Abstract: Birmele [J Graph Theory 2003] proved that every graph with circumference $t$ has treewidth at most $t - 1$. Under the additional assumption of 2-connectivity, such graphs have bounded pathwidth, which is a qualitatively stronger conclusion. Birmele’s theorem was extended by Birmele et al. [Combinatorica 2007] who showed that every graph without $k$ disjoint cycles of length at least $t$ has treewidth $O(tk^2)$. Our main result states that, under the additional assumption of $(k + 1)$-connectivity, such graphs have bounded pathwidth. In fact, they have pathwidth $O(t^3 + tk^2)$. Moreover, examples show that $(k + 1)$-connectivity is required for bounded pathwidth to hold. These results suggest the following general question: for which...
values of $k$ and graphs $H$ does every $k$-connected $H$-minor-free graph have bounded pathwidth? We discuss this question and provide a few observations. © 2014 Wiley Periodicals, Inc. J. Graph Theory 00: 1–11, 2014

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1. INTRODUCTION

Birmele [7] proved that every graph with circumference $t$ has treewidth at most $t - 1$, and this bound is tight for the complete graph $K_t$. Nešetřil and Ossona de Mendez [16, p. 118] showed that under the additional assumption of 2-connectivity, such graphs have treedepth at most $1 + (t - 2)^2$. Since pathwidth is at most treedepth minus 1, every 2-connected graph with circumference $t$ has pathwidth at most $(t - 2)^2$. Our first result strengthens this bound.

**Theorem 1.** Every 2-connected graph with circumference $t$ has pathwidth at most \( \left\lfloor \frac{t}{2} \right\rfloor (t - 1) \).

The 2-connectivity assumption is needed in Theorem 1 since complete binary trees have unbounded pathwidth. In particular, the complete binary tree of height $h$ has pathwidth $\left\lceil \frac{h}{2} \right\rceil$.

Birmele’s theorem was extended by Birmele et al. [6], who showed that graphs without $k$ disjoint cycles of length at least $t$ have treewidth $O(tk^2)$. Under the additional assumption of $(k + 1)$-connectivity, we prove that such graphs have bounded pathwidth.

**Theorem 2.** Every $(k + 1)$-connected graph without $k$ disjoint cycles of length at least $t$ has pathwidth at most $O(t^3 + tk^2)$.

We now show that the assumption of $(k + 1)$-connectivity is needed in Theorem 2. Suppose on the contrary that every $k$-connected graph without $k$ disjoint cycles of length at least $t$ has pathwidth at most $f(k, t)$ for some function $f$. Let $G$ be the graph obtained from the complete binary tree of height $h$ by adding $k - 1$ dominant vertices. Observe that $G$ is $k$-connected. Since every cycle in $G$ uses at least one of the dominant vertices, $G$ contains no $k$ disjoint cycles. Thus $G$ has pathwidth at most $f(k, t)$ for all $t \geq 3$. On the other hand, the pathwidth of $G$ equals $\left\lceil \frac{h}{2} \right\rceil + k - 1$. We obtain a contradiction by choosing $h > 2 \cdot f(k, t)$.

The proofs of Theorems 1 and 2 are given in Sections 3 and 4, respectively. We conclude in Section 5 by reinterpreting these results in terms of excluded minors. In general, we observe that highly connected $H$-minor-free graphs have bounded pathwidth. Determining the minimum connectivity required for this behavior to occur is an interesting line of future research.

2. DEFINITIONS

Let $G$ be an (undirected, simple, finite) graph. The circumference of $G$ is the length of the longest cycle in $G$, or is 0 if $G$ is acyclic. A tree decomposition $(T, \{B_x \subseteq V(G) : x \in V(T)\})$ of $G$ consists of a tree $T$ and a set $\{B_x \subseteq V(G) : x \in V(T)\}$ of sets of vertices of $G$ indexed by the nodes of $T$, such that:
for each vertex \( v \in V(G) \), the set \( \{ x \in V(T) : v \in B_x \} \) induces a nonempty (connected) subtree of \( T \), and

- for each edge \( uv \in E(G) \), there is some \( x \in V(T) \) such that \( u, v \in B_x \).

We refer to the sets \( B_x \) in the decomposition as bags. The width of a decomposition is the maximum size of a bag minus 1. The treewidth of a graph \( G \), denoted by \( tw(G) \), is the minimum width over all tree decompositions of \( G \). A path decomposition of \( G \) is a tree decomposition whose underlying tree is a path. The pathwidth of a graph \( G \), denoted by \( pw(G) \), is the minimum width over all path decompositions of \( G \). For simplicity, we describe a path decomposition by \( (B_1, B_2, \ldots, B_n) \), where \( B_i \) is the bag associated with the \( i \)-th vertex in the path. In such a decomposition, for each vertex \( v \) of \( G \), let \( L(v) \) be the bag \( B_i \) containing \( v \) with \( i \) minimum. If \( L(v) = L(w) = B_i \) for distinct \( v, w \in V(G) \), then replace \( B_i \) by the two bags \( B_i \setminus \{ v \} \) and \( B_i \). Now \( L(v) \neq L(w) \). Repeat this step until \( L(v) \neq L(w) \) for all distinct \( v, w \in V(G) \). Such a path decomposition is said to be normalized. Hence, every graph has a normalized path decomposition with width \( pw(G) \).

