On the Upward Planarity of Mixed Plane Graphs

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Abstract. A mixed plane graph is a plane graph whose edge set is partitioned into a set of directed edges and a set of undirected edges. An orientation of a mixed plane graph \(G\) is an assignment of directions to the undirected edges of \(G\) resulting in a directed plane graph \(G\). In this paper, we study the computational complexity of testing whether a given mixed plane graph \(G\) is upward planar, i.e., whether it admits an orientation resulting in a directed plane graph \(G\) such that \(G\) admits a planar drawing in which each edge is represented by a curve monotonically increasing in the \(y\)-direction according to its orientation.

Our contribution is threefold. First, we show that the upward planarity testing problem is solvable in cubic time for mixed outerplane graphs. Second, we show that the problem of testing the upward planarity of mixed plane graphs reduces in quadratic time to the problem of testing the upward planarity of mixed plane triangulations. Third, we exhibit linear-time testing algorithms for two classes of mixed plane triangulations, namely mixed plane 3-trees and mixed plane triangulations in which the undirected edges induce a forest.

1 Introduction

Upward planarity is the natural extension of planarity to directed graphs. When visualizing a directed graph, one usually requires an upward drawing, that is a drawing in which the edges flow monotonically in the \(y\)-direction according to their orientation. A drawing is upward planar if it is planar and upward. Testing the upward planarity of a directed graph \(G\) is \(NP\)-hard [9], however it is polynomial-time solvable if \(G\) has a fixed planar embedding [3], if it has a single-source [2,13], if it is outerplanar [15], or if it is a series-parallel graph [7]. Exponential-time algorithms [1] and FPT algorithms [12] for upward planarity testing are known.

In this paper we deal with mixed graphs. A mixed graph is a graph whose edge set is partitioned into a set of directed edges and a set of undirected edges. Mixed graphs unify the expressive power of directed and undirected graphs, as they allow one to simultaneously represent hierarchical and non-hierarchical relationships. A number of
problems on mixed graphs have been studied, e.g., coloring mixed graphs [11,17] and orienting mixed graphs to satisfy connectivity requirements [5,6].

Upward planarity generalizes to mixed graphs as follows. A drawing of a mixed graph is upward planar if it is planar, if every undirected edge is monotone in the $y$-direction, and if every directed edge is monotone in the $y$-direction according to its orientation. Hence, testing the upward planarity of a mixed graph is equivalent to testing whether its undirected edges can be oriented to produce an upward planar directed graph. Since the upward planarity testing problem is $NP$-hard for directed graphs [9], it is $NP$-hard for mixed graphs as well. However, the question was raised by Binucci and Didimo [4] of determining the time complexity of testing the upward planarity of mixed plane graphs, that are mixed graphs with a prescribed embedding in the plane. Binucci and Didimo describe an ILP formulation for the problem and present experiments showing the efficiency of their solution. Different graph drawing questions on mixed graphs (related to crossing and bend minimization) have been studied in [8,10].

We show the following results.

In Section 3 we show that the upward planarity testing problem can be solved in $O(n^3)$ time for $n$-vertex mixed outerplane graphs, that are mixed plane graphs in which all the vertices are incident to the outer face. Our algorithm is based on a recursive approach that uses a characterization for the upward planarity of directed plane graphs due to Bertolazzi et al. [3] in order to decide the upward planarity of a mixed outerplane graph $G$ based on the upward planarity of two subgraphs of $G$.

In Section 4 we show that the problem of testing the upward planarity of an $n$-vertex mixed plane graph can be reduced to the problem of testing the upward planarity of an $O(n^2)$-vertex mixed plane triangulation, that is a mixed plane graph whose every face has three incident vertices. As a consequence, the problems of testing the upward planarity of mixed plane triangulations and of testing the upward planarity of general mixed plane graphs have the same computational complexity, up to a polynomial factor.

In Section 5, motivated by the previous result, we present linear-time algorithms to test the upward planarity of two classes of mixed plane triangulations, namely plane 3-trees and mixed plane triangulations in which the undirected edges induce a forest. The former result uses induction, while the approach for the latter result consists of repeatedly reducing the number of undirected edges in the mixed plane triangulation.

Because of space limitations, some proofs are omitted or sketched in the main text and presented in their complete version in the appendix.

2 Preliminaries

A planar drawing of a graph determines a circular ordering of the edges incident to each vertex. Two planar drawings of the same graph are equivalent if they determine the same circular orderings around each vertex. A planar embedding is an equivalence class of planar drawings. A planar drawing partitions the plane into topologically connected regions, called faces. The unbounded face is the outer face and the bounded faces are the internal faces. An edge of $G$ incident to the outer face (not incident to the outer face) is called external (resp. internal). Two planar drawings with the same planar embedding have the same faces. However, they could still differ in their outer
faces. A plane embedding is a planar embedding together with a choice for the outer face. A plane graph is a graph with a given plane embedding. An outerplane graph is a plane graph whose vertices are all incident to the outer face. A plane triangulation is a plane graph whose faces are delimited by 3-cycles. An outerplane triangulation is an outerplane graph whose internal faces are delimited by 3-cycles.

A graph is connected if every pair of vertices is connected by a path. A k-connected graph $G$ is such that removing any $k - 1$ vertices leaves $G$ connected. A cutvertex is a vertex whose removal disconnects the graph. A block of a graph $G(V, E)$ is a maximal (both in terms of vertices and in terms of edges) 2-connected subgraph of $G$; in particular, an edge of $G$ whose removal disconnects $G$ is considered as a block of $G$.

In this paper, when talking about the connectivity of mixed graphs or directed graphs, we always refer to the connectivity of their underlying undirected graphs.

A vertex $v$ in a directed graph is a sink (source) if every edge incident to $v$ is incoming at $v$ (resp. outgoing at $v$). A vertex $v$ in a directed plane graph is bimodal if the incoming edges at $v$ are consecutive in the cyclic ordering of edges incident to $v$ (which implies that the outgoing edges at $v$ are also consecutive). A directed plane graph is bimodal if every vertex is bimodal. A vertex $v$ in a 2-connected directed outerplane graph is a sink-switch (source-switch) if the two external edges incident to $v$ are both incoming (resp. outgoing) at $v$.

Bertolazzi et al. [3] characterized the directed plane graphs that are upward planar. In this paper, we will use such a characterization when dealing with two specific classes of directed plane graphs, namely directed outerplane triangulations and directed plane triangulations. Thus, we state such a characterization directly for such graph classes.

Theorem 1 ([3]). A directed outerplane triangulation $G$ is upward planar if and only if it is acyclic, it is bimodal, and the number of sources plus the number of sinks in $G$ equals the number of sink-switches (or source-switches) plus one.

Theorem 2 ([3]). A directed plane triangulation $G$ is upward planar if and only if it is acyclic, it is bimodal, and $G$ has exactly one source and one sink that are adjacent on the outer face of $G$.

An orientation $G$ of an undirected graph $G$ or of a mixed graph $G$ is an assignment of directions to the undirected edges of $G$. With a slight abuse of notation we denote by $G$ both the orientation of $G$ and the resulting directed graph. An orientation of a (plane) graph $G$ is upward planar if the resulting directed (plane) graph is upward planar. Testing the upward planarity of a mixed graph $G$ coincides with testing whether $G$ admits an upward planar orientation. The orientation of an edge in a mixed graph $G$ is prescribed if the edge is directed in $G$.

A mixed plane graph is upward planar if and only if each of its connected components is upward planar. Thus, without loss of generality, we only consider connected mixed plane graphs. In the following lemma, we show that a stronger condition can in fact be assumed for each considered plane graph $G$, namely that $G$ is 2-connected.

Lemma 1. Every $n$-vertex mixed plane graph $G$ can be augmented with new edges and vertices to a 2-connected mixed plane graph $G'$ with $O(n)$ vertices such that $G$ is upward planar if and only if $G'$ is. If $G$ is outerplane, than $G'$ is also outerplane. Moreover, $G'$ can be constructed from $G$ in $O(n)$ time.
3 Upward Planarity Testing for Mixed Outerplane Graphs

This section is devoted to the proof of the following theorem.

**Theorem 3.** The upward planarity of an \( n \)-vertex mixed outerplane graph can be tested in \( O(n^3) \) time.

Let \( G \) be any \( n \)-vertex mixed outerplane graph. By Lemma 1, an \( O(n) \)-vertex 2-connected mixed outerplane graph \( G^* \) can be constructed in \( O(n) \) time such that \( G \) is upward planar if and only if \( G^* \) is.

We introduce some notation and terminology. Let \( u \) and \( v \) be distinct vertices of \( G^* \).

