Three-Dimensional Grid Drawings with Sub-Quadratic Volume

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Abstract

A three-dimensional (straight-line grid) drawing of a graph represents the vertices by distinct gridpoints in 3-space, and represents the edges by non-crossing straight line-segments. An \(O(n^{3/2})\) volume bound is proved for three-dimensional drawings of graphs with bounded degree, graphs with bounded genus, and graphs with no bounded complete graph as a minor. The previous best bound for these graph families was \(O(n^2)\). These results (partially) solve open problems due to Felsner, Wismath, and Liotta [Graph Drawing 2001] and Pach, Thiele, and Toth [Graph Drawing 1997].

1 Introduction

A three-dimensional straight-line grid drawing of a graph, henceforth called a 3D drawing, represents the vertices by distinct points in \(\mathbb{Z}^3\) (called gridpoints), and represents each edge as a straight line-segment between its endpoints, such that distinct edges only intersect at common endpoints, and each edge only intersects a vertex that is an endpoint of that edge. In contrast to the case in the plane, it is well known that every graph has a 3D drawing. We therefore are interested in optimising certain measures of the aesthetic quality of such drawings.

The bounding box of a 3D drawing is the minimum axis-aligned box containing the drawing. If the bounding box has side lengths \(X - 1\), \(Y - 1\) and \(Z - 1\), then we speak of an \(X \times Y \times Z\) drawing with volume \(X \cdot Y \cdot Z\). That is, the volume of a 3D drawing is the number of gridpoints in the bounding box. This definition is formulated so that 2D drawings have positive volume. We are interested in 3D drawings with small volume. The volume of 3D drawings has been widely studied [3, 6, 9, 11, 14, 16, 19, 34, 35, 37]. Three-dimensional graph drawings with the vertices having real coordinates have also been studied [5, 7, 8, 10, 17, 21, 24–27, 30, 33]. Aesthetic criteria besides volume which have been considered include symmetry [24–27], aspect ratio [8,
The authors [15] have also established bounds on the volume of three-dimensional polyline grid drawings (where bends in the edges are also at gridpoints). Table 1 summarises the best known upper bounds on the volume of 3D drawings, including those established in this paper.

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Cohen et al. [9] proved that every graph has a 3D drawing with $O(n^3)$ volume, and that this bound is asymptotically optimal for complete graphs $K_n$. Our edge-sensitive bounds of $O(m^{4/3} n)$ and $O(\Delta mn)$ are greater than $O(n^3)$ in the worst case. It is an open problem whether there are edge-sensitive bounds that match the $O(n^3)$ bound in the case of complete graphs (see Section 3).

Pach et al. [34] proved that every $O(1)$-colourable graph has a 3D drawing with $O(n^2)$ volume, and that this bound is asymptotically optimal for $K_n$. This lower bound has been generalised to all graphs by Bose et al. [3], who proved that 3D drawings have volume at least $\frac{1}{8}(n + m)$. Graphs with constant maximum degree are $O(1)$-colourable, and thus, by the result of Pach et al. [34], have 3D drawings with $O(n^2)$ volume. Pach et al. [34] conjectured that graphs with constant maximum degree have 3D drawings with $o(n^2)$ volume. We verify this conjecture. In particular, it is proved that graphs with constant maximum degree have 3D drawings with $O(n^{3/2})$ volume.

The first $O(n)$ upper bound on the volume of 3D drawings was established by Felsner et al. [19] in the case of outerplanar graphs. This result was generalised by the authors for graphs with constant tree-width [16]. Felsner et al. [19] proposed the following inviting open problem: does every planar graph have a 3D drawing with $O(n)$ volume? In this paper we provide a partial solution to this problem, by proving that planar graphs have 3D drawings with $O(n^{3/2})$ volume. Note that $O(n^2)$ is the optimal area for plane 2D grid drawings, and $O(n^2)$ was the previous best upper bound on the volume of 3D drawings of planar graphs.

