ROOTED $K_4$-MINORS

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ABSTRACT. Let $a, b, c, d$ be four vertices in a graph $G$. A $K_4$-minor rooted at $a, b, c, d$ consists of four pairwise-disjoint pairwise-adjacent connected subgraphs of $G$, respectively containing $a, b, c, d$. We characterise precisely when $G$ contains a $K_4$-minor rooted at $a, b, c, d$ by describing six classes of obstructions, which are the edge-maximal graphs containing no $K_4$-minor rooted at $a, b, c, d$. The following two special cases illustrate the full characterisation: (1) A 4-connected non-planar graph contains a $K_4$-minor rooted at $a, b, c, d$ for every choice of $a, b, c, d$. (2) A 3-connected planar graph contains a $K_4$-minor rooted at $a, b, c, d$ if and only if $a, b, c, d$ are not on a single face.

1. INTRODUCTION

Let $G$ and $H$ be graphs. An $H$-minor in $G$ is a set $\{G_x : x \in V(H)\}$ of pairwise disjoint connected subgraphs of $G$ indexed by the vertices of $H$, such that if $xy \in E(H)$ then some vertex in $G_x$ is adjacent to some vertex in $G_y$. Each subgraph $G_x$ is called a branch set of the minor. A complete graph $K_t$-minor in $G$ is rooted at distinct vertices $v_1, \ldots, v_t \in V(G)$ if $v_1, \ldots, v_t$ are in distinct branch sets. For brevity, we say that a $K_t$-minor rooted at $\{v_1, \ldots, v_t\}$ is a $\{v_1, \ldots, v_t\}$-minor. Rooted minors are a significant tool in Robertson and Seymour’s graph minor theory [12], and a number of recent papers have studied rooted minors in their own right [4, 7, 21, 22]. Rooted minors are analogous to $H$-linked graphs for subdivisions; see [2, 8, 9].

This paper considers the question:

When does a given graph contain a $K_4$-minor rooted at four nominated vertices?

Theorem 15 answers this question by describing six classes of obstructions, which are the edge-maximal graphs containing no $K_4$-minor rooted at four nominated vertices. The flavour of this result is best introduced by first considering the 3- and 4-connected cases, which are addressed in Sections 3 and 4. First, we survey some definitions and results from the literature that will be employed later in the paper.

2. BACKGROUND

The question of when does a graph contain a $K_3$-minor rooted at three nominated vertices was answered by Wood and Linusson [22].

Lemma 1 ([22]). For distinct vertices $a, b, c$ in a graph $G$, either:

- $G$ contains an $\{a, b, c\}$-minor, or
- for some vertex $v \in V(G)$ at most one of $a, b, c$ are in each component of $G - v$. 

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1We consider finite, simple, undirected graphs.

2This definition of minor is a more concrete version of the standard definition: $H$ is a minor of $G$ if $H$ is isomorphic to a graph obtained from a subgraph of $G$ by contracting edges.
Note that in this lemma it is possible that \( v \in \{a, b, c\} \).

For distinct vertices \( s_1, t_1, s_2, t_2 \) in a graph \( G \), an \((s_1t_1, s_2t_2)\)-linkage consists of an \( s_1t_1 \)-path and an \( s_2t_2 \)-path that are disjoint. Seymour [14] and Thomassen [17] independently proved that there is essentially one obstruction for the existence of a linkage, as we now describe; see [3, 5, 6, 10, 15, 16, 18, 20] for related results.

For a graph \( H \), let \( H^+ \) denote a graph obtained from \( H \) as follows: for each triangle \( T \) of \( H \), add a possibly empty clique \( X_T \) disjoint from \( H \) and adjacent to each vertex in \( T \). We consider \( H^+ \) to be implicitly defined by the graph \( H \) and the cliques \( X_T \). An \((a, b, c, d)\)-web is a graph \( H^+ \), where \( H \) is an embedded planar graph with outerface \((a, b, c, d)\), such that each internal face of \( H \) is a triangle, and each triangle of \( H \) is a face. An \( \{a, b, c, d\}\)-web is an \((a, b, c, d)\)-web for some linear ordering \((a, b, c, d)\). That is, in an \( \{a, b, c, d\}\)-web the vertex ordering around the outerface is not specified.

Lemma 2 ([14, 17]). For distinct vertices \( s_1, t_1, s_2, t_2 \) in a graph \( G \), either:

- \( G \) contains an \((s_1t_1, s_2t_2)\)-linkage, or
- \( G \) is a spanning subgraph of an \((s_1, s_2, t_1, t_2)\)-web.

Lemma 2 implies the following result, first proved by Jung [5].

Lemma 3 ([5]). For distinct vertices \( s_1, s_2, t_1, t_2 \) in a \( 4 \)-connected graph \( G \), either:

- \( G \) contains an \((s_1t_1, s_2t_2)\)-linkage, or
- \( G \) is planar and \( s_1, s_2, t_1, t_2 \) are on some face in this order.

Lemma 3 makes sense since every \( 3 \)-connected planar graph has a unique planar embedding up to the choice of outerface [19]. We implicitly use this fact throughout the paper.

We now describe our first obstruction for a graph to contain a rooted \( K_4 \)-minor.

Lemma 4. Every \((a, b, c, d)\)-web \( G \) contains no \( \{a, b, c, d\}\)-minor.

First proof. Since \( G \) is an \((a, b, c, d)\)-web, \( G \) contains no \((ac, bd)\)-linkage [14, 17]. But if \( G \) contains a \( K_4 \)-minor \( A, B, C, D \) respectively rooted at \( a, b, c, d \), then some \( ac \)-path (contained in \( A \cup C \)) is disjoint from some \( bd \)-path (contained in \( B \cup D \)). Thus \( G \) contains no \( \{a, b, c, d\}\)-minor. □

Second proof. Suppose \( G \) contains an \( \{a, b, c, d\}\)-minor. Since \( G \) is connected, we may assume that every vertex is in some branch set. Contracting each edge with both endpoints in the same branch set produces an outerplanar \( K_4 \), which is a contradiction. □

We will need the following result by Dirac [1].