We denote by \( G^* + (u, v) \) the graph obtained from \( G^* \) by adding edge \((u, v)\) if it is not already in \( G^* \), and by \( G^* - u \) the graph obtained from \( G^* \) by deleting \( u \) and its incident edges. Consider an orientation \( G^* \) of \( G^* \). A vertex is sinky (sourcey) in \( G^* \) if it is a sink-switch but not a sink (if it is a source-switch but not a source, resp.). A vertex that is neither a sink, a source, sinky, nor sourcey is ordinary; that is, \( v \) is ordinary if the two external edges incident to \( v \) are one incoming at \( v \) and one outgoing at \( v \) in \( G^* \). We say the status of a vertex of \( G^* \) in \( G^* \) is sink, source, sinky, sourcey, or ordinary.

First note that \( G^* \) is upward planar if and only if there is an upward planar directed outerplane triangulation \( T \) of \( G^* \), that is, if and only if \( G^* \) can be augmented to a mixed outerplane triangulation, and the undirected edges of such a triangulation can be oriented in such a way that the resulting directed outerplane triangulation \( T \) is upward planar. The approach of our algorithm is to determine if there is such a \( T \) using recursion. The algorithm can be easily modified to produce \( T \) if it exists.

We observe that a directed outerplane triangulation \( T \) is acyclic if and only if every 3-cycle in \( T \) is acyclic. One direction is trivial. Conversely, suppose that \( T \) contains a directed cycle. Let \( C \) be a shortest directed cycle of \( T \). If \( C \) is a 3-cycle, then we are done. Otherwise, an edge \((x, y) \notin C \) exists in \( T \) between two vertices \( x \) and \( y \) both in \( C \). Thus, \( C + (x, y) \) contains two shorter cycles, one of which is a directed cycle, contradicting the choice of \( C \). Hence, to ensure the acyclicity of a directed outerplane triangulation, it suffices to ensure that its internal faces are acyclic.

A potential edge of \( G^* \) is a pair of distinct vertices \( x \) and \( y \) in \( G^* \) such that \( G^* + (x, y) \) is outerplane, which is equivalent to saying that \( x \) and \( y \) are incident to a common external face of \( G^* \) (notice that an edge of \( G^* \) is a potential edge of \( G^* \)). Fix some external edge \( r \) of \( G^* \), called the root edge. Let \( e = \{x, y\} \) be an internal potential edge of \( G^* \). Then \( \{x, y\} \) separates \( G^* \), that is, \( G^* \) contains two subgraphs \( G_1^* \) and \( G_2^* \), such that \( G^* = G_1^* \cup G_2^* \) and \( V(G_1^* \cap G_2^*) = \{x, y\} \). (Thus, there is no edge between \( G_1^* - x - y \) and \( G_2^* - x - y \).) W.l.o.g., \( r \in E(G_1^*) \). Let \( G_e^* := G_2^* + (x, y) \). Observe that \( G_e^* \) is a 2-connected mixed outerplane graph with \( e \) incident to the outer face. Also, let \( e = \{x, y\} \neq r \) be an external potential edge of \( G^* \). Then, we define \( G_e^* \) to be the 2-vertex graph containing the single edge \((x, y)\). Further, let \( G_e^* := G_e^* \). For any (internal or external) potential edge \( e = \{x, y\} \) of \( G_e^* \) and for an orientation \( \vec{xy} \) of \( e \), let \( G_{xy} = G_e^* \) be \( G_e^* \) with \( e \) oriented \( \vec{xy} \). Define a partial order \( \prec \) on the potential edges of \( G_e^* \) as follows.

For distinct potential edges \( e \) and \( f \) of \( G^* \), say \( e \prec f \) if both end-vertices of \( f \) are in \( G_e^* \). Loosely speaking, \( e \prec f \) if \( G^* + e + f \) is outerplane and \( e \) is “between” \( r \) and \( f \).

A potential arc of \( G^* \) is a potential edge that is assigned an orientation preserving its orientation in \( G^* \). So if \( e \) is an undirected edge of \( G^* \) or a potential edge not in

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$G^*$, then there are two potential arcs associated with $e$, while if $e$ is a directed edge of $G^*$, then there is one potential arc associated with $e$. If a potential arc $\overrightarrow{x'y}$ is part of a triangulation $T$ of $G^*$, then $x$ is a source, sourcey, or ordinary, and $y$ is a sink, sinky, or ordinary in $G^*_z$. We define the status of $\overrightarrow{x'y}$ in $G^*_z$ as an ordered pair $S$ of $S(x) \in \{\text{source, sourcey, ordinary}\}$ and $S(y) \in \{\text{sink, sinky, ordinary}\}$.

We now define a function $\text{UP}(\overrightarrow{x'y}, S)$, that takes as input a potential arc $\overrightarrow{x'y}$ and a status $S$ of $\overrightarrow{x'y}$, and has value “true” if and only if there is an upward planar directed outerplane triangulation $T_{\overrightarrow{x'y}}$ of $G^*_z$ that respects $S(x)$ and $S(y)$; notice that, if $\overrightarrow{x'y}$ is external and does not correspond to $r$, then $T_{\overrightarrow{x'y}}$ is a single edge. First, the values of $\text{UP}(\overrightarrow{x'y}, S)$ can be computed in total $O(n)$ time for all the external potential arcs $\overrightarrow{x'y}$ of $G^*$ not corresponding to $r$ and for all status of $\overrightarrow{x'y}$. Indeed, $\text{UP}(\overrightarrow{x'y}, S)$ is true if and only if $S(x) = \text{source}$ and $S(y) = \text{sink}$.

We show below that, for each potential arc $\overrightarrow{x'y}$ in $G^*$ that is internal or that is external and corresponds to $r$, and for each status $S$ of $\overrightarrow{x'y}$, the value of $\text{UP}(\overrightarrow{x'y}, S)$ can be computed in $O(n)$ time from values associated to potential arcs corresponding to potential edges $e$ with $\{x, y\} \not\sim e$. Since there are at most $n(n+1)$ potential arcs and nine status for each potential arc, all the values of $\text{UP}(\overrightarrow{x'y}, S)$ can be computed in total $O(n^3)$ time by dynamic programming in reverse order to a linear extension of $\prec$. Then, there is an upward planar directed outerplane triangulation of $G^*$ if and only if $\text{UP}(\overrightarrow{x'y}, S)$ is true for some orientation $\overrightarrow{x'y}$ of $r$ and some status $S$ of $\overrightarrow{x'y}$.

Let $\overrightarrow{x'y}$ be a potential arc that is internal to $G^*$ or that corresponds to $r$. Let $S$ be a status of $\overrightarrow{x'y}$. Suppose that $\text{UP}(\overrightarrow{x'y}, S)$ is true. Then, there is an upward planar directed outerplane triangulation $T_{\overrightarrow{x'y}}$ of $G^*_z$ that respects $S(x)$ and $S(y)$. Such a triangulation contains a vertex $z \in V(G^*_z) - x - y$ such that $(x, y, z)$ is an internal face of $T_{\overrightarrow{x'y}}$. Since $T_{\overrightarrow{x'y}}$ has edge $(x, y)$ oriented from $x$ to $y$, then edges $(x, z)$ and $(y, z)$ cannot be simultaneously incoming at $x$ and outgoing at $y$, respectively, as otherwise $T_{\overrightarrow{x'y}}$ would contain a directed cycle, which is not possible by Theorem 1. Hence, edges $(x, z)$ and $(y, z)$ in $T_{\overrightarrow{x'y}}$ are either outgoing at $x$ and incoming at $y$, or outgoing at $x$ and outgoing at $y$, or incoming at $x$ and incoming at $y$, respectively.

Now, for any status $S$ of $\overrightarrow{x'y}$ and for a particular vertex $z \in V(G^*_z) - x - y$, we characterize the conditions for which an upward planar directed outerplane triangulation $T_{\overrightarrow{x'y}}$ exists that respects $S(x)$ and $S(y)$ and that contains edges $(x, z)$ and $(y, z)$ oriented according to the each of the three orientations described above.

**Lemma 2.** There is an upward planar directed outerplane triangulation $T_{\overrightarrow{x'y}}$ that respects $S(x)$ and $S(y)$, that contains edge $(x, z)$ outgoing at $x$, and that contains edge $(z, y)$ incoming at $y$, if and only if $\overrightarrow{xz}$ and $\overrightarrow{zy}$ are potential arcs of $G^*$ and there are statuses $S_1$ of $\overrightarrow{xz}$ and $S_2$ of $\overrightarrow{zy}$ such that the following conditions hold: (a) $S_1(x) = S_2(y) = \text{source, sourcey, ordinary}$, (b) $S_2(y) = S_3(y) \in \{\text{sink, sinky, ordinary}\}$, (c) $S_1(z) = \{\text{sink, ordinary}\}$, (d) $S_2(z) \in \{\text{sink, ordinary}\}$, (e) $S_3(z) = \text{sink or source}$, and (f) both $\text{UP}(\overrightarrow{xz}, S_1)$ and $\text{UP}(\overrightarrow{zy}, S_2)$ are true.