A graph $H$ is a minor of a graph $G$ if $H$ is isomorphic to a graph obtained from a subgraph of $G$ by contracting edges. The genus of a graph $G$ is the minimum $\gamma$ such that $G$ can be embedded in the orientable surface with $\gamma$ handles. Of course, planar graphs have genus 0 and no $K_5$-minor. A generalisation of our result for planar graphs is that every graph with constant genus or with no $K_h$-minor for constant $h$ has a 3D drawing with $O(n^{3/2})$ volume.
2 Track Layouts

We consider undirected, finite, and simple graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of $G$ are respectively denoted by $n = |V(G)|$ and $m = |E(G)|$. A vertex $c$-colouring of $G$ is a partition $\{V_i : 1 \leq i \leq c\}$ of $V(G)$, such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. Each $i \in \{1, 2, \ldots, c\}$ is a colour, and each set $V_i$ is a colour class. At times it will be convenient to write $\text{col}(v) = i$ rather than $v \in V_i$. If $G$ has a vertex $c$-colouring then $G$ is $c$-colourable. The chromatic number of $G$, denoted by $\chi(G)$, is the minimum $c$ such that $G$ is $c$-colourable. The span of an edge $vw$ in a vertex colouring is $|\text{col}(v) - \text{col}(w)|$. That there is a fixed ordering of the colours in the vertex colouring is implicit in this definition of span.

Let $\{V_i : 1 \leq i \leq c\}$ be a vertex $c$-colouring of a graph $G$. If $<_i$ is a total order of a colour class $V_i$, then the pair $(V_i, <_i)$ is called a track. If $(V_i, <_i)$ is a track for each colour $i$, then $\{(V_i, <_i) : 1 \leq i \leq t\}$ is a $t$-track assignment of $G$. To ease the notation, at times we say that $\{V_i : 1 \leq i \leq c\}$ is a track assignment, where the ordering of each colour class $V_i$ is implicit. An $X$-crossing in a track assignment consists of two edges $vw$ and $xy$ such that $v <_i x$ and $y <_j w$, for distinct tracks $V_i$ and $V_j$. A $t$-track layout of $G$ consists of a $t$-track assignment of $G$ with no $X$-crossing. The track-number of $G$ is the minimum $t$ such that $G$ has a $t$-track layout, and is denoted by $\tn(G)$. Track layouts were introduced in [14, 16] although they are implicit in many previous works.

We have the following upper bounds on the track-number. Note that the authors also proved that track-number is bounded by (a doubly exponential function of) tree-width [16].

**Lemma 1.** Let $G$ be a graph with $n$ vertices, maximum degree $\Delta$, path-width $p$, tree-width $w$, genus $\gamma$, and with no $K_{1,\gamma}$-minor. Then the track-number of $G$ satisfies: (a) $\tn(G) \leq p+1$, (b) $\tn(G) \leq 72w\Delta$, (c) $\tn(G) \in O(\gamma^{1/2}n^{1/2})$, (d) $\tn(G) \in O(h^{3/2}n^{1/2})$.

**Proof.** Part (a) is by Dujmović et al. [14]. Part (b) is by the authors [16]. Gilbert et al. [22] and Djidjev [13] independently proved that $G$ has a $O(\gamma^{1/2}n^{1/2})$-separator, and thus has an $O(\gamma^{1/2}n^{1/2})$ path-width (see [2, Theorem 20(iii)]). Hence (c) follows from (a). Similarly (d) follows from the result by Alon et al. [1] that $G$ has a $O(h^{3/2}n^{1/2})$-separator.

The next result is the main contribution of this section.

**Theorem 1.** Every graph $G$ with $m$ edges and maximum degree $\Delta$ has track-number $\tn(G) \leq 14\sqrt{\Delta m}$.

To prove Theorem 1 we introduce the following concept. A vertex colouring is a strong star colouring if between every pair of colour classes, all edges (if any) are incident on a single vertex. That is, each bichromatic subgraph consists of a star and possibly some isolated vertices. The strong star chromatic number of a graph $G$, denoted by $\chi_{ss}(G)$, is the minimum number of colours in a strong star colouring of $G$. Note that star colourings, in which each bichromatic subgraph is a forest of stars, have also been studied (see [20, 32] for example). The star chromatic number of a graph $G$, denoted by $\chi_{st}(G)$, is the minimum number of colours in a star-colouring of $G$.

With an arbitrary order of each colour class in a strong star colouring, there is no $X$-crossing. Thus track-number $\tn(G) \leq \chi_{st}(G)$ for every graph $G$, and Theorem 1 is an immediate corollary of the next lemma.
Lemma 2. Every graph $G$ with $m$ edges and maximum degree $\Delta \geq 1$ has strong star chromatic number $\chi_{st}(G) \leq 14\sqrt{\Delta m}$.

To prove Lemma 2 we use the weighted version of the Lovász Local Lemma [18].