Lemma 5 ([1]). For every set \( S \) of \( k \) vertices in a \( k \)-connected graph \( G \), there is a cycle in \( G \) containing \( S \).

3. The 4-Connected Case

The following result characterises when a \( 4 \)-connected graph contains a rooted \( K_4 \)-minor. It is analogous to Lemma 3.

Theorem 6. For distinct vertices \( a, b, c, d \) in a \( 4 \)-connected graph \( G \), either:

- \( G \) contains an \( \{a, b, c, d\}\)-minor, or
- \( G \) is planar and \( a, b, c, d \) are on a common face.
Proof. Lemma 4 implies that if $G$ contains an $\{a, b, c, d\}$-minor, then the second outcome does not occur. To prove the converse, assume that $G$ is non-planar, or if $G$ is planar then $a, b, c, d$ are not on a common face. Since $G$ is 4-connected, by Lemma 5, $G$ contains a cycle $C$ through $a, b, c, d$. Without loss of generality, $a, b, c, d$ appear in this order in $C$. By Lemma 3, $G$ contains an $(ac, bd)$-linkage. The result follows from Lemma 7 below. \hfill \Box

**Lemma 7.** Let $C$ be a cycle in a graph $G$ containing vertices $a, b, c, d$ in this order. If $G$ contains an $(ac, bd)$-linkage then $G$ contains an $\{a, b, c, d\}$-minor.

*Proof.* Let $G$ be a counterexample firstly with $|V(G)|$ minimum and then with $|E(G)|$ minimum. If $V(G) = \{a, b, c, d\}$ then $G \cong K_4$. Now assume that $|V(G)| \geq 5$, and the result holds for graphs with less than $|V(G)|$ vertices, or with $|V(G)|$ vertices and less than $|E(G)|$ edges.

Let $P$ be an $ac$-path disjoint from some $bd$-path $Q$. Let $R_{ab}$ be the $ab$-path contained in $C$ avoiding $c$ and $d$. Similarly define $R_{bc}, R_{cd}$ and $R_{da}$. If some vertex or edge $x$ is not in $P \cup Q \cup C$, then $G - x$ is not a counterexample, and thus contains an $\{a, b, c, d\}$-minor. Now assume that $G = P \cup Q \cup C$. We show that contracting some edge gives a graph that satisfies the hypothesis.

Suppose that some vertex $v$ has degree 2. For at least one edge $e$ incident to $v$, the endpoints of $e$ are not both in $\{a, b, c, d\}$. Thus the contraction $G/e$ satisfies the hypothesis, and $G/e$ and hence $G$ contains an $\{a, b, c, d\}$-minor. Now assume that every vertex has degree at least 3. Thus $V(G) = V(C) = V(P \cup Q)$.

Colour $P$ red, and colour $Q$ blue. Suppose that consecutive vertices $u$ and $v$ in $C$ receive the same colour. Then $G/uv$ satisfies the hypothesis, as illustrated in Figure 1 in the case that $u$ and $v$ are red. By the choice of $G$, $G/uv$ and thus $G$ contains an $\{a, b, c, d\}$-minor. Now assume that the colours alternate around $C$. In particular, $|V(P)| = |V(Q)|$. If $P = ac$ then $Q = bd$ and we are done. Now assume that $P$ contains some internal vertex.

![Figure 1](image1.png)

**Figure 1.** If consecutive vertices $u$ and $v$ in $C$ receive the same colour then contract $uv$.

Let $v$ be the neighbour of $a$ in $P$, and let $w$ be the neighbour of $c$ in $P$. If $v$ is in $R_{da} \cup R_{ab}$, then $G/av$ satisfies the hypothesis, as illustrated in Figure 2. By the choice of $G$, $G/av$ and thus $G$ contains an $\{a, b, c, d\}$-minor. Now assume that $v \in R_{bc} \cup R_{cd}$. Similarly, $w \in R_{da} \cup R_{ab}$. Since $P$ and $Q$ are disjoint, $v \in R_{bc} \cup R_{cd} \setminus \{b, d\}$ and $w \in R_{da} \cup R_{ab} \setminus \{b, d\}$. Thus $v \neq w$. That is, $P$ (and $Q$ also) contains at least two internal vertices. Label $v$ and $a$ by “$a$”. Label every other vertex in $P$ by “$c$”. \hfill \Box
Let $x$ be the neighbour of $v$ between $v$ and $c$ in $R_{bc} \cup R_{cd}$. Let $y$ be the neighbour of $a$ between $w$ and $a$ in $R_{da} \cup R_{ab}$. Since the colours around $C$ alternate, $x$ and $y$ are in $Q$. Without loss of generality, $b, x, y, d$ appear in this order in $Q$. Label the $yd$-subpath of $Q$ by “$d$”, and label the remaining vertices in $Q$ (including $x$) by “$b$”. Thus $x$, which is labelled “$b$”, is adjacent to some vertex in $Q$ labelled “$d$”. The neighbours of $x$ in $C$ are labelled “$a$” and “$c$”, and the neighbours of $y$ in $C$ are labelled “$a$” and “$c$”. The sets of vertices labelled “$a$”, “$b$”, “$c$”, “$d$” form pairwise disjoint subpaths of $P$ or $Q$ respectively containing $a, b, c, d$. Thus contracting the vertices with the same label into a single vertex gives an $\{a, b, c, d\}$-minor in $G$, as illustrated in Figure 3. □

Let $x$ be the neighbour of $v$ between $v$ and $c$ in $R_{bc} \cup R_{cd}$. Let $y$ be the neighbour of $a$ between $w$ and $a$ in $R_{da} \cup R_{ab}$. Since the colours around $C$ alternate, $x$ and $y$ are in $Q$. Without loss of generality, $b, x, y, d$ appear in this order in $Q$. Label the $yd$-subpath of $Q$ by “$d$”, and label the remaining vertices in $Q$ (including $x$) by “$b$”. Thus $x$, which is labelled “$b$”, is adjacent to some vertex in $Q$ labelled “$d$”. The neighbours of $x$ in $C$ are labelled “$a$” and “$c$”, and the neighbours of $y$ in $C$ are labelled “$a$” and “$c$”. The sets of vertices labelled “$a$”, “$b$”, “$c$”, “$d$” form pairwise disjoint subpaths of $P$ or $Q$ respectively containing $a, b, c, d$. Thus contracting the vertices with the same label into a single vertex gives an $\{a, b, c, d\}$-minor in $G$, as illustrated in Figure 3. □

Figure 3. Construction of a rooted $K_4$-minor in Lemma 7.

