) *$S_1, \dots, S_t$ are pairwise disjoint, and $|S_i| = \ell$ for $1 \leqslant i \leqslant t$.*
(P1) *At most $5\nu t$ of the S_i are bad.*
(P2) *At most $5\mu t$ of the S_i are disjointed.*
(P3) *At most $\frac{5}{2}t$ edges of H are problematic.*
(P4) *For all vertices $v, w \in V(G)$,*

$$|(N(v) \cap N(w)) \setminus (S_1 \cup \dots \cup S_t)| \;\geqslant\; \left(1 - \frac{\ell t}{n} - \frac{\epsilon}{2}\right) |N(v) \cap N(w)|.$$

(P5) *For each vertex $v \in V(G)$,*

$$|N(v) \setminus (S_1 \cup \dots \cup S_t)| \;\geqslant\; \left(1 - \frac{\ell t}{n} - \frac{\epsilon^2}{\lambda(1+\epsilon)}\right) |N(v)|.$$

Claim 2. *Given subsets $S_1, \dots, S_t$ of $V(G)$ that satisfy* (P0), (P1), (P2), (P3), (P4) *and* (P5)*, there exist subsets $T_1, \dots, T_t$ of $V(G)$ satisfying the following properties:*

(Q0) *$S_1, \dots, S_t, T_1, \dots, T_t$ are pairwise disjoint, and for $1 \leqslant i \leqslant t$,*

$$|T_i| = \begin{cases} \ell^2 & \text{if } S_i \text{ is bad,} \\ 0 & \text{if } S_i \text{ is good.} \end{cases}$$

(Q1) *At most $\frac{t}{2}$ edges of H are nasty.*

Before proving these claims we show that they imply the lemma. By (P0),

$$|S_1 \cup \dots \cup S_t| = \ell t, \tag{7}$$

and by (P1) and (Q0),

$$|T_1 \cup \dots \cup T_t| \leqslant 5\nu t \cdot \ell^2. \tag{8}$$

Mark each vertex in $\bigcup_i S_i \cup T_i$ as *used*.

For $i = 1, 2, \dots, t$, choose a set U_i of less than r_i vertices in G as follows, where r_i is the number of components of $G[S_i \cup T_i]$. Note that if S_i is good and not disjointed, then $|S_i \cup T_i| = \ell$ and each component of $G[S_i \cup T_i]$ has more than $\frac{\ell}{6}$ vertices, implying $r_i \leqslant 5$. Otherwise (if S_i is bad or disjointed) all we need is that $r_i \leqslant |S_i| + |T_i| \leqslant \ell + \ell^2$. For $1 \leqslant j \leqslant r_i$, let x_j be an arbitrary vertex in the j-th component of $G[S_i \cup T_i]$. For $j = 1, \dots, r_i - 1$, choose an unused common neighbour z of x_j and x_{j+1}, add z to U_i, and mark z as used.

To prove that such a vertex z exists, we first estimate $|\bigcup_i U_i|$. By (P1) and (P2), at most $5(\nu+\mu)t$ of the S_i are bad or disjointed. Each of these contribute at most $\ell+\ell^2$ vertices to $\bigcup_i U_i$. For each S_i that is good and not disjointed, at most 5 vertices are added to $\bigcup_i U_i$. In total, by (8),

$$|\bigcup_i U_i| \leqslant 5(\nu+\mu)(\ell+\ell^2)t + 5t \quad \text{and} \tag{9}$$

$$|\bigcup_i T_i \cup U_i| \leqslant 5(\nu+\mu)(\ell+\ell^2)t + 5t + 5\nu\ell^2 t = (\theta-3)t. \tag{10}$$

By (P4) and (4) and (10), and since $n \geqslant (1+\epsilon)\ell t$,

$$\begin{aligned}
|(N(x_j) \cap N(x_{j+1})) \setminus \bigcup_i (S_i \cup T_i \cup U_i)| \;&\geqslant\; \left(1 - \frac{\ell t}{n} - \frac{\epsilon}{2}\right)(2\lambda-1)n - (\theta-3)t \\
&\geqslant\; \left(1 - \frac{1}{1+\epsilon} - \frac{\epsilon}{2}\right)(2\lambda-1)(1+\epsilon)\ell t - (\theta-3)t \\
&= \tfrac{\epsilon}{2}(1-\epsilon)(2\lambda-1)\ell t - (\theta-3)t \\
&\geqslant 3t \;>\; 0.
\end{aligned}$$

The used vertices are precisely $\bigcup_i (S_i \cup T_i \cup U_i)$. Thus the above inequality says that there is an unused common neighbour z of x_j and x_{j+1}, as claimed. By construction, each subgraph $G[S_i \cup T_i \cup U_i]$ is connected.

