ON MULTIPLICATIVE SIDON SETS

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Received: 9/20/12, Accepted: 4/21/13, Published: 5/10/13

Abstract
Fix integers \( b > a \geq 1 \) with \( g := \gcd(a, b) \). A set \( S \subseteq \mathbb{N} \) is \( \{a, b\}\)-multiplicative if \( ax \neq by \) for all \( x, y \in S \). For all \( n \), we determine an \( \{a, b\}\)-multiplicative set with maximum cardinality in \([n]\), and conclude that the maximum density of an \( \{a, b\}\)-multiplicative set is \( \frac{b}{b+g} \). For \( A, B \subseteq \mathbb{N} \), a set \( S \subseteq \mathbb{N} \) is \( \{A, B\}\)-multiplicative if for all \( a \in A \) and \( b \in B \) and \( x, y \in S \), the only solutions to \( ax = by \) have \( a = b \) and \( x = y \). For \( 1 < a < b < c \) and \( a, b, c \) coprime, we give a \( O(1) \) time algorithm to approximate the maximum density of an \( \{\{a\}, \{b, c\}\}\)-multiplicative set to arbitrary given precision.

1. Introduction

Erdős [3], Erdős [4], Erdős [5] defined a set \( S \subseteq \mathbb{N} \) to be multiplicative Sidon\(^2\) if \( ab = cd \) implies \( \{a, b\} = \{c, d\} \) for all \( a, b, c, d \in S \); see [9, 10, 11]. In a similar direction, Wang [14] defined a set \( S \subseteq \mathbb{N} \) to be double-free if \( x \neq 2y \) for all \( x, y \in S \), and proved that the maximum density of a double-free set is \( \frac{2}{3} \); see [1] for related results. Here \( \mathbb{N} := \{1, 2, \ldots\} \), \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \([n] := \{1, 2, \ldots, n\} \), and the density of \( S \subseteq \mathbb{N} \) is

\[
\lim_{n \to \infty} \frac{|S \cap [n]|}{n}.
\]

Motivated by some questions in graph colouring, Pór and Wood [8] generalised the notion of double-free sets as follows. For \( k \in \mathbb{N} \), a set \( S \subseteq \mathbb{N} \) is \( k\)-multiplicative

\(^1\)Research supported by the Australian Research Council.

\(^2\)Additive Sidon sets have been more widely studied; see the classical papers [6, 12, 13] and the survey by O’Bryan [7].
(Sidon) if \( ax = by \) implies \( a = b \) and \( x = y \) for all \( a, b \in [k] \) and \( x, y \in S \). Pór and Wood [8] proved that the maximum density of a \( k \)-multiplicative set is \( \Theta\left(\frac{1}{\log k}\right) \).

Here we study the following alternative generalization of double-free sets. For distinct \( a, b \in \mathbb{N} \), a set \( S \subseteq \mathbb{N} \) is \( \{a, b\}\)-multiplicative if \( ax \neq by \) for all \( x, y \in S \). Our first result is to determine the maximum density of an \( \{a, b\} \)-multiplicative set. Assume that \( a < b \) throughout.

Say \( x \in \mathbb{N} \) is an \( i \)-th subpower of \( b \) if \( x = b^{y} \) for some \( y \neq 0 \) (mod \( b \)). If \( x \) is an \( i \)-th subpower of \( b \) for some even/odd \( i \) then \( x \) is an even/odd subpower of \( b \). The following table gives the even subpowers of \( b \in \{2, 3, 4\} \) and the corresponding entry in The On-Line Encyclopedia of Integer Sequences.

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<thead>
<tr>
<th>( b )</th>
<th>Entries</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>1, 3, 4, 5, 7, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, ... [A003159]</td>
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<td>3</td>
<td>1, 2, 4, 5, 7, 8, 9, 10, 11, 13, 14, 16, 17, 18, 19, 20, 22, ... [A007417]</td>
</tr>
<tr>
<td>4</td>
<td>1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 21, ... [A171948]</td>
</tr>
</tbody>
</table>

We prove the following result:

**Theorem 1.** Fix integers \( b > a \geq 1 \). Let \( g := \gcd(a, b) \). Then for every integer \( n \in \mathbb{N} \), the even subpowers of \( \frac{b}{g} \) in \( [n] \) are an \( \{a, b\}\)-multiplicative set in \( [n] \) with maximum cardinality. And the even subpowers of \( \frac{b}{g} \) are an \( \{a, b\}\)-multiplicative set with density \( \frac{b}{e^{1}g} \), which is maximum.