4. THE 3-CONNECTED CASE

We have the following characterisation for 3-connected graphs.

**Theorem 8.** The following are equivalent for distinct vertices $a, b, c, d$ in a 3-connected graph $G$:

1. $G$ contains an $\{a, b, c, d\}$-minor,
2. $G$ is not a spanning subgraph of an $\{a, b, c, d\}$-web,
3. $G$ contains an $(ab, cd)$-linkage, an $(ac, bd)$-linkage, and an $(ad, bc)$-linkage.

**Proof.** Lemma 4 implies (1) $\implies$ (2). Lemma 2 implies (2) $\implies$ (3). It remains to prove (3) $\implies$ (1). First suppose that some cycle $C$ contains $a, b, c, d$. Without loss of generality assume that the order of the vertices in $C$ is $(a, b, c, d)$. Since $G$ contains an $(ac, bd)$-linkage, by Lemma 7, $G$ contains an $\{a, b, c, d\}$-minor. Now assume that no cycle contains $a, b, c, d$. By Lemma 5, since $G$ is 3-connected, $G$ contains a cycle $C$ through $a, b, c$. Colour red the vertices in the $ab$-path.
in $C$ that avoids $c$. Likewise colour blue the vertices in the $bc$-path in $C$ that avoids $a$. And colour green the vertices in the $ca$-path in $C$ that avoids $b$. Note that $a, b$ and $c$ each receive two colours. By Menger’s Theorem there exists three paths from $d$ to $C$, such that each path intersects $C$ in one vertex, and any two of the paths only intersect at $d$. Colour each path with the colour of its vertex in $C$. If two paths receive the same colour, then we obtain a cycle through $a, b, c, d$, as illustrated in Figure 4(a). Now assume that no two paths receive the same colour. In this case we obtain an $\{a, b, c, d\}$-minor, as illustrated in Figure 4(b).

![Figure 4](image)

**Figure 4.** Finding a rooted $K_4$-minor in a 3-connected graph.

Note that Theorem 8 does not hold for 2-connected graphs. For example, $K_{2,3}$ with colour classes $\{a, b, c\}$ and $\{d, v\}$ contains an $(ab, cd)$-linkage, an $(ac, bd)$-linkage, and an $(ad, bc)$-linkage, but contains no $\{a, b, c, d\}$-minor.

Theorem 8 can be strengthened for 3-connected planar graphs.

**Theorem 9.** For distinct vertices $a, b, c, d$ in a 3-connected planar graph $G$, either:

- $G$ contains an $\{a, b, c, d\}$-minor, or
- $a, b, c, d$ are on a common face.

**Proof.** If $a, b, c, d$ are on a common face, then $G$ is a spanning subgraph of an $\{a, b, c, d\}$-web; thus $G$ contains no $\{a, b, c, d\}$-minor by Lemma 4. For the converse, assume that $G$ contains no $\{a, b, c, d\}$-minor. By Theorem 8, $G$ is a spanning subgraph of $H^+$ for some planar graph $H$ with outerface $\{a, b, c, d\}$, such that every internal face of $H$ is a triangle. Suppose that for some triangular face $T = (u, v, w)$ of $H$, at least two vertices $x, y \in X_T$ are adjacent in $G$ to each of $u, v, w$. Let $z$ be a vertex of $H$ outside of $T$. There is such a vertex since the outerface has four vertices. Since $G$ is 3-connected, there are three internally disjoint $xz$-paths, respectively passing through $u, v, w$. Thus $G$ contains a subdivision of $K_{3,3}$ with colours classes $\{u, v, w\}$ and $\{x, y, z\}$. This contradiction proves that for each triangular face $T = (u, v, w)$ of $H$, at most one vertex in $X_T$ is adjacent to each of $u, v, w$ in $G$. If there is such a vertex $x \in X_T$ then move $x$ into $H$. Observe that $H$ remains planar: the face $uvw$ is replaced by the faces $T_w = (u, v, x)$, $T_v = (u, w, x)$ and $T_u = (v, w, x)$. Each remaining vertex in $X_T$ is now adjacent to at most two of $u, v, w$ (and possibly $x$). Assign such a vertex to one of $X_{T_u}, X_{T_v}, X_{T_w}$ according to its neighbours in $T$. Repeat this step until $X_T = \emptyset$ for each triangle $T$ of $H$. In this case, $G$ is a spanning subgraph of $H$ (not $H^+$), and $a, b, c, d$ are on a common face of $G$. □

**Corollary 10.** A planar triangulation contains an $\{a, b, c, d\}$-minor for all distinct vertices $a, b, c, d$. 5
5. Reductions

This section describes a number of operations that simplify the search for rooted $K_4$-minors. The first motivates the definition of $H^+$.

Lemma 11. Let $a, b, c, d$ be distinct vertices in a graph $H$. For each graph $H^+$, we have $H^+$ contains an $\{a, b, c, d\}$-minor if and only if $H$ contains an $\{a, b, c, d\}$-minor.