Suppose that there is no edge in G between $S_i \cup T_i \cup U_i$ and $S_j \cup T_j \cup U_j$ for some edge $ij \in E(H)$. If S_i or S_j is good, then ij is problematic, otherwise ij is nasty. Thus, by (P3) and (Q1) there are at most $3t$ such edges. Choose an unused common neighbour z of some vertex in $S_i \cup T_i \cup U_i$ and some vertex in $S_j \cup T_j \cup U_j$, add z to U_i, and mark z as used. This step increases $|\bigcup_i U_i|$ by at most $3t$, implying that $|\bigcup_i T_i \cup U_i| \leqslant \theta t$ by (10). By the argument above, such a vertex z exists. Now $S_1, \dots, S_t, T_1, \dots, T_t, U_1, \dots, U_t$ are pairwise disjoint, $G[S_i \cup T_i \cup U_i]$ is connected for each i, and for each edge $ij \in E(H)$, there is an edge in G between $S_i \cup T_i \cup U_i$ and $S_j \cup T_j \cup U_j$. Thus G contains H as a minor (by contracting each set $S_i \cup T_i \cup U_i$). It remains to prove Claims 1 and 2.

Proof of Claim 1. Choose $S_1, \dots, S_t \subseteq V(G)$ satisfying (P0) uniformly at random. Since $n > \ell t = |S_1 \cup \cdots \cup S_t|$, such subsets exist. We now bound the probability that each of (P1), (P2), (P3), (P4) and (P5) fail.

(P1): Consider a subset S_i and a vertex v in $G - S_i$. Since v has degree at least λn in G, and since S_i is chosen at random in $V(G)$, for each vertex $x \in S_i$, the probability that v is not adjacent to x is at most $1-\lambda$. Thus the probability that v is a non-neighbour of S_i is at most $(1-\lambda)^\ell$. By the linearity of expectation, the expected number of non-neighbours of S_i is at most $(n-\ell)(1-\lambda)^\ell$. Recall that S_i is bad if S_i has at least $(n-\ell)(1-\lambda+\epsilon)^\ell$ non-neighbours. Markov's inequality implies that the probability that S_i is bad is at most

$$(n-\ell)\,(1-\lambda)^\ell\,/(n-\ell)(1-\lambda+\epsilon)^\ell = \nu.$$

Thus the expected number of bad S_i is at most νt. Since (P1) fails if the number of bad S_i is more than $5\nu t$, Markov's inequality implies that (P1) fails with probability less than $\frac{1}{5}$.

(P2): Consider a disjointed set S_i. The number of subsets of S_i with at most $\frac{\ell}{6}$ vertices is

$$\sum_{j=0}^{\lfloor \ell/6 \rfloor} \binom{\ell}{j} \leqslant 2^{h(1/6)\ell} < (1.5692)^\ell,$$

where $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function. Let v be a vertex in a component of $G[S_i]$ with at most $\frac{\ell}{6}$ vertices. Thus v is not adjacent to the at least $\frac{5}{6}\ell$ vertices in the other components of $G[S_i]$. These other vertices were chosen randomly. Thus the probability that S_i is disjointed is less than $(1.5692)^\ell(1-\lambda)^{5\ell/6} = \mu$, and the expected number of disjointed S_i is at most μt. Since (P2) fails if the number of disjointed S_i is more than $5\mu t$, Markov's inequality implies that (P2) fails with probability less than $\frac{1}{5}$.

(P3): Consider a problematic edge ij in H, where S_i is good. Thus S_i has at most $(n-\ell)(1-\lambda+\epsilon)^\ell$ non-neighbours. Since there is no edge between S_i and S_j, every vertex in S_j is one of these at most $(n-\ell)(1-\lambda+\epsilon)^\ell$ non-neighbours of S_i. Since S_j is chosen randomly out of the $n-\ell$ vertices in $G-S_i$, the probability that each of the ℓ vertices in S_j is a non-neighbour of S_i is at most $(1-\lambda+\epsilon)^{\ell^2} \leqslant \frac{1}{d}$. (This is the key inequality in the whole proof.) Thus the probability that $ij \in E(H)$ is problematic is at most $\frac{1}{d}$. By the linearity of expectation, the expected number of problematic edges (out of a total of $\frac{dt}{2}$) is at most $\frac{t}{2}$. Since (P3) fails if the number of problematic edges is more than $\frac{5}{2}t$, Markov's inequality implies that the probability that (P3) fails is less than $\frac{1}{5}$.