Lemma 3. [29, p. 221] Let $E = \{A_1, \ldots, A_n\}$ be a set of ‘bad’ events. Let $0 < p = \frac{1}{q} \leq \frac{1}{2}$ be a real number, and let $t_1, \ldots, t_n \geq 1$ be integers. Suppose that for all $A_i \in E$,

(a) the probability $P(A_i) \leq p^{t_i}$,

(b) $A_i$ is mutually independent of $E \setminus (\{A_i\} \cup D_i)$ for some $D_i \subseteq E$, and

(c) $\sum_{A_i \in D_i} (2p)^{t_i} \leq \frac{t_i}{2}$.

Then with positive probability, no event in $E$ occurs.

Proof of Lemma 2. Let $c \geq 4$ be a positive integer to be specified later. Let $p = \frac{1}{\sqrt{c}}$. Then $0 < p \leq \frac{1}{2}$. For each vertex $v \in V(G)$, randomly and independently choose $\text{col}(v)$ from $\{1, 2, \ldots, c\}$.

For each edge $vw \in E(G)$, let $A_{vw}$ be the type-I event that $\text{col}(v) = \text{col}(w)$. Let $E'$ be the set of arcs $E' = \{(v, w), (w, v) : vw \in E(G)\}$. For each pair of arcs $(v, w), (x, y) \in E'$ with no endpoint in common, let $B_{(v, w), (x, y)}$ be the type-II event that $\text{col}(v) = \text{col}(x)$ and $\text{col}(w) = \text{col}(y)$.

We will apply Lemma 3 to obtain a colour assignment such that no type-I event and no type-II event occurs. No type-I event implies that we have a (proper) vertex colouring. No type-II event implies that no two disjoint edges share the same pair of colours; that is, we have a strong star colouring.

For each type-I event $A$, $P(A) = \frac{1}{c}$. Let $t_A = 1$. Then $P(A) = p^{t_A}$. For each type-II event $B$, $P(B) = \frac{1}{\sqrt{c}}$. Let $t_B = 2$. Then $P(B) = p^t_B$. Thus condition (a) of Lemma 3 is satisfied.

An event involving a set of vertices $S$ is dependent only on other events involving at least one of the vertices in $S$. Each vertex is involved in at most $\Delta$ type-I events, and at most $2\Delta |E'| = 4\Delta m$ type-II events. A type-I event involves two vertices, and is thus mutually independent of all but at most $2\Delta$ type-I events and at most $8\Delta m$ type-II events. A type-II event involves four vertices, and is thus mutually independent of all but at most $4\Delta$ type-I events and at most $16\Delta m$ type-II events.

For condition (c) of Lemma 3 to hold we need $2\Delta \left(\frac{1}{\sqrt{c}}\right)^1 + 8\Delta m \left(\frac{1}{\sqrt{c}}\right)^2 \leq \frac{1}{2}$ for the type-I events, and $4\Delta \left(\frac{1}{\sqrt{c}}\right)^1 + 16\Delta m \left(\frac{1}{\sqrt{c}}\right)^2 \leq 1$ for the type-II events. It is a happy coincidence that these two equations are equivalent, and it is easily verified that $c = \lceil 4(\Delta + \sqrt{\Delta(1 + 4m)}) \rceil \geq 4$ is a solution.

Thus by Lemma 3, with positive probability no type-I event and no type-II event occurs. Thus for every vertex $v \in V(G)$, there exists $\text{col}(v) \in \{1, \ldots, c\}$ such that no type-I event and no type-II event occurs. As proved above such a colouring is a strong star colouring. Since $\Delta \leq \sqrt{\Delta m}$, the number of colours $c \leq \lceil 4(1 + \sqrt{5})\sqrt{\Delta m} \rceil \leq 14\sqrt{\Delta m}$. \hfill $\Box$

We have the following corollary of Lemma 2.

Lemma 4. Every graph $G$ with $m$ edges has strong star chromatic number $\chi_{st}(G) \leq 15m^{2/3}$.

Proof. Let $X$ be the set of vertices of $G$ with degree greater than $\frac{1}{3}m^{1/3}$. Let $H$ be the subgraph of $G$ induced by $V(G) \setminus X$. Thus $H$ has maximum degree at most $\frac{1}{3}m^{1/3}$. By Lemma 2, $H$ has a strong star colouring at most $14(\frac{1}{3}m^{1/3})^{2/3} = 7m^{2/3}$. Now $|X| \leq 2m/(\frac{1}{3}m^{1/3})^3 = 8m^{2/3}$. Adding each vertex in $X$ to its own colour class we obtain a strong star colouring of $G$ with at most $15m^{2/3}$ colours. \hfill $\Box$
Since \( \text{tn}(G) \leq \chi_{\text{st}}(G) \) for every graph \( G \), the next result follows immediately from Lemma 4.