Proof. Since $H$ is a subgraph of $H^+$, if $H$ contains an $\{a, b, c, d\}$-minor then so does $H^+$. For the converse, say $A, B, C, D$ is a $K_4$-minor in $H^+$ rooted at $a, b, c, d$. Let $A' := A \cap H$. Define $B', C', D'$ similarly. Suppose that $A'$ intersects the clique $X_T$ associated with some triangle $T$ of $H$. Since $T$ separates $a$ and $X_T$, $A'$ intersects $T$. Since the vertices in $A \cap T$ are pairwise adjacent, $A \cap H$ is a connected subgraph of $H$. If two branch sets, say $A$ and $B$, are adjacent in $X_T$, then they both contain a vertex in $T$, and $A'$ and $B'$ are adjacent in $H$. Thus $A', B', C', D'$ is a $K_4$-minor in $H$ rooted at $a, b, c, d$. □

A separation of a graph $G$ is an ordered pair $(G_1, G_2)$ of subgraphs of $G$ such that $G = G_1 \cup G_2$, and $G_1 \not\subseteq G_2$ and $G_2 \not\subseteq G_1$. So there is no edge between $G_1 - G_2$ and $G_2 - G_1$. The order of $(G_1, G_2)$ is $|V(G_1 \cap G_2)|$. If certain vertices in $G$ are nominated, and there are $s$ nominated vertices in $G_1$ and $t$ nominated vertices in $G_2$, then $(G_1, G_2)$ is an $(s, t)$-separation.

Lemma 12. Let $a, b, c, d$ be four nominated vertices in a 2-connected graph $G$. Let $(G_1, G_2)$ be a $(2, 2)$-separation of $G$ of order 2, such that $a, b \in V(G_1)$ and $c, d \in V(G_2)$. Let $\{u, v\} := V(G_1) \cap V(G_2)$. Let $G'_1$ be the graph obtained from $G_1$ by adding the edge $uv$. Then $G$ contains an $\{a, b, c, d\}$-minor if and only if $G'_1$ contains an $\{a, b, u, v\}$-minor or $G_2$ contains a $\{u, v, c, d\}$-minor.

Proof. Since $G$ is 2-connected, $G'_2$ can obtained from $G$ by contracting $G_1$ onto the edge $uv$, and $G'_1$ can obtained from $G$ by contracting $G_2$ onto $uv$. Thus, if $G'_1$ contains an $\{a, b, u, v\}$-minor or $G'_2$ contains a $\{u, v, c, d\}$-minor, then $G$ contains an $\{a, b, c, d\}$-minor. For the converse, assume that $G$ contains a $K_4$-minor $A, B, C, D$ containing $a, b, c, d$ respectively. Grow the branch sets until $u$ and $v$ are in $A \cup B \cup C \cup D$. Without loss of generality, $u$ is in $A$. Thus $v$ separates $b$ from $\{c, d\}$ in $G - A$. Hence $v$ is in $B$. Therefore $A \cap G_2, B \cap G_2, C, D$ is a $\{u, v, c, d\}$-minor of $G_2$. □

Lemma 13. Let $G$ be a graph with four nominated vertices $a, b, c, d$, such that $N_G(a) = N_G(b) = \{u, v\}$ for some vertices $u, v \in V(G) \setminus \{a, b, c, d\}$. Let $G'$ be the graph obtained from $G$ by deleting $a$ and $b$, and adding the edge $uv$. Then $G$ contains an $\{a, b, c, d\}$-minor if and only if $G'$ contains a $\{u, v, c, d\}$-minor.

Proof. If $G'$ contains a $\{u, v, c, d\}$-minor, then contracting the edges $au$ and $bv$ gives an $\{a, b, c, d\}$-minor in $G$. For the converse, say $A, B, C, D$ is a $K_4$-minor in $G$ respectively rooted at $a, b, c, d$. Grow the branch sets until $u$ and $v$ are in $A \cup B \cup C \cup D$. If $u$ is in $C$ then $v$ separates $\{a, b\}$ and $D$, implying $v$ is in $D$, in which case $A = \{a\}$ and $B = \{b\}$, and $A$ and $B$ are not adjacent. By symmetry, $\{u, v\} \cap (C \cup D) = \emptyset$. Thus $u, v \in A \cup B$. If $u, v \in A$ then $A$ separates $b$ and $C \cup D$. Thus $u \in A$ and $v \in B$, without loss of generality. Hence $A - a, B - b, C, D$ is a $\{u, v, c, d\}$-minor in $G'$. □

6. Obstructions

Consider the following classes of graphs, each of which contains no $K_4$-minor rooted at the four nominated vertices. Each graph in each class is called an obstruction: see Figure 5.
Class $A$: Let $H$ be the graph consisting of an edge $pq$ with $p$ nominated, and three nominated vertices adjacent to both $p$ and $q$. Let $A$ be the class of all graphs $H^+$.

Class $B$: Let $H$ be the graph consisting of an edge $pq$, and four nominated vertices adjacent to both $p$ and $q$. Let $B$ be the class of all graphs $H^+$.

Class $C$: Let $H$ be the graph consisting of a triangle $uvw$, plus two nominated vertices adjacent to $u$ and $v$, and two nominated vertices adjacent to $v$ and $w$. Let $C$ be the class of all graphs $H^+$.

Class $D$: Let $H$ be a planar graph with an outerface of four nominated vertices, such that every internal face is a triangle, and every triangle is a face. Let $D$ be the class of all graphs $H^+$. (These are the webs.)

Class $E$: Let $H$ be a planar graph with outerface $(p,q,r,s)$ where $p$ and $q$ are nominated, every internal face is a triangle, and every triangle is a face. Add to $H$ two nominated vertices $v$ and $w$ adjacent to $r$ and $s$. Let $E$ be the class of all graphs $H^+$.

Class $F$: Let $H$ be a planar graph with outerface $(p,q,r,s)$ where every other face is a triangle and every triangle is a face. Add to $H$ two nominated vertices adjacent to $p$ and $q$, and two nominated vertices adjacent to $r$ and $s$. Let $F$ be the class of all graphs $H^+$.

The type of a nominated vertex $x$ in one of the above obstructions $H^+$ is defined as follows:

**Type-1:** $H^+ \in D \cup E$, and $x$ is adjacent to some other nominated vertex in $H$.

**Type-2:** $H^+ \in A$, and $x$ has degree 4 in $H$.

**Type-3:** $H^+ \in A \cup B \cup C \cup D \cup E \cup F$, and $x$ is neither type-1 nor type-2; such a vertex $x$ has degree 2 in $H$.

**Lemma 14.** Every graph in $A \cup B \cup C \cup D \cup E \cup F$ contains no $K_4$-minor rooted at the four nominated vertices.