(P4): Consider a pair of distinct vertices $v, w \in V(G)$. Let X be the random variable $|(N(v)\cap N(w)) \setminus (S_1\cup\cdots\cup S_t)|$. Since $S_1\cup\cdots\cup S_t$ consists of ℓt vertices chosen randomly from the n vertices in G, we have $\mathbb{E}(X) = (1-\frac{\ell t}{n})|N(v)\cap N(w)|$. If (P4) fails for v, w then $X - \mathbb{E}(X) < -\frac{\epsilon}{2}|N(v)\cap N(w)|$. Hence

$$\mathbb{P}(\text{(P4) fails for } v,w) \;\leqslant\; \mathbb{P}(|X-\mathbb{E}(X)| \geqslant \tfrac{\epsilon}{2}|N(v)\cap N(w)|).$$

The selection of $S_1,\dots,S_t$ may be considered as ℓt trials, each choosing a random vertex from the vertices not already chosen. Changing the outcome of any one trial changes $\mathbb{E}(X)$ by at most 1. Thus by Azuma's inequality[3] with $x = \frac{\epsilon}{2}|N(v)\cap N(w)|$,

$$\mathbb{P}(\text{(P4) fails for } v,w) \;\leqslant\; \mathbb{P}(|X-\mathbb{E}(X)| > x) \;\leqslant\; 2\exp\left(\frac{-(\frac{\epsilon}{2}|N(v)\cap N(w)|)^2}{2\ell t}\right).$$

[3] Azuma's inequality [1] says that if X is a random variable determined by n trials $R_1,\dots,R_n$, such that

Since $|N(v) \cap N(w)| \geqslant (2\lambda - 1)n$ and $n > \ell t$ and by (5),

$$\mathbb{P}(\text{(P4) fails for } v, w) \;\leqslant\; 2\exp\!\left(\frac{-\epsilon^2(2\lambda-1)^2n^2}{8\ell t}\right) \;<\; 2\exp\!\left(\frac{-\epsilon^2(2\lambda-1)^2n}{8}\right) \;\leqslant\; \left(5\binom{n}{2}\right)^{-1}.$$

By the union bound, the probability that (P4) fails (for some pair of distinct vertices in G) is less than $\frac{1}{5}$.

(P5): Consider a vertex $v \in V(G)$. Let X be the random variable $|N(v) \setminus (S_1 \cup \cdots \cup S_t)|$. Since $S_1 \cup \cdots \cup S_t$ consists of ℓt vertices chosen randomly from the n vertices in G, we have $\mathbb{E}(X) = (1 - \frac{\ell t}{n})|N(v)|$. If (P5) fails for v then $X - \mathbb{E}(X) < -\frac{\epsilon^2}{\lambda(1+\epsilon)}|N(v)|$. Hence

$$\mathbb{P}(\text{(P5) fails for } v) \;\leqslant\; \mathbb{P}(|X - \mathbb{E}(X)| \geqslant \frac{\epsilon^2}{\lambda(1+\epsilon)}|N(v)|).$$

As before, Azuma's inequality is applicable with $x = \frac{\epsilon^2}{\lambda(1+\epsilon)}|N(v)|$, giving

$$\mathbb{P}(\text{(P5) fails for } v) \;\leqslant\; 2\exp\!\left(-\left(\frac{\epsilon^2|N(v)|}{\lambda(1+\epsilon)}\right)^2 / \, 2\ell t\right).$$

Since $|N(v)| \geqslant \lambda n$ and $n > \ell t$, and by (6),

$$\mathbb{P}(\text{(P5) fails for } v) \;\leqslant\; 2\exp\!\left(\frac{-\epsilon^4 n^2}{2(1+\epsilon)^2\ell t}\right) \;<\; 2\exp\!\left(\frac{-\epsilon^4 n}{2(1+\epsilon)^2}\right) \;\leqslant\; (5n)^{-1}.$$

By the union bound, the probability that (P5) fails (for some vertex in G) is less than $\frac{1}{5}$.