A graph \( H \) is a minor of a graph \( G \) if \( H \) is isomorphic to a graph formed from a subgraph of \( G \) by contracting edges. When \( H \) is a minor of \( G \), for each vertex \( v \in V(H) \) there is a connected subgraph \( C \) of \( G \) that contracts to form \( v \) in the minor. We call \( C \) the branch set of \( v \).

In a rooted forest \( F \), the height of a vertex \( v \) in \( F \) is the distance between \( v \) and the root of the component of \( F \) that contains \( v \). The height of \( F \) is the maximum height over all vertices of \( F \). The closure of \( F \), denoted \( clos(F) \), is the graph with vertex set \( V(F) \) and edge set \( \{ xy : x \text{ is an ancestor of } y, x \neq y \} \). The treedepth of a graph \( G \), denoted \( td(G) \), is the minimum height plus 1 of a forest \( F \) such that \( G \subseteq clos(F) \). Treedepth is equivalent to several other notions including minimal elimination tree height and is closely related to a number of graph invariants including pathwidth and treewidth; see [3], [16].

3. PROOF OF THEOREM 1

Lemma 3. Every 2-connected graph \( G \) with circumference \( t \) has treedepth at most \( \lfloor \frac{t}{2} \rfloor (t - 1) + 1 \).

Proof. Let \( T \) be a depth-first spanning subtree of \( G \) rooted at some vertex \( r \). Thus \( G \subseteq clos(F) \). Say an edge \( vw \) of \( T \) has span \( |i - j| \), where \( v \) and \( w \) are respectively at height \( i \) and \( j \) in \( T \). For each edge \( vw \) of span \( s \geq 2 \), the \( vw \)-path in \( T \) plus \( vw \) forms a cycle of length \( s + 1 \). Thus \( s \leq t - 1 \). Consider a vertex \( v \) in \( G \). By Menger’s Theorem, there are two internally disjoint \( vr \)-paths in \( G \). Their union is a cycle of length at most \( t \). Thus there is a \( vr \)-path \( P \) in \( G \) of length at most \( \lfloor \frac{t}{2} \rfloor \). Since each edge in \( P \) has span at most \( t - 1 \), the height of \( v \) is at most \( \lfloor \frac{t}{2} \rfloor (t - 1) \). Hence the height of \( T \) is at most \( \lfloor \frac{t}{2} \rfloor (t - 1) \). The result follows.

Theorem 1 follows directly from Lemma 3 since \( pw(G) \leq td(G) - 1 \) (see [16]).

4. PROOF OF THEOREM 2

A block in a graph \( G \) is a maximal 2-connected subgraph of \( G \), or the subgraph of \( G \) induced by a bridge edge or an isolated vertex. It is well known that the blocks of \( G \) form
FIGURE 1. Example of a block-cut tree: $P$ is the path $(b_1; v_1; b_2; v_2; b_3; v_3; b_4; v_4; b_5)$. The subgraphs $G_0$ and $G'$ respectively consist of the blocks above and below the dotted line.

a proper partition of $E(G)$. The block-cut-forest $T$ of a graph $G$ is defined as follows: $V(T)$ is the set of cut-vertices and blocks of $G$, where a cut-vertex $v$ is adjacent to a block $B$ whenever $v \in B$. It is well known that $T$ is a forest, and if $G$ is connected, then $T$ is a tree called the block-cut-tree.

**Lemma 4.** Let $T$ be the block-cut-forest of a graph $G$. Assume that $\text{pw}(T) \leq n$ and $\text{pw}(B) \leq m$ for each block $B$ of $G$. Then $\text{pw}(G) \leq (m + 3)(n + 1) - 3$.

**Proof.** We proceed by induction on $\text{pw}(T)$. For the base case, say $\text{pw}(T) = 0$. Then $T$ has no edges, and each component of $G$ is 2-connected. Clearly, the pathwidth of $G$ equals the maximum pathwidth of the components of $G$. Thus $\text{pw}(G) \leq m = (m + 3)(n + 1) - 3$.