Note that if \( g = a \) then \( b \geq 2g \) and \( b + g \leq \frac{3}{2}b \), and if \( g < a \) then \( a \geq 2g \) and \( b + g \leq b + a < \frac{3}{2}b \). In both cases the density bound \( \frac{b}{e^{1}g} \) in Theorem 1 is at least \( \frac{2}{e} \), which is the bound obtained by Wang [14] for the \( a = 1 \) and \( b = 2 \) case.

We propose a further generalization of double-free sets. Let \( A, B \subseteq \mathbb{N} \). Say \( S \subseteq \mathbb{N} \) is \( \{A, B\}\)-multiplicative if \( ax = by \) implies \( a = b \) and \( x = y \) for all \( a \in A \) and \( b \in B \), and \( x, y \in S \). One case is easily dealt with. If \( B := \{b\} \) and \( b \) is coprime to each element of \( A \), and there is some element \( a \in A \) such that \( a < b \), then, by the reasoning above, the even subpowers of \( b \) form an \( \{A, B\}\)-multiplicative set of (maximum) density \( \frac{b}{e^{1}T} \).

The simplest nontrivial case (not covered by Theorem 1) is \( \{A, B\}\)-multiplicativity for \( A = \{a\}, B = \{b, c\}, 1 < a < b < c \), with \( a, b, c \) pairwise coprime. We have the following theorem:

**Theorem 2.** Consider \( a, b, c \in \mathbb{N} \) pairwise coprime, with \( 1 < a < b < c \). For all fixed \( \epsilon > 0 \), there is a \( O(1) \) time algorithm that computes the maximum density of an \( \{\{a\}, \{b, c\}\}\)-multiplicative set to within \( \epsilon \).
2. Proof of Theorem 1

First suppose that gcd\( (a, b) \) = 1. Let \( T \) be the set of even subpowers of \( b \). We now prove that \( T \) is an \( \{a, b\} \)-multiplicative set with maximum density. In fact, for all \( [n] \), we prove that \( T_n := T \cap [n] \) has maximum cardinality out of all \( \{a, b\} \)-multiplicative sets contained in \( [n] \).

The key to our proof is to model the problem using a directed graph. Let \( G \) be the directed graph with \( V(G) := [n] \) where \( (x, y) \in E(G) \) whenever \( bx = ay \) (implying \( x < y \)). Thus \( S \subseteq [n] \) is \( \{a, b\} \)-multiplicative if and only if \( S \) is an independent set in \( G \). If \( (x, y, z) \) is a directed path in \( G \), then \( x = \frac{a}{b} y \) and \( z = \frac{a}{b} y \). Thus each vertex \( y \) has indegree and outdegree at most 1. Since \( (x, y) \in E(G) \) implies \( x < y \), \( G \) contains no directed cycles. Thus \( G \) is a collection of disjoint directed paths. Hence a maximum independent set in \( G \) is obtained by taking all the vertices at even distance from the source vertices\(^3\), where a vertex \( y \) is a source (indegree 0) if and only if \( \frac{a}{b} y \) is not an integer: that is, if \( y \neq 0 \pmod{b} \).

We now prove that the vertices at distance \( d \) from a source vertex are precisely the \( d \)-th subpowers of \( b \). We proceed by induction on \( d \geq 0 \). Each vertex \( y \) of \( G \) has an incoming edge if and only if \( \frac{a}{b} y \in \mathbb{N} \), which occurs if and only if \( y \equiv 0 \pmod{b} \) since gcd\( (a, b) = 1 \). Thus the source vertices of \( G \) are precisely the 0-th subpowers of \( b \). This proves the \( d = 0 \) case of the induction hypothesis. Now consider a vertex \( y \) at distance \( d \) from a source vertex. Thus \( y = \frac{a}{b^d} x \) for some \( x \) at distance \( d - 1 \) from a source vertex. By induction, \( x \) is a \((d - 1)\)-th subpower of \( b \). That is, \( x = b^{d-1} z \) for some \( z \neq 0 \pmod{b} \). Thus \( y = b^d \frac{z}{a} \), which, since gcd\( (a, b) = 1 \), implies that \( \frac{z}{a} \) is an integer. Hence \( \frac{z}{a} \neq 0 \pmod{b} \) and \( y \) is a \( d \)-th subpower of \( b \), as claimed.