We have shown that each of (P1)–(P5) fail with probability less than $\frac{1}{5}$. By the union bound, the probability that at least one of (P1)–(P5) fails is less than 1. Thus the probability that none of (P1)–(P5) fails is greater than 0. Thus there exists $S_1, \dots, S_t$ such that all of (P1)–(P5) hold. $\square$

Proof of Claim 2. Let $W := V(G) \setminus (S_1 \cup \cdots \cup S_t)$. By (P5), since G has minimum degree at least λn, and since $n \geqslant (1+\epsilon)\ell t$, the subgraph $G[W]$ has minimum degree at least

$$\left(1 - \frac{\ell t}{n} - \frac{\epsilon^2}{\lambda(1+\epsilon)}\right)\lambda n \;=\; \lambda(n - \ell t) - \frac{\epsilon^2 n}{1+\epsilon} \;\geqslant\; (\lambda - \epsilon)(n - \ell t) \;=\; (\lambda - \epsilon)|W|.$$

Choose $T_1, \dots, T_t \subseteq W$ satisfying (Q0) uniformly at random. Such subsets exist, since by (8) and (4) (which implies that $\epsilon \geqslant 5\nu\ell$),

$$|W| \;=\; n - \ell t \;\geqslant\; \epsilon\ell t \;\geqslant\; 5\nu\ell^2 t \;\geqslant\; |T_1 \cup \cdots \cup T_t|.$$

for each i, and any two possible sequences of outcomes $r_1, \dots, r_i$ and $r_1, \dots, r_{i-1}, r_i'$,

$$|\mathbb{E}(X \mid R_1 = r_1, \dots, R_i = r_i) - \mathbb{E}(X \mid R_1 = r_1, \dots, R_{i-1} = r_{i-1}, R_i = r_i')| \leqslant c_i,$$

then $\mathbb{P}(|X - \mathbb{E}(X)| > x) \;\leqslant\; 2\exp(-x^2/(2\sum_i c_i^2))$. In all our applications, $c_i = 1$.

If $ij \in E(H)$ is a nasty edge, and v is any vertex in T_j, then v is adjacent to no vertex in T_i. Since v and T_i were chosen randomly in W, and v is adjacent to at least $(\lambda-\epsilon)|W|$ vertices in W, the probability that v is adjacent to no vertex in T_i is at most

$$(1-(\lambda-\epsilon))^{|T_i|} \;=\; b^{-\ell^2} \;\leqslant\; b^{-\log_b d} \;=\; d^{-1}.$$

Thus the probability of an edge in H being nasty is at most d^{-1}. Hence the expected number of nasty edges (out of a total of $\frac{dt}{2}$) is at most $\frac{t}{2}$. With positive probability the number of nasty edges is at most $\frac{t}{2}$. Hence there exists $T_1,\dots,T_t$ such that (Q1) holds. $\square$

This completes the proof of Lemma 20. $\square$

Proof of Theorem 1. Define $\epsilon := 0.00001$ and $\lambda := 0.6518$ and $b := (1-\lambda+\epsilon)^{-1} > 2.8718$. Let H be a graph with t vertices and average degree $d \geqslant d_0$, where d_0 is sufficiently large compared to ϵ and λ (and thus an absolute constant). Let G be a graph with average degree at least $3.895\sqrt{\ln d}\,t$. Define $k := \lceil (1+\epsilon)\lceil \sqrt{\log_b d}\rceil\, t\rceil$. Now

$$3.895\sqrt{\ln b} > 3.895\sqrt{\ln 2.8718} > 4(1+\epsilon).$$

Let $\eta := 3.895 - 4(1+\epsilon)/\sqrt{\ln b}$, which is positive. Thus $3.895-\eta = 4(1+\epsilon)/\sqrt{\ln b}$ and

$$(3.895-\eta)\sqrt{\ln d}\,t = 4(1+\epsilon)\sqrt{\ln d}\,t/\sqrt{\ln b} = 4(1+\epsilon)\sqrt{\log_b d}\,t.$$

For sufficiently large d_0 and $d \geqslant d_0$, we have $\eta\sqrt{\ln d}\,t \geqslant 4(1+\epsilon)t + 4$. Adding these two inequalities gives

$$3.895\sqrt{\ln d}\,t \geqslant 4(1+\epsilon)\sqrt{\log_b d}\,t \;+\; 4(1+\epsilon)t + 4 \geqslant 4\lceil (1+\epsilon)\lceil \sqrt{\log_b d}\rceil\, t\rceil = 4k.$$

Thus G has average degree at least $4k$. By Lemma 7, either G contains K_k as a minor or G contains a minor G' with $n > k$ vertices and minimum degree at least λn. In the first case, G contains H as a minor (since $k \geqslant t$ for sufficiently large d_0 and $d \geqslant d_0$). In the second case, by Lemma 20, there exists d_0 depending only on ϵ and λ, such that G', and thus G, contains H as a minor (assuming $d \geqslant d_0$). $\square$

6 Open Problems

We conclude with a number of open problems that focus on $f(H)$ for various well-structured (non-random) graphs H.