**Proof:** (\Longrightarrow) Let $T_{\overrightarrow{x'y}}$ be an upward planar directed outerplane triangulation of $G^*_z$ that respects $S(x)$ and $S(y)$, that contains edge $(x, z)$ outgoing at $x$, and that contains edge $(z, y)$ incoming at $y$. Then, $\overrightarrow{xz}$ and $\overrightarrow{zy}$ are potential arcs of $G^*$. Further,
Lemma 3. There is an upward planar directed outerplane triangulation $T_{xy}$ that respects $S(x)$ and $S(y)$ and that contains edges $(x,z)$ and $(y,z)$ incoming at $z$ if and only if $xz$ and $yz$ are potential arcs of $G^*$ and there are status $S_1$ of $xz$ and $S_2$ of $yz$ such that the following conditions hold: (a) $S_1(x) = S(x)$ ∈ {sink, source, ordinary}, (b) $S(y) ∈ \{\text{sinky, ordinary}\}$, (c) $S_2(y) ∈ \{\text{source, ordinary}\}$, (d) $S(y)$ = ordinary if and only if $S_2(y) = \text{sinky}$ and only if $S_2(y) = \text{ordinary}$, (e) $S_1(z) ∈ \{\text{sink, sinky, ordinary}\}$, (f) $S_2(z) ∈ \{\text{sink, ordinary, sinky}\}$, and (j) both $UP(xz, S_1)$ and $UP(yz, S_2)$ are true.

Lemma 4. There is an upward planar directed outerplane triangulation $T_{xy}$ that respects $S(x)$ and $S(y)$ and that contains edges $(x,z)$ and $(y,z)$ outgoing at $z$ if and only if $xz$ and $yz$ are potential arcs of $G^*$ and there are status $S_1$ of $xz$ and $S_2$ of $yz$ such that the following conditions hold: (a) $S_2(y) = S(y)$ ∈ {sink, sinky, ordinary}, (b) $S(x) ∈ \{\text{source, ordinary}\}$, (c) $S_1(x) ∈ \{\text{sink, ordinary}\}$, (d) $S(x) = \text{ordinary}$ if and only if $S_1(x) = \text{sink}$, (e) $S(x) = \text{source}$ if and only if $S_1(x) = \text{ordinary}$,
For any status $S$ of $\bar{x}_y$ and for a particular vertex $z \in V(G^*_{xy}) - x - y$, it can be checked in $O(1)$ time whether an upward planar directed outerplane triangulation $T_{\bar{x}_y}$ exists that respects $S(x)$ and $S(y)$ and that contains edges $(x, z)$ and $(y, z)$ by checking whether the conditions in at least one of Lemmata 2-4 are satisfied. Further, $\text{UP}(\bar{x}_y, S)$ is true if and only if there exists a vertex $z \in V(G^*_{xy}) - x - y$ such that an upward planar directed outerplane triangulation $T_{\bar{x}_y}$ exists that respects $S(x)$ and $S(y)$ and that contains edges $(x, z)$ and $(y, z)$. Thus, we can determine $\text{UP}(\bar{x}_y, S)$ in $O(n)$ time since there are less then $n$ possible choices for $z$.

This completes the proof of Theorem 3. The time complexity analysis can be strengthened as follows. Suppose that every internal face of $G^*$ has at most $t$ vertices. Then each vertex $v$ is incident to less than $t \cdot \text{deg}_{G^*}(v)$ potential edges and the total number of potential arcs is less than $2 \sum t \cdot \text{deg}_{G^*}(v) \leq 8tn$. Since each potential arc has nine status, and since there are less than $t$ choices for $z$, the time complexity is $O(t^2n)$. In particular, if $G^*$ is an outerplane triangulation, then the time complexity is $O(n)$.

### 4 Reducing Mixed Plane Graphs To Mixed Plane Triangulations

This section is devoted to the proof of the following theorem.

**Theorem 4.** Let $G$ be an $n$-vertex mixed plane graph. There exists an $O(n^2)$-vertex mixed plane triangulation $G'$ such that $G$ is upward planar if and only if $G'$ is. Moreover, $G'$ can be constructed from $G$ in $O(n^2)$ time.

**Proof:** By Lemma 1, an $O(n)$-vertex 2-connected mixed plane graph $G^*$ can be constructed in $O(n)$ time such that $G$ is upward planar if and only if $G^*$ is.

We show how to construct a graph $G'$ satisfying the statement of the theorem. In order to construct $G'$, we augment $G^*$ in several steps. At each step, vertices and edges are inserted inside a face $f$ of $G^*$ delimited by a cycle $C_f$ with $n_f \geq 4$ vertices. Such an insertion is done in such a way that one of the faces that are created by the insertion of vertices and edges into $f$ has $n_f - 1$ vertices, while all the other such faces have 3 vertices. The repetition of such an augmentation yields the desired graph $G'$.

We now describe how to augment $G^*$. Consider any face $f$ of $G^*$ delimited by a cycle $C_f$ with $n_f \geq 4$ vertices. Let $(u_1, u_2, \ldots, u_{n_f})$ be the clockwise order of the vertices along $C_f$ starting at any vertex. Insert a cycle $C'_f$ inside $f$ with $n_f - 1$ vertices $v_1, v_2, \ldots, v_{n_f-1}$ in this clockwise order along $C'_f$. For any $1 \leq i \leq n_f - 1$, insert edges $(v_i, u_i)$ and $(v_i, u_{i+1})$ inside $C_f$ and outside $C'_f$; also, insert edge $(v_{n_f}, u_{n_f})$ inside cycle $(u_n, v_1, v_2, \ldots, v_{n_f-1})$. All the edges inserted in $f$ are undirected. See Fig. 1. Denote by $G'_f$ the graph consisting of cycle $C_f$ together with the vertices and edges inserted in $f$. Observe that the face of $G'_f$ delimited by $C'_f$ has $n_f - 1$ vertices, while all the other faces into which $f$ is split by the insertion of $x_f$ and of its incident edges have 3 vertices.

We have that $G^*$ before the augmentation is upward planar if and only if $G^*$ after the augmentation is upward planar. One implication is trivial, given that $G^*$ before the
augmentation is a subgraph of $G^*$ after the augmentation. For the other implication, it suffices to prove that, for any upward planar orientation $C_f$ of $C_f$, there exists an upward planar orientation $G'_f$ of $G'_f$ that coincides with $C_f$ when restricted to $C_f$.

Consider an upward planar drawing $G_f$ of $C_f$ with orientation $C_f$. We are going to place the vertices of $C'_f$ inside $f$ in $G_f$, thus obtaining a drawing $G'_f$ of $G'_f$.

Pach and Tóth [14] proved that any planar drawing of a graph $G$ in which all the edges are $y$-monotone can be triangulated by the insertion of $y$-monotone edges inside the faces of $G$ (the result in [14] states that the addition of a vertex might be needed to triangulate the outer face of $G$, which however is not the case if the outer face is bounded by a simple cycle, as in our case). Hence, there exists an index $j$, with $1 \leq j \leq n_f$, such that a $y$-monotone curve can be drawn connecting $u_{j-1}$ and $u_{j+1}$ inside $f$.