This proves that the even subpowers of \( b \) form a maximum independent set in \( G \). That is, \( T_n \) is an \( \{a, b\} \)-multiplicative set of maximum cardinality in \( [n] \). To illustrate this proof, the following table shows two examples of the graph \( G \) with \( b = 3 \). Observe that the \( i \)-th row consists of the \( i \)-th subpowers of 3 regardless of \( a \).

\[
\begin{array}{cccccccccccccccccccc}
\hline
\text{ } & a = 1 \text{ and } b = 3 & \hline
1 & \downarrow & \downarrow & 2 & \downarrow & 3 & \downarrow & 4 & \downarrow & 5 & \downarrow & 6 & \downarrow & 7 & \downarrow & 8 & \downarrow & 9 & \downarrow & 10 & \downarrow & 11 & \downarrow & \cdots \\
\hline
\text{ } & a = 2 \text{ and } b = 3 & \hline
1 & \downarrow & \downarrow & 2 & \downarrow & 3 & \downarrow & 4 & \downarrow & 5 & \downarrow & 6 & \downarrow & 7 & \downarrow & 8 & \downarrow & 9 & \downarrow & 10 & \downarrow & 11 & \downarrow & 12 & \downarrow & \cdots \\
\hline
\end{array}
\]

\( ^3 \)Note that this is not necessarily the only maximum independent set—for a path component with odd length, we may take the vertices at odd distance from the source of this path. This observation readily leads to a characterization of all maximum independent sets in \( G \), and thus of all \( \{a, b\} \)-multiplicative sets in \( [n] \) with maximum cardinality. Details omitted.
We now bound $|T_n|$ from above. Observe that

$$T_n = \left\{ b^{2^i} y : 0 \leq i \leq \frac{1}{2} \log_b n, 1 \leq y \leq \frac{n}{b^{2^i}}, y \not\equiv 0 \pmod{b} \right\} .$$

Thus

$$|T_n| \leq \sum_{i=0}^{\lfloor (\log_b n)/2 \rfloor} \left\lfloor \frac{b-1}{b} \frac{n}{b^{2^i}} \right\rfloor$$

$$\leq 1 + \frac{1}{2} (\log_b n) + \frac{(b-1)n}{b} \sum_{i \geq 0} \frac{1}{b^{2^i}}$$

$$\leq 1 + \frac{1}{2} (\log_b n) + \frac{(b-1)n}{b} \frac{b^2}{b^2 - 1}$$

$$= 1 + \frac{1}{2} (\log_b n) + \frac{b}{b+1} n .$$

We now bound $|T_n|$ from below. Observe that

$$T_n = \left\{ b^{2^i+1} y : 0 \leq i \leq \frac{1}{2} (\log_b n) - 1, 1 \leq y \leq \frac{n}{b^{2^i+1}}, y \not\equiv 0 \pmod{b} \right\} .$$

Thus

$$|T_n| \geq n - \sum_{i=0}^{\lfloor (\log_b n) - 1/2 \rfloor} \left\lfloor \frac{b-1}{b} \frac{n}{b^{2^i+1}} \right\rfloor$$

$$\geq n - \frac{1}{2} (\log_b n) + 1) - \frac{(b-1)n}{b^2} \sum_{i \geq 0} \frac{1}{b^{2^i}}$$

$$\geq n - \frac{1}{2} (\log_b n) + 1) - \frac{(b-1)n}{b^2} \frac{b^2}{b^2 - 1}$$

$$= n - \frac{1}{2} (\log_b n) + 1) - \frac{n}{b+1}$$

$$= \frac{b}{b+1} n - \frac{1}{2} (\log_b n) + 1) .$$

These upper and lower bounds on $|T_n|$ imply that

$$|T_n| = \frac{b}{b+1} n + \Theta(\log_b n) .$$

Hence the density of $T$ is $\frac{b}{b+1}$, and because $T_n$ is optimal for each $n$, no $\{a,b\}$-multiplicative set has density greater than $\frac{b}{b+1}$.