**Proof.** Lemma 4 implies the result for a class $D$ obstruction. Let $H^+$ be an obstruction in some other class. By Lemma 11, it suffices to prove that $H$ contains no $\{a,b,c,d\}$-minor, where $a,b,c,d$ are the four nominated vertices.

If $H^+ \in A$ then $H \cong K_{1,1,3}$, in which case contracting an edge incident to the one non-nominated vertex produces $K_4 - e$ or $K_{1,3}$, neither of which are $K_4$.

For $H^+ \in B \cup C \cup E \cup F$, Lemma 13 is applicable. In particular, $N_H(a) = N_H(b) = \{u,v\}$ for some vertices $u,v \in V(H) \setminus \{a,b,c,d\}$. Thus if $H'$ is the graph obtained from $H$ by deleting $a$ and $b$, and adding the edge $uv$, then $H^+$ contains an $\{a,b,c,d\}$-minor if and only if $H$ contains an $\{a,b,c,d\}$-minor if and only if $H'$ contains a $\{u,v,c,d\}$-minor.

If $H^+ \in B$ then $H' \cong K_4 - e$. Thus in each case, $H'$ contains no $\{u,v,c,d\}$-minor, implying that $H$ contains no $\{a,b,c,d\}$-minor. If $H^+ \in C$ then $H' \in A$, which has no $\{u,v,c,d\}$-minor as proved above. If $H^+ \in E$ then $H' \in D$, which has no $\{u,v,c,d\}$-minor by Lemma 4. If $H^+ \in F$ then $H' \in E$, which has no $\{u,v,c,d\}$-minor as proved above.

7. Main Theorem

We now state and prove the main result of the paper. It characterises when a given graph contains a $K_4$-minor rooted at four nominated vertices.

**Theorem 15.** For every graph $G$ with four nominated vertices, either:

- $G$ contains a $K_4$-minor rooted at the nominated vertices, or
- $G$ is a spanning subgraph of a graph in $A \cup B \cup C \cup D \cup E \cup F$
Proof. Lemma 14 proves that both outcomes are not simultaneously possible. Suppose on the contrary that for some graph $G$ neither outcome occurs. That is, $G$ contains no $K_4$-minor rooted at the nominated vertices, and $G$ is not a spanning subgraph of a graph in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F}$. Choose $G$ firstly with $|V(G)|$ minimum, and then with $|E(G)|$ maximum. Let $a, b, c, d$ be the nominated vertices in $G$. If $|V(G)| = 4$ then $G$ contains an $\{a, b, c, d\}$-minor if and only if $G \cong K_4$. Otherwise, $G$ is a subgraph of $K_4$ minus an edge, which is in class $\mathcal{D}$. Now assume that $|V(G)| \geq 5$ and the result holds for every graph $G'$ with $|V(G')| < |V(G)|$, or $|V(G')| = |V(G)|$ and $|E(G')| > |E(G)|$. We proceed by considering the possible separations in $G$.

- Suppose there is a $(0, 4)$-separation $(G_1, G_2)$ of order 0: If $G_2$ contains a $K_4$-minor rooted at the nominated vertices, then so does $G$. Otherwise, by the choice of $G$, $G_2$ is
a spanning subgraph of an obstruction $H^+$. Adding $V(G_1)$ to $X_T$ for some triangle $T$ of $H$, we obtain an obstruction containing $G$ as a spanning subgraph, as desired.

- Suppose there is a $(1, 3)$-separation $(G_1, G_2)$ of order 0: Let $a$ be the nominated vertex in $G_1$. Let $b, c, d$ be the nominated vertices in $G_2$. Thus $G$ contains no $ab$-path. Hence $G$ contains no $\{a, b, c, d\}$-minor. Let $H := K_4 - ad$ with $V(H) := \{a, b, c, d\}$. Let $X_{abc} := V(G_1) \setminus \{a\}$ and $X_{bcd} := V(G_2) \setminus \{b, c, d\}$. Hence $G$ is a spanning subgraph of $H^+$, a class $D$ obstruction.

- Suppose there is a $(2, 2)$-separation $(G_1, G_2)$ of order 0: Then as in the proof of the previous case, $G$ contains no $\{a, b, c, d\}$-minor and $G$ is a spanning subgraph of a class $D$ obstruction.

Now assume that $G$ is connected.

- Suppose that $(G_1, G_2)$ is a $(0, 4)$-separation of order 1: Let $\{u\} := V(G_1 \cap G_2)$. If $G_2$ contains an $\{a, b, c, d\}$-minor then so does $G$, and we are done. Otherwise, by the choice of $G$, $G_2$ is a spanning subgraph of an obstruction $H^+$. Now, $u$ is in $T \cup X_T$ for some triangle $T$ of $H$. Add $V(G_1) \setminus \{u\}$ to $X_T$. The resulting graph $H^+$ is in the same class as the original $H^+$ and contains $G$ as a spanning subgraph.

- Suppose that $(G_1, G_2)$ is a $(1, 3)$-separation of order 1: Let $\{u\} := V(G_1 \cap G_2)$. Let $a$ be the nominated vertex in $G_1 - G_2$. If $G_2$ contains an $\{u, b, c, d\}$-minor, then adding $G_1$ to the branch set that contains $u$ gives an $\{a, b, c, d\}$-minor in $G$, and we are done. Otherwise, by the choice of $G$, $G_2$ is a spanning subgraph of an obstruction $H^+$, where $u, b, c, d$ are nominated in $G_2$.

If $u$ is type-1, then $u$ is in the outerface of $H$ (as embedded in Figure 5). Let $x$ and $y$ be the two neighbours of $u$ in this outerface. Add $a$ into the outerface of $H$, adjacent to $x, u$ and $y$. Thus $axu$ and $ayu$ become internal faces of $H$. Let $X_{axu} := V(G_1) \setminus \{a, u\}$. The resulting graph $H^+$ contains $G$ as a spanning subgraph, and is in the same class as the original $H^+$.