Now assume that $\text{pw}(T) \geq 1$. Since the pathwidth of $G$ equals the maximum pathwidth of the components of $G$, we may assume that $G$ is connected. Thus $T$ is connected. Let $(X_1, X_2, \ldots, X_i)$ be a path decomposition of $T$ with width at most $n$. Choose vertices $x \in X_1$ and $y \in X_i$. Let $P$ be a maximal path in $T$ that contains an $xy$-path. Then $V(P) \cap X_i = \emptyset$ for all $i$. Let $X'_i = X_i - V(P)$; then $|X'_i| \leq |X_i| - 1$. Now $(X'_1, X'_2, \ldots, X'_i)$ is a path decomposition of $T - V(P)$ with width at most $n - 1$. By the maximality of $P$, the endpoints of $P$ are leaf vertices of $T$. No cut-vertex of $G$ is a leaf of $T$. Thus the endpoints of $P$ correspond to blocks. Say $P = b_1v_1b_2v_2 \ldots b_{s-1}v_{s-1}b_s$, where $b_i$ represents the block $B_i$ in $G$, and $v_i$ is a cut-vertex in $G$. For each $v_i$, let $C_{i,1}, C_{i,2}, \ldots, C_{i,t_i}$ be the blocks in $G$ corresponding to neighbors of $v_i$ in $T - V(P)$. Let $G_0 := \bigcup \{B_i : 1 \leq i \leq s\} \cup \bigcup \{C_{i,j} : 1 \leq i \leq s - 1, 1 \leq j \leq t_i\}$. Let $G'$ be the union of the blocks not in $G_0$. Then $G = G_0 \cup G'$, as illustrated in Figure 1.

Let $T'$ be the forest obtained from $T - V(P)$ by removing the leaf vertices that correspond to cut-vertices in $G$. This step removes all cut-vertices in $G$ that are not cut-vertices in $G'$, and the blocks that remain are blocks in $G'$. Thus $T'$ is the block-cut-forest of $G'$. Since $T'$ is a subgraph of $T - V(P)$, we have $\text{pw}(T') \leq \text{pw}(T - V(P)) \leq n - 1$. Furthermore, since each block $B$ of $G'$ is also a block of $G$, $\text{pw}(B) \leq m$. Let $G'_1, G'_2, \ldots, G'_r$ be the components of $G'$. By induction, $\text{pw}(G'_j) \leq (m + 3)n - 3$ for $1 \leq j \leq r$. Let $(H_{i,1}, H_{i,2}, \ldots, H_{i,k_i})$ be a path decomposition of $G'_j$ with $|H_{i,j}| \leq (m + 3)n - 2$.

We now construct a path decomposition of $G_0$. For $1 \leq i \leq s$, let $(X_{i,1}, X_{i,2}, \ldots, X_{i,k_i})$ be a path decomposition of $B_i$ with $|X_{i,j}| \leq m + 1$ for $1 \leq j \leq k_i$. Define $X_{i,j}^- := X_{i,j} \cup \{v_1\}$ and $X_{i,j}^+ := X_{i,j} \cup \{v_{i-1}\}$ and $X_{i,j}^{++} := X_{i,j} \cup \{v_{i-1}, v_i\}$ for $1 < i < s$. For each $C_{i,j}$, let

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(S_{i,j,1}, S_{i,j,2}, \ldots, S_{i,j,k_{i,j}}) be a path decomposition with |S_{i,j,\ell}| \leq m + 1. Define S_{i,j,\ell}^+ := S_{i,j,\ell} \cup \{v_i\}. Denote by T_{i,j} the sequence of bags (S_{i,j,1}^+, S_{i,j,2}^+, \ldots, S_{i,j,k_{i,j}}^+). It is easily proved that

\[(X_{1,1}^+, \ldots, X_{1,k_1}^+, T_{1,1}, \ldots, T_{1,\ell_1}, X_{2,1}^+, \ldots, X_{2,k_2}^+, T_{2,1}, \ldots, T_{2,\ell_2}, \ldots, T_{r-1,\ell_{r-1}}, X_{s,1}^+, \ldots, X_{s,k_s}^+)\]

is a path decomposition of G_0. The maximum bag size is at most m + 3. Let (Y_1, Y_2, \ldots, Y_p) be a normalized path decomposition of G_0 with |Y_i| \leq m + 3 for 1 \leq i \leq p. Then L(v) \neq L(w) for v \neq w.

We now construct a path decomposition of G. For each component G'_j of G', let w_j be the cut-vertex in G_0 \cap G'_j. Note that w_j is distinct for each G'_j. Replace the bag L(w_j) with the bags

\[(L(w_j) \cup H_{j,1}, L(w_j) \cup H_{j,2}, \ldots, L(w_j) \cup H_{j,k_j}).\]

The bag size is at most m + 3 + (m + 3)n - 2 = (m + 3)(n + 1) - 2. For simplicity, rename the decomposition (Z_1, \ldots, Z_q). It remains to show that (Z_1, \ldots, Z_q) is a path decomposition of G. For each edge xy in G, we have x, y \in Z_i for some i. Suppose v \in Z_i \cap Z_j for j > i + 1. Furthermore, assume v \in V(G' - G_0) and without loss of generality, v \in V(G'_i). Then by construction, H_{1,r} \subset Z_i, H_{1,r+1} \subset Z_{i+1}, \ldots, H_{1,r+j-1} \subset Z_j for some r, v \in H_{1,r} for r \leq t \leq r + j - i so v \in Z_s for i \leq s \leq j. Now instead assume v \in V(G_0). Then by construction, Y_r \subset Z_i and Y_t \subset Z_j for some r, s with s \geq r. v \in Y_r for r \leq t \leq s so v \in Z_t for i \leq \ell \leq j.