If $j < n_f$, then for $1 \leq i \leq j-1$, we place $v_i$ inside $f$ close to $u_i$, with $y(v_i) \neq y(u_i)$, so that $y$-monotone curves can be drawn inside $f$ connecting $v_i$ with $u_{i-1}$, with $u_i$, and with $u_{i+1}$ (we draw $y$-monotone curves corresponding to edges of $G'_f$). Then, we place $v_j$ inside $f$ close to $u_{j+1}$, with $y(v_j) \neq y(u_{j+1})$, so that $y$-monotone curves can be drawn inside $f$ connecting $v_j$ with $u_{j-1}$, with $u_j$, with $u_{j+1}$, and with $u_{j+2}$ (we in fact draw $y$-monotone curves corresponding to edges of $G'_f$). This is possible, since a $y$-monotone curve can be drawn inside $f$ connecting $v_j$ and $u_j$, by construction, and since a $y$-monotone curve can be drawn inside $f$ connecting $u_{j-1}$ and $u_{j+1}$, by assumption, hence a $y$-monotone curve can be drawn inside $f$ connecting $v_j$ and $u_{j-1}$. Then, for $j+1 \leq i \leq n_f-1$, we place $v_i$ inside $f$ close to $u_{i+1}$, with $y(v_i) \neq y(u_{i+1})$, so that $y$-monotone curves can be drawn inside $f$ connecting $v_i$ with $u_i$, with $u_{i+1}$, and with $u_{i+2}$ (we in fact draw $y$-monotone curves corresponding to edges of $G'_f$). For any $1 \leq i \leq n_f-1$, since $y$-monotone curves can be drawn inside $f$ connecting $v_i$ with the vertices of $C_f$ to which $v_{i-1}$ and $v_{i+1}$ are close, $y$-monotone curves can be drawn inside $f$ representing the edges of $C_f$ (we in fact draw such curves). If $j = n_f$, the drawing is constructed analogously by placing $v_i$ inside $f$ close to $u_i$, for any $1 \leq i \leq n_f-1$.

The number of vertices of the mixed plane triangulation $G'$ resulting from the augmentation is $O(n^2)$. Namely, the number of vertices inserted inside a face $f$ of $G^*$ with $n_f$ vertices is $(n_f - 1) + (n_f - 2) + \cdots + 3$, hence the number of vertices of $G'$ is $\sum_{f} (n_f(n_f - 1)/2 - 3) = O(n^2)$, given that $\sum_{f} n_f \in O(n)$ (where the sums are over all the faces of $G^*$). Finally, the augmentation of $G^*$ to $G'$ can be easily performed in a time that is linear in the size of $G'$, hence quadratic in the size of the input graph. □

**Corollary 1.** The problem of testing the upward planarity of mixed plane graphs is polynomial-time equivalent to the problem of testing the upward planarity of mixed plane triangulations.

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**Fig. 1.** Augmentation of a face $f$. 
5 Upward Planarity Testing of Mixed Plane Triangulations

In this section we show how to test in linear time the upward planarity of two classes of mixed plane triangulations.

A plane 3-tree is a plane triangulation that can be constructed as follows. Denote by $H_{abc}$ a plane 3-tree whose outer face is delimited by a cycle $(a, b, c)$, with vertices $a$, $b$, and $c$ in this clockwise order along the cycle. A cycle $(a, b, c)$ is the only plane 3-tree $H_{abc}$ with three vertices. Any plane 3-tree $H_{abc}$ with $n > 3$ vertices can be constructed from three plane 3-trees $H_{abd}$, $H_{bcd}$, and $H_{cad}$ by identifying the vertices incident to their outer faces with the same label. See Fig. 2(a).

Theorem 5. The upward planarity of an $n$-vertex mixed plane 3-tree can be tested in $O(n)$ time.

Consider an $n$-vertex mixed plane 3-tree $H_{uvw}$. We define a function $UP(x, H_{abc})$ as follows. For each graph $H_{abc}$ in the construction of $H_{uvw}$ and for any distinct $x, y \in \{a, b, c\}$ we have that $UP(x, H_{abc})$ is true if and only if there exists an upward planar orientation of $H_{abc}$ in which cycle $(a, b, c)$ has $x$ as a source and $y$ as a sink.

Observe that $H_{uvw}$ is upward planar if and only if $UP(x, H_{uvw})$ is true for some $x, y \in \{u, v, w\}$ with $x \neq y$. The necessity comes from the fact that, in any upward planar orientation of $H_{uvw}$, the cycle delimiting the outer face of $H_{uvw}$ has exactly one source $x$ and one sink $y$, by Theorem 2. The sufficiency is trivial.

We show how to compute the value of $UP(x, H_{abc})$, for each graph $H_{abc}$ in the construction of $H_{uvw}$.

If $|H_{abc}| = 3$, then let $x, y, z \in \{a, b, c\}$ with $x \neq y$, $x \neq z$, and $y \neq z$. Then, $UP(x, H_{abc})$ is true if and only if edges $(x, y)$, $(x, z)$, and $(z, y)$ are not prescribed to be outgoing at $y$, outgoing at $z$, and outgoing at $y$, respectively. Hence, if $|H_{abc}| = 3$ the value of $UP(x, H_{abc})$ can be computed in $O(1)$ time.

Second, if $|H_{abc}| > 3$, denote by $H_{abd}$, $H_{bcd}$, and $H_{cad}$ the three graphs that compose $H$. We have the following:

Lemma 5. For any distinct $x, y, z \in \{a, b, c\}$, $UP(x, H_{abc})$ is true if and only if: (1) $UP(xy, H_{xyd})$, $UP(xd, H_{zxd})$, and $UP(zy, H_{yzd})$ are all true; or (2) $UP(xy, H_{xyd})$, $UP(xz, H_{zxd})$, and $UP(dy, H_{yzd})$ are all true.

Proof sketch: ($\implies$) Assume that $H_{abc}$ has an upward planar orientation $H_{abc}$ with $x$ and $y$ as a source and sink in $\{a, b, c\}$, respectively, (let $z \in \{a, b, c\}$ with $z \neq x, y$).

Fig. 2. (a) Construction of a plane 3-tree $H_{abc}$ with $n > 3$ vertices. (b)-(c) Distinct orientations of edge $(z, d)$ in two upward planar orientations of $H_{abc}$.
Edge \((z, d)\) might be outgoing or incoming at \(z\), as in Figs. 2(b) and 2(c), respectively. In the first case, \(\text{UP}(xy, H_{xyd})\), \(\text{UP}(zy, H_{yzd})\), and \(\text{UP}(xd, H_{xzd})\) are all true, while in the second case \(\text{UP}(xy, H_{xyd})\), \(\text{UP}(dy, H_{yzd})\), and \(\text{UP}(xz, H_{xzd})\) are all true.

\(\leftarrow\) Consider the case in which \(\text{UP}(xy, H_{xyd})\), \(\text{UP}(xd, H_{xzd})\), and \(\text{UP}(zy, H_{yzd})\) are all true, the other case is analogous. Then, there exist upward planar orientations \(H_{xyd}, H_{xzd}, H_{yzd}\) of \(H_{xyd}, H_{xzd}, H_{yzd}\) with \(x\) and \(y\), with \(x\) and \(d\), and with \(z\) and \(y\) as a source and sink, respectively. Orientations \(H_{xyd}, H_{xzd}, H_{yzd}\) together yield an orientation \(\text{UP}(xy, H_{xyd})\) of \(H_{xyd}\), which is upward planar by Theorem 2.

For each graph \(H_{ab}e\) in the construction of \(H_{xyz}\) and for any distinct \(x, y \in \{a, b, c\}\), the conditions in Lemma 5 can be computed in \(O(1)\) time by dynamic programming. Thus, the running time of the algorithm is \(O(n)\). This concludes the proof of Theorem 5.

We now deal with mixed plane triangulations with no cycle of undirected edges.

**Theorem 6.** The upward planarity of an \(n\)-vertex mixed plane triangulation in which the undirected edges induce a forest can be tested in \(O(n)\) time.

**Proof:** Let \(G\) be an \(n\)-vertex mixed plane triangulation. Let \(F\) be the set of undirected edges of \(G\). We assume that \(F\) contains no external edge of \(G\). Indeed, \(F\) contains at most two external edges: We can guess the orientation of all the external edges in \(F\), and for each of the four possibilities, independently, test the upward planarity for the mixed graph \(G\) in which only the internal edges of \(F\) are undirected.

We prove the statement by induction on the size of \(F\).

If \(|F| = 0\), then \(G\) is a directed plane triangulation and its upward planarity can be tested in linear time by checking whether \(G\) satisfies the conditions in Theorem 2.

If \(|F| > 0\), consider a leaf \(v\) in the forest whose edge set is \(F\). Denote by \((v, w)\) the only undirected edge of \(G\) incident to \(v\). By the assumptions, \((v, w)\) is an internal edge of \(G\). Let \((v, w, x_1)\) and \((v, w, x_2)\) be the internal faces of \(G\) incident to edge \((v, w)\).

Suppose that both edges \((x_1, v)\) and \((x_2, v)\) are incoming at \(v\). If \(v\) has an outgoing incident edge, then by the bimodality condition in Theorem 2, edge \((v, w)\) is incoming at \(v\) in every upward planar orientation of \(G\). Suppose that \(v\) has no outgoing incident edge. If \(v\) is the sink of \(G\) (recall that the edges incident to the outer face of \(G\) are directed), then edge \((v, w)\) is incoming at \(v\) in every upward planar orientation of \(G\), by the single sink condition in Theorem 2. Otherwise, edge \((v, w)\) is outgoing at \(v\) in every upward planar orientation of \(G\), again by the single sink condition in Theorem 2.