We now drop the assumption that $\gcd(a,b) = 1$. Let $g := \gcd(a,b)$. Since $ax = by$ if and only if $\frac{a}{g} x = \frac{b}{g} y$, a set $S$ is $\{a,b\}$-multiplicative if and only if $S$ is $\left\{ \frac{a}{g}, \frac{b}{g} \right\}$-multiplicative. Since $\frac{\frac{a}{g}}{\frac{b}{g} + 1} = \frac{\frac{b}{g}}{\frac{b}{g} + 1}$, the theorem is proved.
3. Proof of Theorem 2

Fix \( A = \{a\} \) and \( B = \{b, c\} \), where \( 1 < a < b < c \), and \( a, b, c \) are pairwise coprime. For convenience, we use the infinite graph \( G \) with vertex set \( \mathbb{N} \) and edge set
\[
E(G) = \{(x, y) : bx = ay \text{ or } cx = ay, \text{ and } x, y \in \mathbb{N}\}.
\]
Let \( G_n \) denote the subgraph of \( G \) induced by the vertex set \([n]\). Let \( \delta \) be the maximum density of an \( \{\{a\}, \{b, c\}\}\)-multiplicative set. Then
\[
\delta = \lim_{n \to \infty} \frac{\alpha(G_n)}{n},
\]
where \( \alpha(G_n) \) is the size of a maximum independent set in \( G_n \).

The infinite graph \( G \) has components \( C_{p,q} \) with vertex set
\[
V(C_{p,q}) = \{a^{p-x-y}b^q c^q : x, y \in \mathbb{N}_0\}
\]
for all \( p \in \mathbb{N}_0, q \in \mathbb{N}, \) and \( q \) not divisible by \( a, b, \) or \( c \). Note that each \( C_{p,q} \) is finite. Define \( p \) as the height of the component, and subsets of constant \( x + y \) as rows. Note that the maximum and minimum vertices in \( C_{p,q} \) are \( c^q \) and \( a^p q \) respectively. The first few components of \( G \) for \( a = 2, b = 3, \) and \( c = 5 \) are shown below:

For \( a, b, c \) as above and fixed \( \epsilon > 0 \), let \( d \) be a non-negative integer \( d \in \mathbb{N}_0 \), to be specified later. Basically, \( d \) is a cutoff height which allows us to partition the components of \( G_n \) into three types, for any given \( n \in \mathbb{N} \). The first are complete components \( C_{p,q} \) where \( n > c^d q \). The second are small incomplete components \( S_{p,q} \) where \( p \leq d \) and \( a^p q \leq n < c^d q \). The third are large incomplete components \( L_{p,q} \) with \( p > d \) and \( a^p q \leq n < c^d q \).

Let \( \alpha_T(G_n) \) denote the size of a maximum independent set in the components of type \( T \) in \( G_n \), for \( T \in \{C, S, L\} \). We clearly have
\[
\alpha(G_n) = \alpha_C(G_n) + \alpha_S(G_n) + \alpha_L(G_n) .
\]
Thus,
\[
\delta = \lim_{n \to \infty} \frac{\alpha_C(G_n)}{n} + \lim_{n \to \infty} \frac{\alpha_S(G_n)}{n} + \lim_{n \to \infty} \frac{\alpha_L(G_n)}{n} = \delta_C + \delta_S + \delta_L
\]
where
\[
\delta_T = \lim_{n \to \infty} \frac{\alpha_T(G_n)}{n}.
\]
Below we show that these limits exist, and we determine $\delta_C$ and $\delta_S$ explicitly. Then we show that, for any $\epsilon > 0$, we can choose $d$ so that $\delta_L < \epsilon$. Hence, we can calculate $\delta$ to arbitrary precision.

### 3.1. Complete Components

We require the following lemma about independent sets in grid-like graphs by Cassaigne and Zimmerman [2].

**Lemma 1.** Define a graph $H$ by $V(H) := \mathbb{N}_0 \times \mathbb{N}_0$ and

$$E(H) := \{\{v, w\} : v, w \in V(H), |v_1 - w_1| + |v_2 - w_2| = 1\}.$$ 

Suppose that $F$ is a finite subgraph of $H$ such that $(x, y) \in V(F)$ implies $(x - 1, y) \in V(F)$ unless $x = 0$, and $(x, y - 1) \in V(F)$ unless $y = 0$. Then one of the sets

$$O := \{(x, y) \in V(F) : x + y \text{ is odd}\}$$

$$E := \{(x, y) \in V(F) : x + y \text{ is even}\}$$

is a maximum independent set in $F$.