We conclude that (Z_1, \ldots, Z_q) is a valid path decomposition. Since |Z_i| \leq (m + 3)(n + 1) - 2, we have pw(G) \leq (m + 3)(n + 1) - 3.

Let T be a complete binary tree embedded in the plane as illustrated in Figure 2. Vertices at the same distance from the root are at the same level. Number the leaf vertices from left to right; let v_i be the leaf labeled i as shown.

**Lemma 5.** Let T be a complete binary tree with leaf vertices numbered as in Figure 2. Then the path in T between v_a and v_b has length at least \(2 \log_2 (b - a + 1)\) where \(a \leq b\).

**Proof.** Let V_0 be the set of all leaf vertices of T. Let V_i be the set of all vertices u of T such that the shortest path from u to a vertex in V_0 has length i. Since T is a complete binary tree, each \(u \in V_i\) has exactly \(2^i\) descendants in V_0; furthermore, the descendants are \(v_j, v_{j+1}, \ldots, v_{j+2^i-1}\) for some number \(j\). Consider the vertex \(v_u\) and

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CIRCUMFERENCE AND PATHWIDTH OF HIGHLY CONNECTED GRAPHS 5

FIGURE 2. Left-to-right labeling of the leaves of the complete binary tree.
suppose \( u \in V_j \) is an ancestor of \( v_i \). Then if \( v_j \in V_0 \) also has \( u \) as an ancestor, then \( j \in [a - (2^i - 1), a + (2^i - 1)] \).

For all \( b \geq a \), there exists \( k \) such that \( 2^k \leq (b - a + 1) < 2^{k+1} \). Then, for \( i < k \), \( b \notin [a - (2^i - 1), a + (2^i - 1)] \), so \( v_a \) and \( v_b \) do not have a common ancestor in \( V_i \). However, \( b \in [a - (2^i - 1), a + (2^i - 1)] \) for all \( j \geq k \) and there exists some \( j \geq k \) such that \( v_a \) and \( v_b \) have a common ancestor \( u \) in \( V_j \). Then by the definition of \( V_j \), the path \( P_1 \) from \( v_a \) to \( u \) has length \( j \) and the path \( P_2 \) from \( u \) to \( v_b \) has length \( j \). Thus \( P_1P_2 \) is a path of length \( 2j \) from \( v_a \) to \( v_b \). Since \( 2^k \leq b - a + 1 < 2^{k+1} \) and \( j \geq k \), we have \( 2j \geq 2 \log_2 (b - a + 1) \).

**Lemma 6.** Let \( T \) be a forest with \( \text{pw}(T) \geq t \geq 1 \). Then \( T \) contains a complete binary tree of height \( t - 1 \) as a minor. Moreover, for any vertex \( v \in V(T) \), there is such a minor in \( T \) with the property that \( v \) is in the branch set of the root of the binary tree.

**Proof.** Since the pathwidth of a graph equals the maximum pathwidth of its components, we may assume that \( T \) is a tree. For a vertex \( v \) of \( T \), define the *rooted pathwidth* of \( T \) at \( v \), denoted \( \text{rpw}(T, v) \), as the minimum width of a path decomposition of \( T \) such that \( v \) is in the last bag of the decomposition. We say such a decomposition is *rooted* at \( v \).

We prove, by induction on \( |V(T)| \), that if \( \text{rpw}(T, v) \geq t \) for some vertex \( v \) of a tree \( T \), then \( T \) contains a complete binary tree of height \( t - 1 \) as a minor with \( v \) in the branch set of the root. Since \( \text{rpw}(T, v) \geq \text{pw}(T) \), the result follows when \( \text{pw}(T) \geq t \).

In the base case with \( |V(T)| = 2 \), the rooted pathwidth at a given vertex is 1 and the tree trivially contains a complete binary tree of height 0 rooted at the given vertex.

Now suppose \( |V(T)| \geq 3 \) and let \( v \) be such that \( \text{rpw}(T, v) \geq t \). Let \( w_1, w_2, \ldots, w_d \) be the neighbors of \( v \) and let \( T_i \) be the component of \( T - v \) rooted at \( w_i \) for \( 1 \leq i \leq d \). Let \( r_i = \text{rpw}(T_i, w_i) \). Without loss of generality, \( r_1 \geq r_2 \geq \cdots \geq r_d \).