Analogously, if both \((x_1, v)\) and \((x_2, v)\) are outgoing at \(v\), the orientation of edge \((v, w)\) can be decided without loss of generality.

Assume that \((x_1, v)\) and \((x_2, v)\) are incoming and outgoing at \(v\), respectively, the case in which they are outgoing and incoming at \(v\) is analogous. We have two cases.

**Case 1:** \((x_1, x_2)\) is an edge of \(G\). By the acyclicity condition in Theorem 2, edge \((x_1, x_2)\) is outgoing at \(v\) in every upward planar orientation of \(G\).

If \(\deg(v) = 3\), then remove \(v\) and its incident edges from \(G\), obtaining a mixed plane triangulation \(G'\) with one less undirected edge than \(G\). Inductively test whether \(G'\) admits an upward planar orientation. If not, then \(G\) does not admit any upward planar orientation as well. If \(G'\) admits an upward planar orientation \(G''\), then construct an upward drawing \(I''\) of \(G''\); insert \(v\) in \(I''\) inside cycle \((w, x_1, x_2)\), so that \(y(v) > y(x_1)\), \(y(v) < y(x_2)\), and \(y(v) \neq y(w)\). Draw \(y\)-monotone curves connecting \(v\) with each
of $w$, $x_1$, and $x_2$. The resulting drawing $\Gamma$ of $G$ provides us with an orientation $G$ of $G$, which is upward planar, given that it coincides with $G'$ when restricted to $G'$, given that edges $(x_1, v)$ and $(x_2, v)$ are drawn as $y$-monotone curves according to their orientations, and given that edge $(v, w)$ is drawn as a $y$-monotone curve.

If $\deg(v) > 3$, then cycle $(w, x_1, x_2)$ contains non-empty sets $V'$ and $V''$ of vertices in its interior and its exterior, respectively. Then, two upward planarity tests can be performed, namely one for the subgraph $G'$ of $G$ induced by $V' \cup \{w, x_1, x_2\}$, and one for the subgraph $G''$ of $G$ induced by $V'' \cup \{w, x_1, x_2\}$. If one of the tests fails, then $G$ admits no upward planar orientation. Otherwise, upward planar orientations $G'$ of $G'$ and $G''$ of $G''$ together provide an upward planar orientation $G$ of $G$, given that each edge of $(w, x_1, x_2)$ has the same orientation in $G'$ and in $G''$.

Case 2: $(x_1, x_2)$ is not an edge of $G$. Remove $(v, w)$ from $G$ and insert a directed edge $(x_1, x_2)$ outgoing at $x_1$ inside face $(x_1, v, x_2, w)$. This results in a graph $G''$ with one less undirected edge than $G$. We show that $G$ is upward planar if and only if $G''$ is.

Suppose that $G$ admits an upward planar orientation $G$. Let $\Gamma$ be an upward planar drawing of $G$. Remove edge $(v, w)$ from $G$ in $\Gamma$. Draw edge $(x_1, x_2)$ inside cycle $C_f = (x_1, v, x_2, w)$, thus ensuring the planarity of the resulting drawing $\Gamma'$ of $G'$, following closely the drawing of path $(x_1, v, x_2)$, thus ensuring the upwardness of $\Gamma'$.

Suppose that $G'$ admits an upward planar orientation $G'$. Let $\Gamma'$ be an upward planar drawing of $G'$. Remove $(x_1, x_2)$ from $\Gamma'$. Since $G'$ is acyclic, $C_f$ has three possible orientations in $G'$. In Orientation 1, $w$ is its source and $x_2$ its sink; in Orientation 2, $x_1$ is its source and $w$ its sink; finally, in Orientation 3, $x_1$ is its source and $x_2$ its sink. If $C_f$ is oriented in $G'$ as in Orientation 1 (as in Orientation 2), then draw edge $(v, w)$ inside $C_f$ in $\Gamma'$, thus ensuring the planarity of the resulting drawing $\Gamma'$ of $G$, following closely the drawing of path $(w, x_1, v)$ (resp. of path $(v, x_2, w)$), thus ensuring the upwardness of $\Gamma$. If $C_f$ is oriented in $G'$ as in Orientation 3, slightly perturb the position of the vertices in $\Gamma'$ so that $y(v) \neq y(w)$. Draw edge $(v, w)$ in $\Gamma'$ as follows. Suppose that $y(v) < y(w)$, the other case being analogous. Draw a line segment inside $C_f$ starting at $v$ and slightly increasing in the $y$-direction, until reaching path $(x_1, w, x_2)$. Then, follow such a path to reach $w$. This results in an upward drawing of edge $(v, w)$ inside $C_f$, hence in an upward planar drawing of $G$.

Finally, the running time of the described algorithm is clearly $O(n)$.

6 Conclusions

We considered the problem of testing the upward planarity of mixed plane graphs.

We proved that the upward planarity testing problem is $O(n^3)$-time solvable for mixed outerplane graphs. It would be interesting to investigate whether our techniques can be strengthened to deal with larger classes of mixed plane graphs, e.g. series-parallel plane graphs. Also, since testing upward planarity is a polynomial-time solvable problem for directed outerplane graphs [15], it might be polynomial-time solvable for mixed outerplane graphs without a prescribed plane embedding as well.

We proved that the upward planarity testing problem for mixed plane graphs is polynomial-time equivalent to the upward planarity testing problem for mixed plane triangulations (and showed two classes of mixed plane triangulations for which the
problem can be solved efficiently). This, together with the characterization of the upward planarity of directed plane triangulations in terms of acyclicity, bimodality, and uniqueness of the sources and sinks (see [3] and Theorem 2), might indicate that a polynomial-time algorithm for testing the upward planarity of mixed plane triangulations should be pursued. On the other hand, Patrignani [16] proved that testing the existence of an acyclic and bimodal orientation for a mixed plane graph is \(NP\)-hard.

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References

Appendix: Omitted Proofs

In this section we show proofs that have been sketched in the main text.

**Lemma 1.** Every n-vertex mixed plane graph $G$ can be augmented with new edges and vertices to a 2-connected mixed plane graph $G'$ with $O(n)$ vertices such that $G$ is upward planar if and only if $G'$ is. If $G$ is outerplane, then $G'$ is also outerplane. Moreover, $G'$ can be constructed from $G$ in $O(n)$ time.

**Proof:** Let $G$ be an $n$-vertex mixed plane graph and let $f$ be any face of $G$. Let $C_f$ be the (possibly non-simple) cycle delimiting $f$ and let $c$ be any vertex of $C_f$. If $c$ is a cutvertex of $G$ that is incident to at least two blocks of $G$ containing edges incident to $f$, then define $k(c, f, G)$ to be the number of blocks of $G$ incident to $c$ and containing edges incident to $f$; otherwise, define $k(c, f, G) = 0$. Observe that, if $G$ is outerplane, then for any internal face $g$ of $G$ and for any vertex $u$ incident to $g$, it holds $k(u, g, G) = 0$.

Consider any face $f$ of $G$ and any cutvertex $c$ of $G$ such that $k(c, f, G) > 0$. Refer to Fig. 3. Denote by $B_1, B_2, \ldots, B_{k(c, f, G)}$ the clockwise order of the blocks of $G$ incident to $c$ and containing edges incident to $f$. For any block $B_i$, that is not a single edge, let $u_i$ and $v_i$ be the vertices following and preceding $c$ in the clockwise order of the vertices as they appear on the cycle of $B_i$ whose edges are incident to $f$. For any block $B_i$ that is a single edge, let $u_i = v_i$ be the vertex of $B_i$ different from $c$. Insert a vertex $w$ and undirected edges $(w, v_1)$ and $(w, u_2)$ in $G$, in such a way that vertex $w$ and edges $(w, v_1)$ and $(w, u_2)$ lie inside $f$. Denote by $G^*$ the resulting mixed plane graph. Also, denote by $C_{f'}$ the cycle obtained from $C_f$ by replacing path $P_c = (v_1, c, u_2)$ with path $P_w = (v_1, w, u_2)$. Denote by $f'$ the face of $G^*$ delimited by $C_{f'}$ and denote by $f''$ the face of $G^*$ delimited by path $P_w$ and $P_c$.