**Theorem 2.** Every graph \( G \) with \( m \) edges has track-number \( \text{tn}(G) \leq 15m^{2/3} \).

\[ \square \]

## 3 Edge-Sensitive Volume Bounds

Vertex colourings and track layouts have previously been used to produce three-dimensional drawings with small volume. Pach et al. [34] proved that a \( c \)-colourable graph has a 3D drawing with \( O(c^2n^2) \) volume, and Dujmović et al. [14] proved that a graph with a \( t \)-track layout has a 3D drawing with \( O(t^2n) \) volume. In fact, track layouts and 3D drawings are inherently related.

**Theorem 3.** [14] Every graph \( G \) with track-number \( \text{tn}(G) \leq t \) has a \( O(t) \times O(t) \times O(n) \) drawing with \( O(t^2n) \) volume. Moreover, for every graph family \( G \), every \( n \)-vertex graph \( G \in G \) has a \( F(n) \times F(n) \times O(n) \) drawing if and only if the track-number \( \text{tn}(G) \in F(n) \), where \( F(n) \) is any family of functions closed under multiplication, such as \( O(1) \) or \( O(\text{polylog} \, n) \).

By Theorems 1 and 3 we have:

**Theorem 4.** Every graph with \( n \) vertices, \( m \) edges and maximum degree \( \Delta \) has a \( O((\Delta m)^{1/2}) \times O((\Delta m)^{1/2}) \times O(n) \) drawing with \( O(\Delta mn) \) volume.

By Theorems 2 and 3 we have:

**Theorem 5.** Every graph with \( n \) vertices and \( m \) edges has a \( O(m^{2/3}) \times O(m^{2/3}) \times O(n) \) drawing with \( O(m^{4/3}n) \) volume.

Consider the following open problems: Does every graph have a 3D drawing with \( O(nm) \) volume? Does every \( O(1) \)-colourable graph have a 3D drawing with \( O(n\sqrt{m}) \) volume? These bounds match the lower bounds for \( K_n \) and \( K_{n,n} \), and would make edge-sensitive improvements to the existing upper bounds of \( O(n^3) \) and \( O(n^2) \), respectively. These edge-sensitive bounds are implied by Theorem 3 and Theorem 6 below if every graph has \( O(\sqrt{m}) \) track-number. In turn, this bound on track-number is implied if strong star chromatic number is \( O(\sqrt{m}) \). As far as the authors are aware, a \( O(\sqrt{m}) \) bound is not even known for star chromatic number. The best known bound in this direction is \( \chi_{\text{st}}(G) \leq 11n^{3/5} \), which can be proved in a similar fashion to Lemma 4, in conjunction with the result of Fertin et al. [20] that \( \chi_{\text{st}}(G) \leq \lceil 20\Delta^{1/2} \rceil \).

## 4 Sub-Quadratic Volume Bounds

In the following sequence of results we combine vertex colourings and track layouts to reduce the quadratic dependence on \( t \) in Theorem 3 to linear. This comes at the expense of a higher dependence on the number of colours \( c \). However, in the intended applications \( c \) will be constant, or at least will be independent of the size of the graph. The proof of the next lemma is a further generalisation of the ‘moment curve’ method for producing three-dimensional graph drawings [9, 14, 34].

**Lemma 5.** Let \( G \) be a graph with a vertex \( c \)-colouring \( \{V_i : 0 \leq i \leq c - 1\} \), and a track layout \( \{T_{i,j} : 0 \leq i \leq c - 1, 1 \leq j \leq t_i\} \), such that each \( T_{i,j} \subseteq V_i \). Then \( G \) has a \( O(c) \times O(t^2) \times O(c^5 \text{tn}') \) drawing, where \( t = \max_i t_i \) and \( n' = \max_{i,j} |T_{i,j}| \).
Proof. Let $p$ be the minimum prime such that $p \geq c$. Then $p < 2c$ by Bertrand’s postulate. Let $v(i, j, k)$ denote the $k^{th}$ vertex in track $T_{i, j}$. Define

$$Y(i, j) = p(2it + j) + (i^2 \mod p), \quad \text{and}$$

$$Z(i, j, k) = p(20c n' \cdot Y(i, j) + k) + (i^3 \mod p).$$

Position each vertex $v(i, j, k)$ at the gridpoint $(i, Y(i, j), Z(i, j, k))$, and draw each edge as a line-segment between its endpoints. Since $Y(i, j) \in O(c^2 t)$ and $Z(i, j, k) \in O(c^3 n' \cdot Y(i, j))$, the drawing is $O(c) \times O(c^2 t) \times O(c^3 t)\cdot n'$.