Let \((X_{1,1}, X_{1,2}, \ldots, X_{1,k_1}), \ldots, (X_{d,1}, \ldots, X_{d,k_d})\) be a path decomposition of \( T_i \) rooted at \( w_i \) with width \( r_i \). For \( 2 \leq i \leq d \), let \( X_{i,j} = X_{i,j} \cup \{v\} \). Then

\[
(X_{1,1}, X_{1,2}, \ldots, X_{1,k_1}, \{w_1, v\}, X_{2,1}^+, X_{2,2}^+, \ldots, X_{2,k_2}^+, X_{3,1}^+, X_{3,2}^+, \ldots, X_{3,k_3}^+, \ldots, X_{d,1}^+, X_{d,2}^+, \ldots, X_{d,k_d}^+)
\]

is a path decomposition of \( T \) rooted at \( v \) with width \( \max\{r_1, r_2 + 1\} \). Here we use the fact that \( w_1 \in X_{1,k_1} \). Thus \( \text{rpw}(T, v) \leq \max\{r_1, r_2 + 1\} \).

First suppose that \( r_1 \geq r_2 + 1 \). Then \( \text{rpw}(T_i, w_i) = r_1 \geq \text{rpw}(T, v) \geq t \). By induction, \( T_1 \) contains a complete binary tree of height \( t - 1 \) rooted at \( w_1 \) as a minor. Extend the branch set containing \( w_1 \) to include \( v \). We obtain a complete binary tree of height \( t - 1 \) rooted at \( v \) as a minor in \( T \).

Now suppose that \( r_2 + 1 > r_1 \). Then \( r_1 = r_2 \geq \text{rpw}(T, v) - 1 \geq t - 1 \). By induction, \( T_1 \) and \( T_2 \) each contain a complete binary tree of height \( t - 2 \) as a minor rooted at \( w_1 \) and \( w_2 \), respectively. Thus \( T \) contains a complete binary tree of height \( t - 1 \) rooted at \( v \) as a minor.

To prove Theorem 2, we need the following. Let \( \mathcal{F} \) be a family of graphs. For a graph \( G \), a *hitting set* \( H \) of \( \mathcal{F} \) is a set of vertices of \( G \) such that \( G - H \) contains no member of \( \mathcal{F} \). The family \( \mathcal{F} \) is said to satisfy the Erdős-Pósa property if there is a function \( f : \mathbb{N} \to \mathbb{N} \) such that for all graphs \( G \), either \( G \) contains \( k \) vertex-disjoint members of \( \mathcal{F} \) or \( G \) contains a hitting set \( H \) of size at most \( f(k) \). Birmelé et al. [6] proved that if \( \mathcal{F} \) is the family of cycles of length at least \( t \), then \( \mathcal{F} \) satisfies the Erdős-Pósa property with \( f(k) = 13(t(k - 1)(k - 2) + (2t + 3)(k - 1)) \).

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Proof of Theorem 2. Since $G$ contains no $k$ vertex-disjoint cycles of length at least $t$, by the above-mentioned result of Birmlele et al. [6], there is a hitting set $H \subseteq V(G)$ such that $|H| \leq h := 13(t(k - 1)(k - 2) + (2t + 3)(k - 1))$. Let $T$ be the block-cut-forest of $G - H$. Define

$$i := \lceil \log_2(k - 1)(2h - 2k + 1) \rceil + 1 \quad \text{and} \quad j := \left\lceil \frac{t}{2} + \log_2(h - k + 1) \right\rceil.$$

Since $G$ is $(k + 1)$-connected, $|H| \geq k$. Hence $h \geq |H| \geq k$, and $i$ and $j$ are well defined.

First suppose that $\text{pw}(T) \leq i + j$. Since $H$ is a hitting set, $G - H$ has circumference at most $t - 1$. Thus the 2-connected blocks of $G - H$ have pathwidth at most $\lceil \frac{t-1}{2} \rceil (t-2)$ by Theorem 1. The blocks that are not 2-connected consist of bridges or isolated vertices, which have pathwidth at most 1. By Lemma 4 with $m = \lfloor \frac{t+1}{2} \rfloor (t-2)$ and $n = i + j$, we have $\text{pw}(G - H) \leq (\lfloor \frac{t+1}{2} \rfloor (t-2) + 3)(i + j + 1) - 3$. Add $H$ to each bag of an optimal path decomposition of $G - H$ to obtain a path decomposition of $G$ with width at most $(\lfloor \frac{t+1}{2} \rfloor (t-2) + 3)(i + j + 1) - 3 + h \in O(t^3 + tk^2)$.

It remains to handle the case when $\text{pw}(T) > i + j$. We claim, however, that this case does not occur. Suppose it does and assume $\text{pw}(T) > i + j$. By Lemma 6, $T$ contains a complete binary tree $T'$ of height $i + j$ as a minor. It is well known and easily proved that if a graph $A$ contains a graph $B$ with maximum degree 3 as a minor, then $A$ contains a subdivision of $B$ as a subgraph. Thus, $T$ contains a subdivision $S$ of $T'$ as a subgraph. By taking $S$ maximal, each leaf of $S$ is a leaf of $T$.