Now, consider a complete component $C_{p,1}$ of $G_n$. Note that every complete component $C_{p,q}$ of height $q$ is isomorphic to $C_{p,1}$, and can be obtained by multiplying each vertex by $q$. Thus, we call $C_{p,q}$ a $q$-copy of $C_{p,1}$. In general, we use this terminology for isomorphic components of any type obtained by multiplying each vertex by $q$.

Observe that we can apply Lemma 1 to $C_{p,1}$, since it is isomorphic to a subgraph of $H$ with the required properties. Define a function $\varphi : V(C_{p,1}) \to \mathbb{N}_0 \times \mathbb{N}_0$ by

$$\varphi((a^{p-x-y}b^x c^y)) = (x, y).$$

If $a^{p-x-y}b^x c^y$ is adjacent to $a^{p-x'-y'}b^{x'} c^{y'}$, then $|x - x'| + |y - y'| = 1$ since they must differ by a factor of $b/a$ or $c/a$. Thus, since $\varphi$ is injective, it defines an isomorphism from $C_{p,1}$ to a subgraph of $H$. Assume $a^{p-x-y}b^x c^y \in V(C_{p,1})$. Then $a^{p-x-y+1}b^{x-1} c^y \in V(C_{p,1})$ unless $x = 0$, and similarly $a^{p-x-y+1}b^{x} c^{y-1} \in V(C_{p,1})$ unless $y = 0$. Under $\varphi$, these are clearly equivalent to the conditions required for Lemma 1.

Hence, by Lemma 1 and the definition of $\varphi$, a maximum independent set in $C_{p,1}$ is given by choosing all rows with $x + y$ even, or all rows with $x + y$ odd. In fact, it is clear that a maximum independent set is obtained by choosing the bottom row first, then alternating between remaining rows. Thus, if $p = 2i - 1$, then $\alpha(C_{p,1}) = i(i + 1)$. If $p = 2i$, then $\alpha(C_{p,1}) = (i + 1)^2$. Since the largest vertex in such a component is $c^p$, we must have $p \leq \log_c n$ for the component $C_{p,1}$ to be complete. Hence, the maximum height of a complete component is $\lfloor \log_c n \rfloor$. 

Now we multiply by the number of components of height \( p \) that are complete. For a given \( p \), we require \( 1 \leq q \leq ne^{-p} \). Since \( a, b, c \) are pairwise coprime, the density of numbers not divisible by \( a, b, \) or \( c \) is

\[
\frac{(a - 1)(b - 1)(c - 1)}{abc},
\]

the number of components of height \( p \) in \( G_n \) is

\[
\frac{(a - 1)(b - 1)(c - 1)n}{c^3abc} + o(n).
\]

Let \( M(n) = \frac{1}{2} \lfloor \log_c n \rfloor \). The total number of vertices in a maximum independent set in complete components is therefore

\[
\alpha_C(G_n) = \frac{(a - 1)(b - 1)(c - 1)n}{abc} \sum_{i=0}^{M(n)} \left[ \frac{i(i + 1)}{c^{2i-1}} + \frac{(i + 1)^2}{c^{2i}} \right] + o(n).
\]

Thus, the density contribution is

\[
\delta_C = \lim_{n \to \infty} \frac{\alpha_C(G_n)}{n} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{(a - 1)(b - 1)(c - 1)n}{abc} \sum_{i=0}^{M(n)} \left[ \frac{i(i + 1)}{c^{2i-1}} + \frac{(i + 1)^2}{c^{2i}} \right]
\]

\[
= \frac{(a - 1)(b - 1)(c - 1)}{abc} \sum_{i=0}^{\infty} \left[ \frac{i(i + 1)}{c^{2i-1}} + \frac{(i + 1)^2}{c^{2i}} \right]
\]

\[
= \frac{(a - 1)(b - 1)(c - 1)}{abc} \cdot \frac{c^4}{(c - 1)^3(c + 1)}
\]

\[
= \frac{(a - 1)(b - 1)e^3}{ab(c - 1)^2(c + 1)}.
\]

### 3.2. Small Incomplete Components

Now we consider the small incomplete components. Let \( C_{p,1}[r] \) be the subgraph of \( C_{p,1} \) induced by \([r]\). Define

\[
f(p, r) := \alpha(C_{p,1}[r])
\]

for \( r \in \mathbb{N} \). We can calculate all \( f \) for \( p \leq d \) in \( O(e^d) \) time with a computer, again using Lemma 1. (In fact, these components have bounded size, so any exponential time maximum independent set algorithm runs in \( O(1) \) time.) Note that \( C_{p,q}[n] \) is a \( q \)-copy of \( C_{p,1}[[n/q]] \), and therefore \( \alpha(C_{p,q}[n]) = f(p, [n/q]) \). So we can find the size of maximum independent sets in the small components using the \( f \)'s.