We have that the following statements hold:

(i) $G^*$ has one vertex more than $G$;

(ii) $k(c, f', G^*) < k(c, f, G)$, $k(u, f'', G^*) = 0$ for any vertex $u$ incident to $f''$, and $k(u, g, G^*) = k(u, g, G)$ for any face $g$ of $G$ different from $f$ and for any vertex $u$ incident to $g$;

![Fig. 3. Mixed plane graph $G$ before (a) and after (b) inserting vertex $w$ and edges $(v_1, w)$ and $(u_2, w)$ inside $f$.](image)
Before proving such statements, we prove that they imply the lemma. Namely, while

$G$ has a face $f$ and a vertex $c$ incident to $f$ such that $k(c, f, G) > 0$, insert a vertex $w$ and directed edges $(w, v_1)$ and $(w, u_2)$ inside $f$. Denote by $G'$ the graph obtained at the end of this process. For every face $g$ of $G'$ and every vertex $u$ incident to $g$ it holds $k(u, g, G') = 0$ (by repeated applications of statement (ii)), that is $G'$ is 2-connected.

Moreover, $G'$ is upward planar if and only if $G$ is (by repeated applications of statement (iii)). Also, $G'$ has $m \in O(n)$ vertices and can be constructed in $O(n)$ time (by repeated applications of statements (i) and (v)). Finally, if $G$ is outerplane, then $G'$ is outerplane (by repeated applications of statement (iv)).

We now prove statements (i)-(v). The truth of statements (i), (ii), (iv), and (v) is trivially proved. Namely, graph $G''$ has one vertex more than $G$, by construction, and it can be constructed in constant time by adding a vertex and two edges to $G$. Also, $k(c, f', G'')$ is either equal to $k(c, f, G) - 1$ or to $k(c, f, G) - 2$, depending on whether $c$ is a cutvertex of $G''$ incident to at least two 2-connected components of $G''$ containing edges incident to $f'$ or not. Further, $k(u, f'', G'') = 0$, given that $f''$ is delimited by simple cycle $(v_1, c, u_2, w)$. Moreover, every face $g$ of $G$ different from $f$ is delimited by the same cycle in $G$ and in $G''$, thus $k(u, g, G'') = k(u, g, G)$ holds for any vertex $w$ incident to $g$. Finally, $C_f'$ contains $w$, by definition, and it contains all the vertices of $C_f$. In particular, it contains $c$ given that $c$ occurs at least twice in $C_f$ and given that $C_f'$ has exactly one less occurrence of $c$ than $C_f$.

Next, we prove statement (iii), namely that $G$ is upward planar if and only if $G''$ is. One implication is trivial, namely $G$ is a subgraph of $G''$, hence if $G''$ is upward planar, then $G$ is upward planar as well. Assume that $G$ is upward planar. We prove that $G''$ is upward planar.

Let $G$ be any upward planar orientation of $G$. We define an orientation $G''$ of $G''$ as follows. Every edge of $G''$ that is also an edge of $G$ is oriented as in $G$; further, edge $(v_1, w)$ is outgoing at $v_1$ in $G''$ if and only if edge $(v_1, c)$ is outgoing at $v_1$ in $G$; also, edge $(u_2, w)$ is outgoing at $u_2$ in $G''$ if and only if edge $(u_2, c)$ is outgoing at $u_2$ in $G$. Consider any upward planar drawing $\Gamma$ of $G$ with orientation $G$. We are going to construct an upward planar drawing $\Gamma''$ of $G''$ with orientation $G''$ so that $\Gamma''$ coincides with $\Gamma$ when restricted to $G$.

Suppose that edge $(v_1, c)$ is outgoing at $v_1$ and edge $(u_2, c)$ is outgoing at $c$ in $G$. Then, path $P_c = (v_1, c, u_2)$ is represented by a $y$-monotone curve incident to $f$. Draw path $P_w = (v_1, u, u_2)$ in $\Gamma$ as a $y$-monotone curve lying inside $f$ and arbitrarily close to the drawing of $P_c$. The resulting drawing $\Gamma''$ of $G''$ is upward because $\Gamma$ is upward (by induction) and because $P_w$ is $y$-monotone (by construction). Also $\Gamma''$ is planar because $\Gamma$ is planar (by induction) and because $P_w$ does not cross any edge of $G$, given that it lies inside $f$ and that it is arbitrarily close to the drawing of $P_c$, which does not cross any edge of $G$ (by induction).

Drawing $\Gamma''$ can be constructed analogously to the previous case if edge $(u_2, c)$ is outgoing at $u_2$ and edge $(v_1, c)$ is outgoing at $c$ in $G$. 

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Suppose that edges \((v_1, c)\) and \((u_2, c)\) are both incoming at \(c\) in \(G\). Draw edges \((v_1, w)\) and \((u_2, w)\) in \(\Gamma^*\) as \(y\)-monotone curves lying inside \(f\) and arbitrarily close to the drawings of edges \((v_1, c)\) and \((u_2, c)\), respectively, in such a way that \(w\) lies in a point arbitrarily close to \(c\). Observe that \(w\) is below \(c\) if, considering a horizontal line \(l_c\) slightly below \(c\), edge \((u_2, c)\) intersects \(l_c\) to the left of edge \((v_1, c)\) (see Fig. 4), while \(w\) is above \(c\) if edge \((u_2, c)\) intersects \(l_c\) to the right of edge \((v_1, c)\) (see Fig. 5). The resulting drawing \(\Gamma^*\) of \(G^*\) is upward because \(\Gamma\) is upward (by induction) and because edges \((v_1, w)\) and \((u_2, w)\) are \(y\)-monotone (by construction). Also \(\Gamma^*\) is planar because \(\Gamma\) is planar (by induction) and because edges \((v_1, w)\) and \((u_2, w)\) do not cross any edge of \(G\), given that they lie inside \(f\) and that they are arbitrarily close to the drawing of edges \((v_1, c)\) and \((u_2, c)\), respectively, which do not cross any edge of \(G\) (by induction).
Lemma 3. There is an upward planar directed outerplane triangulation \( T_{xy} \) that respects \( S(x) \) and \( S(y) \) and that contains edges \((x, z)\) and \((y, z)\) incoming at \( z \) if and only if \( \vec{xz} \) and \( \vec{yz} \) are potential arcs of \( G^* \) and there are status \( S_1 \) of \( \vec{xz} \) and \( S_2 \) of \( \vec{yz} \) such that the following conditions hold:

(a) \( S_1(x) = S(x) \in \{\text{source, sourcey, ordinary}\} \),
(b) \( S(y) \in \{\text{sinky, ordinary}\} \),
(c) \( S_2(y) \in \{\text{source, ordinary}\} \),
(d) \( S(y) = \text{ordinary} \) if and only if \( S_2(y) = \text{source} \),
(e) \( S(y) = \text{sinky} \) if and only if \( S_2(y) = \text{ordinary} \),
(f) \( S_1(z) \in \{\text{sink, sinky, ordinary}\} \),
(g) \( S_2(z) \in \{\text{sink, sinky, ordinary}\} \),
(h) \( S_1(z) \in \{\text{sink, ordinary}\} \) or \( S_2(z) = \text{sink} \),
(i) \( S_2(z) \in \{\text{sink, ordinary}\} \) or \( S_1(z) = \text{sink} \), and
(j) both \( \text{UP}(\vec{xz}, S_1) \) and \( \text{UP}(\vec{yz}, S_2) \) are true.

Proof: \( \rightarrow \) Let \( T_{xy} \) be an upward planar directed outerplane triangulation of \( G^*_{xy} \) that respects \( S(x) \) and \( S(y) \), and that contains edge \((x, z)\) and \((y, z)\) incoming at \( z \). Then \( \vec{xz} \) and \( \vec{yz} \) are potential arcs of \( G^* \). \( T_{xy} \) determines upward planar directed outerplane triangulations \( T_{xz} \) and \( T_{yz} \) respectively of \( G^*_x \) and \( G^*_y \) (where \( T_{xz} \) and \( T_{yz} \) are single edges if \( \vec{xz} \) and \( \vec{yz} \) are external, respectively), as well as status \( S_1 \) and \( S_2 \) of \( \vec{xz} \) and \( \vec{yz} \), respectively, such that (j) both \( \text{UP}(\vec{xz}, S_1) \) and \( \text{UP}(\vec{yz}, S_2) \) are true.

Since \( \vec{xy} \) and \( \vec{zx} \) are consecutive outgoing arcs at \( x \), we have (a) \( S_1(x) = S(x) \in \{\text{source, sourcey, ordinary}\} \). Since \( \vec{xy} \) and \( \vec{zx} \) are incident to \( y \), we have (b) \( S(y) \in \{\text{sinky, ordinary}\} \). Since \( \vec{xy} \) in \( T_{xy} \), we have \( S_1(y) \in \{\text{sink, ordinary, sinky}\} \). Analogously, we have (g) \( S_2(z) \in \{\text{sink, ordinary, sinky}\} \). Moreover, (h) if \( z \) is sinky in \( T_{yz} \), then \( z \) is a sink in \( T_{yz} \), as otherwise \( z \) is not bimodal in \( T_{xy} \). Analogously, (i) if \( z \) is sinky in \( T_{yz} \), then \( z \) is a sink in \( T_{xy} \).