If $u$ is type-2, then $H^+$ is in class $A$. Let $x$ be the degree-4 neighbour of $u$ in $H$. Add $a$ to $H$ adjacent to $u$ and $x$, thus creating the triangle $axu$. Let $X_{axu} := V(G_1) \setminus \{a, u\}$. The resulting graph $H^+$ (with $a$ nominated) is in class $B$, and contains $G$ as a spanning subgraph.

If $u$ is type-3, then $u$ is in a unique triangle $uxy$ in $H$. In $H$, delete $u$, add $a$ adjacent to $x$ and $y$, thus creating the triangle $axy$. Let $X_{axy} := V(X_{uxy}) \cup V(G_1) \setminus \{a\}$. The resulting graph $H^+$ (with $a$ nominated) is in the same class as the original $H^+$, and contains $G$ as a spanning subgraph.

- Suppose that $(G_1, G_2)$ is a $(2, 2)$-separation of order 1: Let $\{u\} := V(G_1 \cap G_2)$. Without loss of generality, $a, b \in V(G_1)$ and $c, d \in V(G_2)$. Let $H$ be the planar graph with outerface $(a, b, c, d)$, and one internal vertex $u$ adjacent to $a, b, c, d$. Let $X_{aub} := V(G_1) \setminus \{a, b, u\}$ and $X_{cdu} := V(G_2) \setminus \{c, d, u\}$. The resulting graph $H^+$ is in class $D$, and contains $G$ as a spanning subgraph.

- Suppose that $(G_1, G_2)$ is a $(1, 4)$-separation of order 1: Without loss of generality, $a \in V(G_1)$ and $a, b, c, d \in V(G_2)$. If $G_2$ contains an $\{a, b, c, d\}$-minor then so does $G$. Otherwise, by the choice of $G$, $G_2$ is a spanning subgraph of an obstruction $H^+$. Now, $a$ is in some triangle $T$ of $H$. Add $V(G_1) \setminus \{a\}$ to $X_T$. The resulting graph $H^+$ is in the same class as the original $H^+$, and contains $G$ as a spanning subgraph.

- Suppose that $(G_1, G_2)$ is a $(2, 3)$-separation of order 1: Without loss of generality, $a, b \in V(G_1)$ and $b, c, d \in V(G_2)$. Let $H := K_4 - ad$ where $V(H) := \{a, b, c, d\}$. Let $X_{abc} :=
Now assume that $G$ is 2-connected.

- Suppose there is a $(0, 4)$-separation $(G_1, G_2)$ of order 2, or a $(1, 4)$-separation $(G_1, G_2)$ of order 2, or a $(2, 4)$-separation $(G_1, G_2)$ of order 2: Let $\{u, v\} := V(G_1 \cap G_2)$. Let $G'$ be the graph obtained by contracting $G_1$ onto the edge $uv$. (This is possible since $G$ is 2-connected.) If $G'$ contains an $\{a, b, c, d\}$-minor then do so does $G$, and we are done. Otherwise, by the choice of $G$, $G'$ is a spanning subgraph of an obstruction $H^+$. Since $uv$ is an edge of $G'$, we have $u, v \in T \cup X_T$ for some triangle $T$ of $H$. Add $V(G_1) \setminus \{u, v\}$ to $X_T$. The resulting graph $H^+$ contains $G$ as a spanning subgraph, and is in the same class as the original $H^+$.

- Suppose there is a $(2, 3)$-separation $(G_1, G_2)$ of order 2: Without loss of generality, $a$ is the nominated vertex in $G_1 - G_2$, $\{u, b\} = V(G_1 \cap G_2)$, and $c$ and $d$ are the nominated vertices in $G_2 - G_1$. Let $G'$ be the graph obtained by contracting $G_1$ onto the edge $ub$, and nominating $u, b, c, d$. (This is possible since $G$ is 2-connected.)

  If $G'$ contains a $\{u, b, c, d\}$-minor, then adding $G_1 - b$ to the branch set containing $u$ gives an $\{a, b, c, d\}$-minor in $G$, and we are done. Otherwise, by the choice of $G$, $G'$ is a spanning subgraph of some obstruction $H^+$. Since $ub$ is an edge of $G'$ and both $u$ and $b$ are nominated in $G'$, $H^+$ is in class $\mathcal{A}$, $\mathcal{D}$ or $\mathcal{E}$.

  If $u$ is type-1, then $ub$ is in the outerface of $H$ (as embedded in Figure 5). Let $x$ be the neighbour of $u$ distinct from $b$ in this outerface. Add $a$ into the outerface of $H$ adjacent to $u, b, x$, and let $X_{a ub} := V(G_1) \setminus \{a, b, u\}$. The resulting graph $H^+$ is in the same class as the original $H^+$, and contains $G$ as a spanning subgraph.

  If $u$ is type-2, then $H^+ \in \mathcal{A}$. Add $a$ to $H$ adjacent to $u$ and $b$, thus creating the triangle $aub$. Let $X_{aub} := V(G_1) \setminus \{a, u, b\}$. The resulting graph $H^+$ is in class $\mathcal{E}$, and contains $G$ as a spanning subgraph.

Now assume that $u$ is type-3. Thus $ub$ is in one triangle $ubx$ in $H$ (since both $u$ and $b$ are nominated in $G'$). In $H$, delete $u$, add $a$ adjacent to $x$ and $b$ creating the triangle $axb$, and let $X_{axb} := V(X_{aub}) \cup V(G_1) \setminus \{a, b\}$. The resulting graph $H^+$ contains $G$ as a spanning subgraph and is in the same class as the original $H^+$.

- Suppose there is a $(3, 3)$-separation $(G_1, G_2)$ of order 2: Without loss of generality, $a \in V(G_1 - G_2)$, $\{b, c\} = V(G_1 \cap G_2)$, and $d \in V(G_2 - G_1)$. Let $H := K_4 - ad$ where $V(H) := \{a, b, c, d\}$. Let $X_{abc} := V(G_1) \setminus \{a, b, c\}$ and $X_{bcd} := V(G_2) \setminus \{b, c, d\}$. The resulting graph $H^+$ is in class $\mathcal{D}$, and contains $G$ as a spanning subgraph.