More precisely, given \( p \leq d \) and \( n \), for how many values of \( q \) is \( C_{p,q}[n] \) a \( q \)-copy of \( C_{p,1}[r] \), where \( r = [n/q] \)? First note that

\[
\frac{n}{r + 1} < q \leq \frac{n}{r}.
\]
Thus, there are
\[
\frac{(a-1)(b-1)(c-1)n}{abc} \left( \frac{1}{r} - \frac{1}{r+1} \right) + o(n) = \frac{(a-1)(b-1)(c-1)n}{abcr(r+1)} + o(n)
\]
\(q\)-copies of \(C_{p,1}[r]\). The only restriction on \(r\) is that \(a^p \leq r \leq c^p - 1\). Hence, the size of a maximum independent set in components of type \(S\) is
\[
\sum_{p=0}^{d} \sum_{r=a^p}^{c^p-1} \frac{(a-1)(b-1)(c-1)n}{abcr(r+1)} f(p, r) + o(n)
\]
As \(n \to \infty\), the density contribution of small components is therefore
\[
\delta_S = \lim_{n \to \infty} \frac{1}{n} \sum_{p=0}^{d} \sum_{r=a^p}^{c^p-1} \frac{(a-1)(b-1)(c-1)n}{abcr(r+1)} f(p, r)
\]}
\[
= \sum_{p=0}^{d} \sum_{r=a^p}^{c^p-1} \frac{(a-1)(b-1)(c-1)}{abcr(r+1)} f(p, r).
\]
Since \(a, b\) and \(c\) are constants and \(d\) will be chosen so that it is bounded by a function of \(a, b\) and \(c\) (see the next section for details), \(\delta_S\) can be computed in \(O(1)\) time.

### 3.3. Large Incomplete Components

Finally, we show that we can choose \(d\) so that the density of a maximum independent set in components of type \(L\) is less than \(c\). For large components,
\[
p > d \quad \text{and} \quad a^p q \leq n < c^p q
\]
The latter implies \(c^{-p} n < q \leq a^{-p} n\). From the density of \(q\), the number of large incomplete components \(L_{p,q}\) for a given \(p > d\) is
\[
\frac{(a-1)(b-1)(c-1)n}{abc} \left( \frac{1}{a^p} - \frac{1}{c^p} \right) + o(n)
\]
Since there are less than \(p^2\) vertices in a component of height \(p\),
\[
\alpha_L(G_n) \leq \sum_{p=d}^{\infty} p^2 \cdot \frac{(a-1)(b-1)(c-1)n}{abc} \left( \frac{1}{a^p} - \frac{1}{c^p} \right)
\]}
\[
\leq (a-1)(b-1)(c-1)n \sum_{p=d}^{\infty} \frac{p^2}{a^p}
\]}
\[
= (a-1)(b-1)(c-1)n \cdot \frac{a^{1-d}(a-1)^2d^2 + 2(a-1)d + a + 1}{(a-1)^3}
\]
\[ \frac{(b-1)(c-1)n}{bc} \cdot a^{-d/2} \]

where the last inequality holds for \( d \geq 22 \). Define \( \beta := (b-1)(c-1)/bc \). Hence,

\[
\delta_L = \lim_{n \to \infty} \frac{\alpha_L(G_n)}{n} \leq \lim_{n \to \infty} \frac{1}{n} \cdot \beta n \cdot a^{-d/2} = \beta a^{-d/2}.
\]

So, to obtain a precision of \( \epsilon \) in the approximation \( \delta \approx \delta_C + \delta_S \), we pick

\[
d = \max\{2 \log_a(\beta/\epsilon), 22\}
\]

which is a function of \( a, b, c, \) and \( \epsilon \). This completes the proof of Theorem 2.

The following table gives approximate values of \( \delta \) for small \( a, b, \) and \( c \):

<table>
<thead>
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<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( \delta )</th>
</tr>
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These results were obtained by incrementing \( d \) and looking for convergence to 4 decimal places. We also approximated \( \delta_S \) using a naive algorithm (based on Lemma 1) for large \( n \). Numerical convergence occurred at values of \( d \) slightly lower than the bound given above.

References