\( \leftarrow \) Let \( T_{xy} \) be an upward planar directed outerplane triangulation of \( G^*_{xy} \) respecting \( S_1 \) (\( T_{xy} \) is a single edge if \( \vec{xz} \) is external). Let \( T_{yz} \) be an upward planar directed outerplane triangulation of \( G^*_{yz} \) respecting \( S_2 \) (\( T_{yz} \) is a single edge if \( \vec{yz} \) is external). Such triangulations exist because \( \text{UP}(\vec{xz}, S_1) \) and \( \text{UP}(\vec{yz}, S_2) \) are true. Let \( T_{xy} \) be the triangulation of \( G^*_{xy} \) determined from \( T_{xz} \) and \( T_{yz} \) by adding the arc \( \vec{xy} \). Since \( T_{xz} \), \( T_{yz} \), and \( (x, y, z) \) are acyclic, \( T_{xy} \) is acyclic.

Since \( x \) is bimodal in \( T_{xz} \), it is bimodal in \( T_{xy} \). Since \( y \) is not source in \( T_{yz} \), it is bimodal in \( T_{yz} \). If \( z \) is not bimodal in \( T_{yz} \), then \( z \) is sinky in \( T_{yz} \) and is not a sink in \( T_{yz} \) (or vice versa, switching the role of \( T_{xz} \) and \( T_{yz} \)). However, this is not possible by...
conditions (h) and (i), hence $z$ is bimodal in $T_{xy}$. Every other vertex is bimodal in $T_{xy}$ because it is bimodal in $T_{x^*}$ or in $T_{y^*}$. Hence, $T_{xy}$ is bimodal.

Let $s_i$, $t_i$ and $w_i$ be the number of sources, sinks, and source-switches in $T_{x^*}$, respectively. Let $s_i$, $t_i$ and $w_i$ be the number of sources, sinks, and source-switches in $T_{y^*}$, respectively. By Theorem 1, $s_i + t_i = w_i + 1$ for $i \in \{1, 2\}$. Let $s$, $t$ and $w$ be the number of sources, sinks, and source-switches in $T_{xy}$, respectively.

We distinguish the following cases: $y$ is a source or is ordinary in $T_{y^*}$. And $z$ is a sink in $T_{x^*}$ or $T_{y^*}$, or $z$ is in ordinary in both $T_{x^*}$ and $T_{y^*}$.

If $y$ is a source in $T_{y^*}$ and $z$ is a sink in $T_{x^*}$ or $T_{y^*}$ (possibly both), then $s = s_1 + s_2 - 1$ (for $y$) and $t = t_1 + t_2 - 1$ (for $z$) and $w = w_1 + w_2 - 1$ (for $y$). (Here, if $z$ is a sink in both $T_{x^*}$ and $T_{y^*}$, then $z$ is a sink in $T_{xy}$, but still $t = t_1 + t_2 - 1$.) If $y$ is a source in $T_{x^*}$ and $z$ is ordinary in both $T_{x^*}$ and $T_{y^*}$, then $s = s_1 + s_2 - 1$ (for $y$) and $t = t_1 + t_2$ and $w = w_1 + w_2 - 1 + 1$ (for $y$ and $z$). If $y$ is ordinary in $T_{x^*}$ and $z$ is a sink in $T_{y^*}$ or in $T_{xy}$, then $s = s_1 + s_2$ and $t = t_1 + t_2 - 1$ (for $z$) and $w = w_1 + w_2$. If $y$ is ordinary in $T_{x^*}$ and $z$ is ordinary in $T_{x^*}$ and $T_{y^*}$, then $s = s_1 + s_2$ and $t = t_1 + t_2$ and $w = w_1 + w_2 + 1$ (for $z$). In all cases, it follows that $s + t = w + 1$.

By Theorem 1, $T_{xy}$ is upward planar. By construction, $T_{xy}$ respects $S(x)$ and $S(y)$ and contains edge $(x, z)$ and $(y, z)$ ingoing at $z$.

**Lemma 4.** There is an upward planar directed outerplane triangulation $T_{xy}$ that respects $S(x)$ and $S(y)$ and that contains edges $(z, x)$ and $(z, y)$ outgoing at $z$ if and only if $\overline{x^*}z$ and $\overline{y^*}z$ are potential arcs of $G^*$ and there are status $S_1$ of $\overline{x^*}z$ and $S_2$ of $\overline{y^*}z$ such that the following conditions hold:

(a) $S_2(y) = S(y) \in \{\text{sink, sinky, ordinary}\}$,
(b) $S(x) \in \{\text{source, ordinary}\}$,
(c) $S_1(x) \in \{\text{sink, ordinary}\}$,
(d) $S(x) = \text{ordinary if and only if } S_1(x) = \text{sink}$,
(e) $S(x) = \text{sourcey if and only if } S_1(x) = \text{ordinary}$,
(f) $S_1(z) \in \{\text{source, sourcey, ordinary}\}$,
(g) $S_2(z) \in \{\text{source, sourcey, ordinary}\}$,
(h) $S_1(z) \in \{\text{source, ordinary}\}$ or $S_2(z) = \text{source}$,
(i) $S_2(z) \in \{\text{source, ordinary}\}$ or $S_1(z) = \text{source}$ and
(j) both $\uparrow \overline{x^*}z, S_1)$ and $\uparrow \overline{y^*}z, S_2)$ are true.

**Proof:** ($\Rightarrow$) Let $T_{xy}$ be an upward planar directed outerplane triangulation of $G^+_xy$ that respects $S(x)$ and $S(y)$ in $S$, and that contains edges $(z, x)$ and $(z, y)$ outgoing at $z$. Then, $\overline{x^*}z$ and $\overline{y^*}z$ are potential arcs of $G^*$. $T_{xy}$ determines upward planar directed outerplane triangulations $T_{x^*}$ and $T_{y^*}$ respectively of $G^+_x$ and $G^+_y$ (where $T_{x^*}$ and $T_{y^*}$ are single edges if $\overline{x^*}z$ and $\overline{y^*}z$ are external, respectively), as well as status $S_1$ and $S_2$ respectively of $\overline{x^*}z$ and $\overline{y^*}z$, such that (j) both $\uparrow \overline{x^*}z, S_1)$ and $\uparrow \overline{y^*}z, S_2)$ are true. Since $\overline{x^*}z$ and $\overline{y^*}z$ are consecutive incoming arcs at $y$, we have (a) $S_2(y) = S(y) \in \{\text{sink, sinky, ordinary}\}$. Since $\overline{x^*}z$ and $\overline{y^*}z$ are incident to $x$, (b) $S(x) \in \{\text{source, ordinary}\}$. Since $\overline{x^*}z$ in $T_{x^*}$, we have $S_1(x) \in \{\text{sink, sinky, ordinary}\}$. Moreover, if $x$ is sinky in $T_{x^*}$, then $x$ is not bimodal in $T_{xy}$. Thus, (c) $S_1(x) \in \{\text{sink, ordinary}\}$. Observe that
(d) $S(x) = \text{ordinary if and only if } S_1(x) = \text{sink (otherwise } x \text{ is not bimodal in } T_{xy}$. 

Similarly, (e) $S(x) = \text{source if and only if } S_1(x) = \text{ordinary. Since } \overrightarrow{xy} \text{ is in } T_{xy}$,

we have (f) $S_1(z) \in \{\text{source, sourcey, ordinary}\}$. Moreover, (h) if $z$ is sourcey in $T_{xy}$,

then $z$ is a source in $T_{xy}^*$, as otherwise $z$ is not bimodal in $T_{xy}$. Analogously, (i) if $z$ is

sourcey in $T_{xy}$, then $z$ is a source in $T_{xy}$.

($(\Leftarrow)$) Let $T_{xy}^*$ be an upward planar directed outerplane triangulation of $G_{xy}^*$ respecting

$S_1 (T_{xy})$ is a single edge if $\overrightarrow{xy}$ is external). Let $T_{xy}^*$ be an upward planar directed

outerplane triangulation of $G_{xy}^*$ respecting $S_2 (T_{xy})$ is a single edge if $\overrightarrow{xy}$ is external).