For each $v \in H$, let $d(v)$ be the number of leaves $u$ of $S$ such that $v$ is adjacent in $G$ to some vertex in the block corresponding to $u$ (in which case we say that $v$ is adjacent to $u$). Since $G$ is $(k + 1)$-connected, each leaf of $S$ has at least $k$ neighbors in $H$. Since $S$ contains $2^{i+j}$ leaves, $\sum_{v \in H} d(v) \geq k 2^{i+j}$. Let $H = \{v_1, v_2, \ldots, v_h\}$ and $d_m := d(v_m)$. Without loss of generality, $d_1 \geq d_2 \geq \cdots \geq d_h$. Since $d_m \leq 2^{i+j}$ for $1 \leq m \leq h$,

$$d_k + d_{k+1} + \cdots + d_h \geq k 2^{i+j} - (d_1 + d_2 + \cdots + d_{k-1}) \geq k 2^{i+j} - (k - 1)2^{i+j} = 2^{i+j}.$$ 

Hence $d_1 \geq d_2 \geq \cdots \geq d_k \geq 2^{i+j}/(h - k + 1)$. Let $X := \{v_1, v_2, \ldots, v_k\}$.

Since $T'$ has height $i + j$, there are $2^i$ pairwise disjoint subtrees $T_1, T_2, \ldots, T_{2^i}$ in $S$, each a subdivision of a complete binary tree of height $j$, such that for $1 \leq m \leq 2^i$, the leaves of $T_m$ are leaves of $S$ and the root of $T_m$ is at height $i$ in $T'$, as illustrated in Figure 3.
For each $v \in X$, we say the pair $(v, T_m)$ is good if $v$ is adjacent to at least $2^{j-1}/(h - k + 1)$ leaves of $T_m$. We claim that each $v \in X$ is in at least $k$ good pairs. Suppose for the sake of contradiction that some $v \in X$ is in at most $k - 1$ good pairs. Then

$$\frac{2^{j+j}}{h - k + 1} \leq d(v) \leq (k - 1)2^j + (2^j - k + 1) \frac{2^j}{2(h - k + 1)}.$$  

Thus $2^j \leq (k - 1)(2h - 2k + 1)$, which contradicts the definition of $i$. Thus each $v \in X$ is in at least $k$ good pairs. Since $|X| = k$, there is a distinct $T_m$ for each $v \in X$ such that $(v, T_m)$ is a good pair.

For each such pair $(v, T_m)$, label the leaf vertices of $T_m$ as in Figure 2. Since $v$ is adjacent to at least $2^{j-1}/(h - k + 1)$ leaf vertices, there are two leaves $x$ and $y$ labeled $a$ and $b$ respectively such that $b - a + 1 \geq \frac{2^{j-1}}{h-k+1}$. Then by Lemma 5, there is a path $P$ of length at least $2 \log_2(\frac{2^{j-1}}{h-k+1}) = 2j - 2 - 2 \log_2(h - k + 1)$ in $T_m$ between $x$ and $y$. Thus $vPv$ is a cycle of length $2j - 2 - 2 \log_2(h - k + 1) \geq t$ in $T_m \cup \{v\}$. Since the $T_m$ are pairwise disjoint, we have $k$ pairwise disjoint cycles $C_1, C_2, \ldots, C_k$ of length at least $t$ in $T \cup H$.

We now construct pairwise disjoint cycles $C'_1, C'_2, \ldots, C'_k$ in $G$. Say $C_1 = v_1B_1v_2B_2\ldots B_nv_1$, where $v_1 \in H$, $v_i$ is a cut-vertex in $G - H$ for $2 \leq i \leq r - 1$, and $B_i$ is a block in $G - H$. The vertex $v_1$ is adjacent to a vertex $x$ in $B_1$. Let $P_i$ be a path from $x$ to $v_2$ in $B_1$. Next, for $2 \leq i \leq r - 1$, let $P_i$ be a path from $v_i$ to $v_{i+1}$ in $B_i$, such that if there is a vertex $v$ in $B_i \cap V(C_j)$ for some $j \neq 1$, then choose $P_i$ such that $v \notin V(P_i)$, as illustrated in Figure 4. Since each vertex in $S$ has degree at most 3, there is at most one such vertex $v$ to be avoided. Therefore, since $B_i$ is 2-connected, such a $P_i$ exists. For $B_r$, let $P_r$ be a path from $v_r$ to $y$ in $B_r$, where $y$ is a neighbor of $v_1$. Let $C'_1 = v_1xP_1v_2P_2v_3\ldots P_yv_1$. From each $C_i$, construct $C'_i$ in $G$ in this same manner. The cycles $C'_1, C'_2, \ldots, C'_k$ by construction are pairwise disjoint with length at least $t$ in $G$, which is a contradiction.