Observe that the tracks from a single colour class are within a distinct $YZ$-plane, each track occupies a distinct vertical line, and the $Z$-coordinates of the vertices within a track preserve the given ordering of that track. In addition, the $Y$-coordinates satisfy the following property.

Claim 1. For all distinct colours $i_1$ and $i_2$ and for all $1 \leq j_1, j_2 \leq t$, we have $2c | (Y(i_1, j_1) - Y(i_2, j_2))$ is greater than the $Y$-coordinate of any vertex.

Proof. Without loss of generality $i_1 > i_2$. Observe that every $Y$-coordinate is less than $p(2(c - 1)t + t) + p = p(2ct - t + 1) \leq 2cpt$. Now $2c | Y(i_1, j_1) - Y(i_2, j_2) | > 2c | p(2i_1t + 1) - p(2i_2t + t + 1) | \geq 2cp | 2i_2 + 1)t - (2i_2 + t) | = 2cpt. \quad \square$

We first prove that the only vertices each edge intersects are its own endpoints. It suffices to prove that if three tracks are collinear in the $XY$-plane then they are all from the same colour class. Loosely speaking, an edge does not pass through any track. Clearly two tracks from the same colour class are not collinear (in the $XY$-plane) with a third track from a distinct colour class. Thus we need only consider tracks $\{T(i_\alpha, j_\alpha) : 1 \leq i_\alpha \leq 3\}$ from three distinct colour classes $\{i_1, i_2, i_3\}$. Let $R$ be the determinant,

$$R = \begin{vmatrix} 1 & i_1 & Y(i_1, j_1) \\ 1 & i_2 & Y(i_2, j_2) \\ 1 & i_3 & Y(i_3, j_3) \end{vmatrix}.$$ 

If $\{T(i_\alpha, j_\alpha) : 1 \leq i_\alpha \leq 3\}$ are collinear in the $XY$-plane then $R = 0$. However $Y(i, j) \equiv i^2 \mod p$, and thus

$$R \equiv \begin{vmatrix} 1 & i_1 & i_1^2 \\ 1 & i_2 & i_2^2 \\ 1 & i_3 & i_3^2 \end{vmatrix} = \prod_{1 \leq \alpha < \beta \leq 3} (i_\alpha - i_\beta) \neq 0 \mod p.$$ 

Since $i_\alpha \neq i_\beta$, and $p$ is a prime greater than any $i_\alpha - i_\beta$, we have $R \neq 0$. Thus $\{T(i_\alpha, j_\alpha) : 1 \leq i_\alpha \leq 3\}$ are not collinear in the $XY$-plane. Hence the only vertices that an edge intersects are its own endpoints.

It remains to prove that there are no edge crossings. Consider two edges $e$ and $e'$ with distinct endpoints $v(i_\alpha, j_\alpha, k_\alpha), 1 \leq \alpha \leq 4$. (Clearly edges with a common endpoint do not cross.) Let $Y_\alpha$ denote $Y(i_\alpha, j_\alpha)$. Consider the following determinant

$$D = \begin{vmatrix} 1 & i_1 & Y_1 & Z(i_1, j_1, k_1) \\ 1 & i_2 & Y_2 & Z(i_2, j_2, k_2) \\ 1 & i_3 & Y_3 & Z(i_3, j_3, k_3) \\ 1 & i_4 & Y_4 & Z(i_4, j_4, k_4) \end{vmatrix}.$$
If $e$ and $e'$ cross then their endpoints are coplanar, and $D = 0$. Thus it suffices to prove that $D \neq 0$. We proceed by considering the number $N = |\{i_1, i_2, i_3, i_4\}|$ of distinct colours assigned to the four endpoints of $e$ and $e'$. Clearly $N \in \{2, 3, 4\}$.

**Case** $N = 4$: Since $Y_a \equiv i_a^2 \pmod{p}$ and $Z(i_a, j_a, k_a) \equiv i_a^3 \pmod{p}$,

$$D \equiv \begin{vmatrix} 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \\ 1 & i_4 & i_4^2 & i_4^3 \end{vmatrix} = \prod_{1 \leq \alpha < \beta \leq 4} (i_\alpha - i_\beta) \neq 0 \pmod{p}.$$ 

This follows since $i_\alpha \neq i_\beta$, and $p$ is a prime greater than any $i_\alpha - i_\beta$. Thus $D \neq 0$.