- Suppose there is a $(2, 2)$-separation $(G_1, G_2)$ of order 2: Let $\{u, v\} := V(G_1 \cap G_2)$. Let $G'_1$ be the graph obtained from $G_1$ by adding the edge $uv$. Since $G$ is 2-connected, by Lemma 12, if $G'_1$ contains an $\{a, b, u, v\}$-minor or $G'_2$ contains a $\{u, v, c, d\}$-minor, then $G$ contains an $\{a, b, c, d\}$-minor, and we are done. Otherwise, by the choice of $G$, each $G'_i$ is a spanning subgraph of an obstruction $H^+_i$. Since the nominated vertices $u$ and $v$ are adjacent in $G'_1$, $G'_2$, $H^+_1$ and $H^+_2$ are class $\mathcal{A}$, $\mathcal{D}$ or $\mathcal{E}$.

  Consider the case in which $H^+_1 \in \mathcal{D}$. Then the edge $uv$ is either on the outerface of $H_1$ or is a diagonal of $H_1$. If $uv$ is a diagonal of $H_1$ then $H_1 \cong K_4 - ab$ since every triangle of $H_1$ is a face of $H_1$. Similarly, if $H^+_2 \in \mathcal{D}$ and $uv$ is a diagonal of $H_2$, then $H_2 \cong K_4 - cd$.

  Let $H^+$ be the graph obtained by identifying $u,v$ in $H^+_1$ with $u,v$ in $H^+_2$. Thus $H^+$ contains $G$ as a spanning subgraph. By adding gray edges to $H^+$ as illustrated in Figure 6, we now show that $H^+$ is an obstruction. Consider the following cases:
Now assume that $G$ is 2-connected and every separation of order 2 is a $(1,3)$-separation. Before addressing this case it will be convenient to first eliminate a particular separation of order 3.

- Suppose there is a separation $(G_1, G_2)$ of order 3 with no nominated vertices in $G_2 - G_1$, such that $|V(G_2)| \geq 5$:
  Let $\{u, v, w\} := V(G_1 \cap G_2)$. We claim that $G_2$ contains a $\{u, v, w\}$-minor. If not, then by Lemma 1, there is a vertex $x$ such that at most one of $u, v, w$ is in each component of $G_2 - x$. Since $|V(G_2)| \geq 5$ there is a vertex $y \in V(G_2) \setminus \{u, v, w, x\}$. If $y$ is in the same component of $G_2 - x$ as $u$, then $\{u, x\}$ is a cut-pair that forms a $(0, 4)$-separation of order 2 in $G$. Thus $y$ is not in the same component of $G_2 - x$ as $u$. Similarly, $y$ is not in the same component of $G_2 - x$ as $v$ or $w$. Thus $x$ is a cut-vertex, which is a contradiction. Hence $G_2$ contains a $\{u, v, w\}$-minor. Let $G'$ be the graph obtained from $G_1$ by adding the triangle $uvw$. Thus $G'$ is a minor of $G$, and $|V(G')| < |V(G)|$. If $G'$ contains an $\{a, b, c, d\}$-minor then so does $G$ and we are done. Otherwise, by the choice of $G$, $G'$ is a spanning subgraph of an obstruction $H^+$. The triangle $uvw$ is contained in $T \cup X_T$ for some triangle $T$ of $H$. Add $V(G_2) \setminus \{u, v, w\}$ to $X_T$. The resulting graph $H^+$ contains $G$ as a spanning subgraph (since the neighbours of each vertex in $G_2 \setminus \{u, v, w\}$ are in $G_2$) and is of the same class as the original $H^+$.

Now assume that if $(G_1, G_2)$ is a separation of order 3 with no nominated vertices in $G_2 - G_1$, then $|V(G_2)| = 4$. We consider the following two types of $(1,3)$-separations.

- Suppose there is a $(1,3)$-separation $(G_1, G_2)$ of order 2, such that $|V(G_1)| \geq 4$, or $|V(G_1)| = 3$ and $G_1 \not\cong K_3$:
  Let $a$ be the nominated vertex in $G_1 - G_2$. Let $\{u, v\} := V(G_1 \cap G_2)$. Let $G'$ be the graph obtained from $G_2$ by adding the edge $uv$ if it does not already exist, and by adding a new vertex $a'$ adjacent to $u$ and $v$, where $a', b, c, d$ are nominated in $G'$. Observe that $|V(G')| < |V(G)|$ or if $|V(G')| = |V(G)|$ then $|E(G')| > |E(G)|$. Thus by the choice of $G$, $G'$ contains an $\{a', b, c, d\}$-minor, or $G'$ is a spanning subgraph of an obstruction $H^+$.

First suppose that $G'$ contains a $K_4$-minor $A', B, C, D$ respectively rooted at $a', b, c, d$. Since $a'$ has degree 2 in $G'$, without loss of generality, $u$ is in $A'$. Now $G_1 - v$ is connected, as otherwise $v$ is a cut-vertex in $G$. Thus $A := (G_1 - v) \cup A'$ is connected and is disjoint from $B \cup C \cup D$. We claim that $A, B, C, D$ is an $\{a, b, c, d\}$-minor in $G$. Clearly $A, B, C, D$ respectively contain $a, b, c, d$. Since the edge $uv$ was added to $G'$, it may be that $G'$ is not a minor of $G$. So this claim is not immediate. However, if $uv$ is in $G$ then $G'$ is a minor of $G$, and $A, B, C, D$ is a $K_4$-minor in $G$, and we are done. It remains to show
Figure 6. Constructions of new obstructions in the case of a $(2, 2)$-separation. Black vertices are nominated. Gray vertices are the cut-pair. White vertices are not nominated. Gray edges are inserted. Gray regions are webs.
that the edge $uv$ is not needed for $A, B, C, D$ to be a $K_4$-minor. Since $u$ is in $A$, and $A$ is connected, the only problem is if $uv$ is the only edge between $A$ and some other branch set, say $B$. But, since $G$ is 2-connected, $v$ has a neighbour in $G_1 - u - v$, which is a subgraph of $A$. This proves that $A, B, C, D$ is an $\{a, b, c, d\}$-minor in $G$.