\section{5. Relationship to Forbidden Minors}

Another way to describe a graph $G$ with circumference $t - 1$ is to say $G$ is $C_t$-minor-free where $C_t$ is a cycle on $t$ vertices. Our two main theorems can thus be restated in terms of minors:

\textbf{Theorem 7.} Let $G$ be a 2-connected $C_t$-minor-free graph. Then $\text{pw}(G) \leq \lfloor \frac{t-1}{2} \rfloor (t - 2)$.

Let $C_{t,k}$ be the graph consisting of $k$ disjoint cycles of length $t$. 

\textit{Journal of Graph Theory DOI 10.1002/jgt}
Let $G$ be a $(k+1)$-connected $C_{t,k}$-minor-free graph. Then $\text{pw}(G) \leq O(t^3 + tk^2)$.

These results suggest the following definition. For a graph $H$, let $g(H)$ be the minimum integer for which there exists a number $c = c(H)$ such that every $g(H)$-connected $H$-minor-free graph has pathwidth at most $c$. Mader [12] exhibited a function $\ell$ such that every $\ell(H)$-connected graph contains $H$ as a minor. (Kostochka [13,14] and Thomason [18] independently proved that if $t = |V(H)|$ then $\ell(H) \leq \ell(K_t) \in \Theta(t^{\sqrt{\log t}})$.) Thus every $H$-minor-free $\ell(H)$-connected graph has bounded pathwidth (since there is no such graph). Hence $g(H)$ is well defined, and $g(H) \leq \ell(H)$. We conclude with some observations about $g(H)$.

For some graphs, $g(H) = \ell(H)$. For example, $g(K_5) = \ell(K_5) = 6$ (since every 6-connected graph contains $K_5$ as a minor, but 5-connected planar (and thus $K_5$-minor-free) graphs have unbounded pathwidth).

On the other hand, $g(H)$ and $\ell(H)$ can be far apart. For example, we showed that $g(C_t) = 2$ but $\ell(C_t) \geq t - 1$ since $K_{t-1}$ is $(t-2)$-connected and contains no $C_t$-minor.

Observe that if $H_1$ is a minor of $H_2$, then $g(H_1) \leq g(H_2)$. Thus, for each integer $c$, the class $\mathcal{H}_c := \{H : g(H) \leq c\}$ is minor closed. By Robertson and Seymour’s graph minor theorem, for each $c$, there is a finite set of minimal excluded minors for $\mathcal{H}_c$.

Bienstock et al. [1] proved that for every forest $F$, every graph with pathwidth at least $|F| - 1$ contains $F$ as a minor. Thus $g(F) = 0$. Moreover, since complete binary trees have unbounded pathwidth, $g(F) = 0$ if and only if $F$ is a forest. And $K_3$ is the only minimal excluded minor for $\mathcal{H}_0$.

There is no graph $H$ with $g(H) = 1$ since the pathwidth of a graph equals the maximum pathwidth of its connected components.

We showed that $g(C_t) = 2$ for all $t \geq 3$. It is an interesting open problem to characterize the graphs $H$ with $g(H) = 2$. (An answer is conjectured below.)

The following example is important. Consider $G_0 := K_3$ embedded in the plane. For $i \geq 0$, construct $G_{i+1}$ from $G_i$ as follows: for each edge $vw$ on the outerface of $G_i$, add one new vertex adjacent to $v$ and $w$. Thus $G_i$ is 2-connected and outerplanar. Hence $G_i$ is $K_4$-minor-free and $K_{2,3}$-minor-free. Observe that the dual of $G_i$ contains a complete binary tree of height $i$ as a minor, which has pathwidth $i$. By a result of Bodlaender and Fomin [2], the class $\{G_i : i \geq 0\}$ has unbounded pathwidth. Hence $g(K_4) \geq 3$ and $g(K_{2,3}) \geq 3$.

Dirac [9] proved that every 3-connected graph has a $K_4$-minor. Thus $g(K_4) = \ell(K_4) = 3$.

An unfinished result of Ding [8] implies that, for some function $f$, every 3-connected $K_{2,t}$-minor-free graph has pathwidth at most $f(t)$, implying $g(K_{2,t}) \leq 3$. Thus $g(K_{2,t}) \geq g(K_{2,3})$ and $g(K_{2,t}) = 3$ for $t \geq 3$ (assuming Ding’s result).

We proved that $g(C_{t,k}) = k + 1$ for all $t \geq 3$, where the lower bound follows from the example given after the statement of Theorem 2. This leads to the following lower bound on $g(H)$: If $H$ contains $k$ disjoint cycles, then $C_{3,k}$ is a minor of $H$, and $g(H) \geq k + 1$. This observation can be strengthened as follows. A transversal in a graph $H$ is a set $X$ of vertices such that $H - X$ is acyclic. Let $\tau(H)$ be the minimum size of a transversal in $H$. Note that if $H$ is a minor of $G$, then $\tau(H) \leq \tau(G)$.