- Let H consist of $k \geqslant 1$ disjoint triangles. Corradi and Hajnal [4] proved that every graph of minimum degree at least $2k$ and order at least $3k$ contains k disjoint cycles, and thus contains H as a minor. Let G be a graph with average degree at least

$4k-2$ for some positive integer k. By Lemma 3, G has a minor with minimum degree at least $2k$ and average degree at least $4k-2$ (implying the number of vertices is at least $4k-1 \geqslant 3k$). By the above result of Corradi and Hajnal [4], G contains H as a minor, and $f(H) \leqslant 4k-2$. (The same conclusion also follows from a result of Justesen [10].) In fact, $f(H) = 4k-2$ since if G is the complete bipartite graph $K_{2k-1,n}$ with $n \gg k$, then the average degree of G tends to $4k-2$ as $n \to \infty$, but G contains no H-minor since each cycle includes at least two vertices on each side. We conjecture the following generalisation: Every graph with average degree at least $\frac{4}{3}t-2$ contains every 2-regular graph on t vertices as a minor.

- Fix integers $d \ll s \ll t$. Let H_0 be a d-regular graph on t vertices. Myers and Thomason [22] prove that $f(H_0) \geqslant c\sqrt{\log d}\,t$. Let H be the graph obtained from H_0 by adding s dominant vertices. Thus H has average degree about $2s$. Hence $c_1\sqrt{\log d}\,t \leqslant f(H_0) \leqslant f(H) \leqslant c_2\sqrt{\log s}\,t$ by Theorem 1. Where $f(H)$ lies between $c\sqrt{\log d}\,t$ and $c\sqrt{\log s}\,t$ is an interesting open problem.

- What is the least function g such that every graph with average degree at least $g(k) \cdot t$ contains every graph with t vertices and treewidth at most k as a minor? Note that "graph with t vertices and treewidth at most k" can be replaced by "k-tree on t vertices" in the above. Since every such k-tree has less than kt edges, Proposition 19 and Theorem 1 respectively imply that $g(k) \leqslant 7.477 + 2.375k$ and $g(k) \in \mathcal{O}(\sqrt{\log k})$. Since every 2-tree is 2-degenerate, $g(2) \leqslant 6.929$ by Lemma 11.

- What is the minimum constant c such that every graph with average degree at least ct^2 contains the $t \times t$ grid as a minor? Since the $t \times t$ grid is 2-degenerate, $c \leqslant 6.929$ by Lemma 11.

- What is the least constant c such that every graph with average degree at least ct contains every planar graph with t vertices as a minor? Since such a planar graph has less than $3t$ edges, Proposition 19 implies that $c \leqslant 14.602$.

- What is the least function g such that every graph with average degree at least $g(k) \cdot t$ contains every K_k-minor-free graph with t vertices as a minor? Since every K_k-minor-free graph has average degree $\mathcal{O}(k\sqrt{\log k})$, Theorem 1 implies that $g(k) \in \mathcal{O}(\sqrt{\log k})$.

- Every graph with average degree at least $10t^2$ contains a subdivision of K_t as a subgraph. A proof of this result is given by Diestel [6] based on results on highly connected subgraphs by Mader [19] and on linkages by Thomas and Wollan [25]. This method immediately generalises to prove that for every graph H with t vertices and q edges, every graph with average degree at least $4t+20q$ contains a subdivision of H as a subgraph. Determining the best constants in such a result is an interesting line of research. Note that there is a linear lower bound for a graph H with t vertices and q edges, such that every set of at least $\frac{t}{2}$ vertices induces a subgraph with at least ϵq edges, for some $\epsilon > 0$. Say $K_{n,n}$ contains a subdivision of H. At least $\frac{t}{2}$ original vertices of H are on one side of $K_{n,n}$. Thus at least ϵq edges have a division vertex on the other side of $K_{n,n}$, implying $n \geqslant \epsilon q$. Hence, average degree at least ϵq is needed to force a subdivision of H.

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