Now assume that $G'$ is a spanning subgraph of some obstruction $H^+$. Thus $a', u, v \in T \cup X_T$ for some triangle $T$ of $H$, and $a' \in T$. Rename $a'$ as $a$ in $H$, and add $V(G_1) \setminus \{a, u, v\}$ to $X_T$. The resulting graph $H^+$ is in the same class as the original $H^+$ and contains $G$ as a spanning subgraph.

Now assume that if $(G_1, G_2)$ is a separation of order 2, then $|V(G_1)| = 3$, the vertex in $G_1 - G_2$ is nominated, and $G_1 \cong K_3$ (since $G$ is 2-connected).

- Suppose there is a $(1, 3)$-separation $(G_1, G_2)$ of order 2: Let $a$ be the nominated vertex in $G_1 - G_2$. Let $\{u, v\} := V(G_1 \cap G_2)$. Thus $G_1 \cong K_3$ with vertex set $\{a, u, v\}$.

  Let $G_u$ be the graph obtained from $G$ by contracting the edge $au$ into $u$, and nominating $u$. Let $G_v$ be the graph obtained from $G$ by contracting the edge $av$ into $v$, and nominating $v$. Each of $G_u$ and $G_v$ have four nominated vertices. Since $a$ has degree 2 in $G$, $G$ contains an $\{a, b, c, d\}$-minor if and only if $G_u$ contains a $\{a, b, c, d\}$-minor or $G_v$ contains a $\{v, b, c, d\}$-minor. Also observe that $G_u \cong G_v$; they only differ in one nominated vertex. For the time being, concentrate on $G_u$; we will return to $G_v$ later.

  If $G_u$ contains a $\{u, b, c, d\}$-minor, then $G$ contains an $\{a, b, c, d\}$-minor, and we are done. Otherwise, by the choice of $G$, $G_u$ is a spanning subgraph of an obstruction $H^+$. Since a class $A$ obstruction has a $(2, 3)$-separation, and a class $B, C, E$ or $F$ obstruction has a $(2, 2)$-separation, $H^+$ is in class $D$.

  If $|X_T| \geq 2$ for some triangle $T$ of $H$ then $(G - X_T, T \cup X_T)$ is a separation of order 3 with no nominated vertices in $X_T$, such that $|V(T \cup X_T)| \geq 5$, which is a contradiction. Thus $|X_T| \leq 1$. If $X_T = \{w\}$ then move $w$ out of $X_T$ into $H$; the resulting graph $H^+$ is in $D$ and contains $G_u$ as a spanning subgraph. Repeat this step until $X_T = \emptyset$ for each triangle $T$ of $H$. Thus $G_u$ is a spanning subgraph of $H$ (not $H^+$), and $G_u$ is planar. Since $G_u$ was obtained from $G$ by deleting a degree-2 vertex whose neighbours are adjacent, $G$ is also planar.

  Since $H \in D$, $u$ is type-1. Let $S$ be the set of degree-2 nominated vertices in $G$. Thus $a \in S \subseteq \{a, b, c, d\}$. Observe that $G$ is almost 3-connected in the sense that the only cut-pairs are the neighbours of vertices in $S$, and in this case the cut-pair are adjacent. As illustrated in Figure 7, let $G^* := G - S$. A separation in $G^*$ is a separation in $G$. Thus $G^*$ is 3-connected and planar. Hence $G^*$ has a unique planar embedding. Moreover, every planar embedding of $G$ is obtained from the unique planar embedding of $G^*$ by drawing each vertex $x \in S$ in one of the two faces that contain the edge between the two neighbours of $x$. In the planar embedding of $G_u$ induced by the planar embedding of $H$, the nominated vertices $u, b, c, d$ are on the outerface. Moreover, the unique planar embedding of $G^*$ is obtained from this embedding of $G_u$ by deleting $S \setminus \{a\}$.

  If the edge $uv$ is on the outerface of $G_u$ (as in Figure 7(a)), then draw $a$ in the outerface of $G_u$ adjacent to $u$ and $v$, and possibly add edges between $a$ and other nominated vertices to obtain an obstruction (in the same class as $H$) that contains $G$ as a spanning subgraph.

  Now assume that $uv$ is not on the outerface of $G_u$ (as in Figure 7(b)). Recall that $G_u \cong G_v$, and $v, b, c, d$ are nominated in $G_v$. Consider this embedding of $G_u$ to be an embedding of $G_v$. The outerface of $G_v$ contains $b, c, d$ but not $v$. 

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For $x \in \{b, c, d\}$, if $x \in S$ then choose a neighbour $x'$ of $x$, otherwise let $x' := x$. If $x$ and $y$ are distinct vertices in $S$, then $N_G(x) \neq N_G(y)$, as otherwise $G$ would contains a $(2, 2)$-separation of order 2. Thus we may choose $b', c', d'$ so that they are distinct. Each of $b', c', d'$ are on the outerface of $G_v$. So $v, b', c', d'$ are all distinct.

Consider $v, b', c', d'$ to be nominated vertices in $G^*$. Consider the embedding of $G^*$ formed from $H$. Then $b', c', d'$ are on the outerface of $G^*$, but $v$ is not. In a 3-connected planar graph, three vertices all appear on at most one face. Thus, no face of $G^*$ contains all of $v, b', c', d'$. Thus by Theorem 9, $G^*$ contains a $\{v, b', c', d'\}$-minor. Given that $G^*$ can be obtained from $G$ by contracting $av, bb', cc'$ and $dd'$, $G$ contains an $\{a, b, c, d\}$-minor. (Here, if $b = b'$ then contracting $bb'$ does nothing.)

Now assume that $G$ is 3-connected. The result follows from Theorem 8, since a web is in class $D$.

8. Algorithmics

Robertson and Seymour [13] presented a $O(n^3)$ time algorithm that (for fixed $t$) tests whether a given $n$-vertex graph contains a $K_t$-minor rooted at $t$ nominated vertices. We conjecture that for $t = 4$ there is a $O(n)$ time algorithm for this problem; see [3, 6, 11, 20] for related linear time algorithms.

References


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