**Proposition 9.** $g(H) \geq \tau(H) + 1$ for every graph $H$ with $\tau(H) \geq 1$. 

*Journal of Graph Theory* DOI 10.1002/jgt
Proof. Suppose on the contrary that \( g(H) \leq \tau(H) \) for some graph \( H \). Let \( G \) be the graph obtained from the complete binary tree of height \( h \) by adding \( \tau(H) - 1 \) dominant vertices. Then \( G \) is \( \tau(H) \)-connected, and \( \tau(G) = \tau(H) - 1 \), implying \( G \) is \( H \)-minor-free. By the definition of \( g(H) \), for some \( c = c(H) \), the pathwidth of \( G \) is at most \( c \). This is a contradiction for \( h > 2c \), since \( G \) has pathwidth \( \lceil \frac{h}{2} \rceil + \tau(H) - 1 \). Therefore \( g(H) \geq \tau(H) + 1 \).

We have described three minor-minimal graphs \( H \) with \( g(H) = 3 \). Namely, \( K_4, K_{2,3}, \) and \( K_3 \cup K_3 \). (It is easily seen that these graphs are minor-minimal.) There is one more key example. Let \( Q \) be the octahedron graph \( K_{2,2,2} \) minus the edges of a triangle. Observe that \( \tau(Q) = 2 \), and thus \( g(Q) \geq 3 \) by Proposition 9. Moreover, \( Q \) contains no \( K_4, K_{2,3}, \) or \( K_3 \cup K_3 \) minor.

Conjecture 10. The minimal excluded minors for \( \mathcal{H}_2 \) are \( \{ K_4, K_{2,3}, K_3 \cup K_3, Q \} \).

It is well known that \( H \) is outerplanar if and only if \( H \) contains no \( K_4 \) or \( K_{2,3} \) minor, and it follows from a result of Lovász [15] that \( \tau(H) \leq 1 \) if and only if \( H \) contains no \( K_4, K_3 \cup K_3 \) or \( Q \) minor. Thus Conjecture 10 is equivalent to saying that \( g(H) \leq 2 \) if and only if \( H \) is outerplanar and \( \tau(H) \leq 1 \).

In the above examples \( H \) is planar. Planarity is significant for these types of questions since the class of \( H \)-minor-free graphs has bounded treewidth if and only if \( H \) is planar [17]. However, \( g(H) \) is well defined for all graphs, and is interesting for certain nonplanar graphs. For example, Böhme et al. [5] proved that there is a function \( n \) such that every \( 7 \)-connected graph with at least \( n(k) \) vertices contains \( K_{3,k} \) as a minor. That is, every \( 7 \)-connected \( K_{3,k} \)-minor-free graph has less than \( n(k) \) vertices, implying \( g(K_{3,k}) \leq 7 \). More generally, Böhme et al. [4] conjectured that for all \( a, k \) there is an integer \( n(a,k) \) such that every \((2a + 1)\)-connected graph on at least \( n(a,k) \) vertices contains \( K_{a,k} \) as a minor. This would imply that \( g(K_{a,k}) \leq 2a + 1 \).

In general, it would be interesting if some function of \( \tau(H) \) was an upper bound on \( g(H) \). Or is there a family of graphs \( H \) with bounded transversals, but with \( g(H) \) unbounded?

NOTES ADDED IN PROOF

Fiorini and Herinckx [11] recently improved the above-mentioned result of Birmele et al. [6] by showing that cycles of length at least \( t \) satisfy the Erdős-Pósa property with \( f(k) = \mathcal{O}(tk \log k) \) (which is optimal for fixed \( k \) or fixed \( t \)). It follows that the \( \mathcal{O}(t^3 + tk^2) \) bound in Theorem 2 can be improved to \( \mathcal{O}(t^3 + tk \log k) \).

In an early version of this paper, the graph \( Q \) was omitted from Conjecture 10. Proposition 9 and the importance of \( Q \) were jointly observed with János Barát and Gwenael Joret.

Gwenael Joret also pointed out the following alternative proof of a slightly weaker version of Theorem 1. Let \( G \) be a \( 2 \)-connected graph with circumference \( t \). Let \( p \) be the number of edges in the longest path in \( G \). Dirac [10] proved that \( t > \sqrt{2p} \). Thus \( p < \lceil \frac{t^2}{2} \rceil \).

That is, \( G \) contains no path on \( \lceil \frac{t^2}{2} \rceil \) edges. Hence \( G \) contains no path on \( \lceil \frac{t^2}{2} \rceil \) edges as a minor. Bienstock et al. [1] proved that every graph that excludes a fixed forest on \( k \) edges as a minor has pathwidth at most \( k - 1 \). Thus \( G \) has pathwidth at most \( \lceil \frac{t^2}{2} \rceil - 1 \).

Thanks János and Gwen.

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