Such triangulations exist because $\text{UP}(\overrightarrow{xy}, S_1)$ and $\text{UP}(\overrightarrow{xy}, S_2)$ are true. Let $T_{xy}^*$ be the

triangulation of $G_{xy}^*$ determined from $T_{xy}$ and $T_{xy}$ by adding the arc $\overrightarrow{xy}$. Since $T_{xy}$,

$T_{xy}^*$, and $xy$ are acyclic, $T_{xy}^*$ is acyclic.

Since $y$ is bimodal in $T_{xy}$, it is bimodal in $T_{xy}$. Since $x$ is not sinky in $T_{xy}$, it is

bimodal in $T_{xy}^*$. If $z$ is not bimodal in $T_{xy}$, then $z$ is sourcey in $T_{xy}$ and is not a source

in $T_{xy}$ (or vice versa, switching the role of $T_{xy}$ and $T_{xy}$). However this is not possible

by conditions (h) and (i), hence $z$ is bimodal in $T_{xy}$. Every other vertex is bimodal in

$T_{xy}$ because it is bimodal in $T_{xy}$ or in $T_{xy}$. Hence, $T_{xy}^*$ is bimodal.

Let $s_1$, $t_1$, and $w_1$ be the number of sources, sinks, and sink-switches in $T_{xy}^*$, respectively.

Let $s_2$, $t_2$, and $w_2$ be the number of sources, sinks, and sink-switches in $T_{xy}$, respectively.

By Theorem 1, $s_1 + t_1 = w_1 + 1$ for $i = 1, 2$. Let $s, t$, and $w$ be the

number of sources, sinks, and sink-switches in $T_{xy}$, respectively.

We distinguish the following cases: $x$ is a sink or is ordinary in $T_{xy}$. And $z$ is a

source in $T_{xy}$ or $T_{xy}$, or $z$ is in ordinary in both $T_{xy}$ and $T_{xy}$.

If $x$ is a sink in $T_{xy}$ and $z$ is a source in $T_{xy}$ or $T_{xy}$ (possibly both), then $t =$ $t_1 + t_2 - 1$ (for $x$) and $s = s_1 + s_2 - 1$ (for $z$) and $w = w_1 + w_2 - 1$ (for $x$). (Here, if

$z$ is a source in both $T_{xy}$ and $T_{xy}$, then $z$ is a source in $T_{xy}$, but still $s = s_1 + s_2 - 1$.)

If $x$ is a sink in $T_{xy}$ and $z$ is ordinary in both $T_{xy}$ and $T_{xy}$, then $t = t_1 + t_2 - 1$ (for $x$)

and $s = s_1 + s_2$ and $w = w_1 + w_2 - 1 + 1$ (for $x$ and $z$). If $x$ is ordinary in $T_{xy}$

and $z$ is a source in $T_{xy}$ or in $T_{xy}$, then $t = t_1 + t_2$ and $s = s_1 + s_2 - 1$ (for $z$) and

and $w = w_1 + w_2$. If $x$ is ordinary in $T_{xy}$ and $z$ is ordinary in $T_{xy}$ and $T_{xy}$, then $t = t_1 + t_2$

and $s = s_1 + s_2$ and $w = w_1 + w_2 + 1$ (for $z$). In all cases, it follows that $s + t = w + 1$.

By Theorem 1, $T_{xy}$ is upward planar. By construction, $T_{xy}$ respects $S(x)$ and $S(y)$

and contains edges $(z, x)$ and $(z, y)$ outgoing at $z$. $\square$

**Lemma 5.** For any distinct $x, y, z \in \{a, b, c\}$, $\text{UP}(\overrightarrow{xy}, H_{abc})$ is true if and only if: (1) $\text{UP}(\overrightarrow{xy}, H_{axy}), \text{UP}(\overrightarrow{xy}, H_{bay}), \text{UP}(\overrightarrow{xy}, H_{aza})$, and $\text{UP}(\overrightarrow{xy}, H_{azy})$ are all true; or (2) $\text{UP}(\overrightarrow{xy}, H_{xy}), \text{UP}(\overrightarrow{xy}, H_{xyz}), \text{UP}(\overrightarrow{xy}, H_{yxd})$, and $\text{UP}(\overrightarrow{xy}, H_{yze})$ are all true.

**Proof:** For the necessity, assume that $H_{abc}$ has an upward planar orientation $H_{abc}$.

Then, by Theorem 2, cycle $(a, b, c)$ has exactly one source and one sink when it is

oriented according to $H_{abc}$. Denote such a source and sink by $x$ and $y$, respectively.

Again by Theorem 2, we have that $x$ and $y$ are a source and a sink for $H_{abc}$. Hence, edges $(x, z)$ and $(y, z)$ are outgoing at $x$ and incoming at $y$ in $H_{abc}$, respectively. On

the other hand, edge $(z, x)$ might be outgoing or incoming at $z$. Refer to Figs. 2(b)

and 2(c), respectively. In the first case, $H_{xyd}, H_{yxd}, H_{xzd}$ admit upward planar orientations

with $x$ and $y$, with $z$ and $y$, and with $x$ and $z$ as a source and sink, respectively,
namely $H_{abc}$ restricted to $H_{xyd}$, $H_{yzd}$, $H_{zxd}$ provides us with such orientations. Hence, $UP(-\rightarrow_{xy}; H_{xyd})$, $UP(-\rightarrow_{zy}; H_{yzd})$, and $UP(-\rightarrow_{zd}; H_{zxd})$ are all true. In the second case, $H_{xyd}$, $H_{yzd}$, and $H_{zxd}$ admit upward planar orientations with $x$ and $y$, with $d$ and $y$, and with $x$ and $z$ as a source and sink, respectively, namely $H_{abc}$ restricted to $H_{xyd}$, $H_{yzd}$, and $H_{zxd}$ provides us with such orientations. Hence, $UP(-\rightarrow_{xy}; H_{xyd})$, $UP(-\rightarrow_{zd}; H_{zxd})$, and $UP(-\rightarrow_{zd}; H_{zxd})$ are all true.

For the sufficiency, consider the case in which $UP(-\rightarrow_{xy}; H_{xyd})$, $UP(-\rightarrow_{zd}; H_{zxd})$, and $UP(-\rightarrow_{zy}; H_{yzd})$ are all true, the other case being analogous. Then, there exist upward planar orientations $H_{xyd}$, $H_{zxd}$, and $H_{yzd}$ of $H_{xyd}$, $H_{zxd}$, and $H_{yzd}$ in which the outer face of $H_{xyd}$ has $x$ and $y$ as a source and sink, in which the outer face of $H_{zxd}$ has $x$ and $d$ as a source and sink, and in which the outer face of $H_{yzd}$ has $z$ and $y$ as a source and sink, respectively. Orientations $H_{xyd}$, $H_{zxd}$, and $H_{yzd}$ coincide on the common edges, hence altogether they yield an orientation $UP(-\rightarrow_{xy}; H_{xyz})$ of $H_{xyz}$.

We confirm that $H_{xyz}$ is upward planar using Theorem 2. Namely, $H_{xyz}$ has a single source and a single sink, namely $x$ and $y$ respectively, that are adjacent in the outer face of $H_{xyz}$. Also, $H_{xyz}$ is bimodal, given that every vertex different from $x$, $y$, $z$, and $d$ is bimodal in $H_{xyd}$, $H_{zxd}$, or $H_{yzd}$, given that $x$ and $y$ are source and a sink in $H_{xyz}$, respectively, given that all the edges incoming at $z$ appear consecutively in $H_{xyz}$ since they appear consecutively in $H_{zxd}$, and given that all the edges outgoing at $d$ appear consecutively in $H_{xyz}$ since they appear consecutively in $H_{xyd}$ and in $H_{yzd}$. Finally, suppose for a contradiction that $H_{xyz}$ has a directed cycle $C$. Assume w.l.o.g. that $C$ is minimal, i.e., no directed cycle $C'$ exists whose vertices are a subset of the vertices of $C$. Since $H_{xyd}$, $H_{zxd}$, and $H_{yzd}$ are acyclic and since the orientation of the subgraph of $H_{xyz}$ induced by $x$, $y$, $z$, and $d$ is acyclic in $H_{xyz}$, it follows that $C$ passes through an internal vertex of one of $H_{xyd}$, $H_{zxd}$, or $H_{yzd}$, say $H_{xyd}$. Then $C$ contains a path internal to $H_{xyd}$ and connecting two vertices out of $x$, $y$, and $d$, say $x$ and $d$. However, either such a path is directed from $d$ to $x$, thus contradicting the acyclicity of $H_{xyd}$, or it is directed from $x$ to $d$, hence it can be replaced by edge $(x, d)$, thus contradicting the minimality of $C$. \qed