**Case** $N = 3$: Without loss of generality $i_1 = i_2$. It follows that $D = 5S_0 + S_1 + S_2 + S_3 + S_4$ where

$$S_0 = 4cpm(i_3 - i_1)(i_4 - i_1)(Y_2 - Y_1)(Y_3 - Y_4)$$

$$S_1 = p(Y_2 - Y_1)(k_3(i_4 - i_1) - k_4(i_3 - i_1))$$

$$S_2 = p(i_4 - i_3)(k_2Y_1 - k_1Y_2)$$

$$S_3 = p(k_2 - k_1)(Y_4(i_3 - i_1) - Y_3(i_4 - i_1))$$

$$S_4 = (Y_2 - Y_1)((i_3 - i_1)(i_4^3 \pmod{p}) - (i_3 - i_1)(i_3^2 \pmod{p}) + (i_4 - i_1)(i_3^2 \pmod{p})).$$

If $Y_1 = Y_2$ then $e$ and $e'$ do not cross, since no three tracks from distinct colour classes are collinear in the $XY$-plane. Assume $Y_1 \neq Y_2$. If $i_3 < i_1 < i_4$ or $i_4 < i_1 < i_3$ then $e$ and $e'$ do not cross, simply by considering the projection in the $XY$-plane. Thus $i_1 < i_3, i_4$ or $i_1 > i_3, i_4$, which implies

$$(i_4 - i_1)(i_3 - i_1) > |i_4 - i_3|.$$ 

(1)

**Claim 2.** If $|S_0| \geq |S_1|$, $|S_0| \geq |S_2|$, $|S_0| \geq |S_3|$ and $|S_0| \geq |S_4|$ then $D \neq 0$.

**Proof.** To prove that $D = 5S_0 + S_1 + S_2 + S_3 + S_4$ is nonzero it suffices to show that $D' = 5|S_0| + |S_1| + |S_2| + |S_3| + |S_4|$ is nonzero for all combinations of pluses and minuses. Consider $X = \pm|S_0| \pm |S_1| \pm |S_2| \pm |S_3| \pm |S_4|$ for some combination of pluses and minuses. Since $|S_1| \leq |S_0|$, $|S_2| \leq |S_0|$, $|S_3| \leq |S_0|$, and $|S_4| \leq |S_0|$, we have $-4|S_0| \leq X \leq 4|S_0|$. Since $S_0 \neq 0$, we have $5|S_0| + X \neq 0$ and $-5|S_0| + X \neq 0$. That is, all values of $D'$ are nonzero. Therefore $D \neq 0$. \qed

Therefore, to prove that $D \neq 0$ it suffices to show that $|S_0| \geq |S_1|$, $|S_0| \geq |S_2|$, $|S_0| \geq |S_3|$ and $|S_0| \geq |S_4|$. We will use the following elementary facts regarding absolute values:

\[ \forall a_1, \ldots, a_k \in \mathbb{R} \quad |a_1 a_2 \ldots a_k| = |a_1| |a_2| \cdots |a_k| \]

\[ |a_1 + a_2 + \cdots + a_k| \leq |a_1| + |a_2| + \cdots + |a_k| \leq k \cdot \max\{ |a_1|, |a_2|, \ldots, |a_k|\}. \]

- First we prove that $|S_0| \geq |S_1|$. That is,

\[ |4cpm(i_3 - i_1)(i_4 - i_1)(Y_2 - Y_1)(Y_3 - Y_4)| \geq |p(Y_2 - Y_1)(k_3(i_4 - i_1) - k_4(i_3 - i_1))| \]

Hence,

\[ |S_0| > |S_1| \iff 2n'|i_3 - i_1||i_4 - i_1||Y_3 - Y_4| \geq |k_3(i_4 - i_1) - k_4(i_3 - i_1)| \]

\[ \iff 2n'|i_3 - i_1||i_4 - i_1||Y_3 - Y_4| \geq 2 \cdot \max\{ |k_4(i_3 - i_1)|, |k_3(i_4 - i_1)| \}. \]

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Since $n' \geq k_3, k_4$ and $|Y_3 - Y_4| \geq 1$,

$$|S_0| > |S_1| \iff |i_3 - i_1| |i_4 - i_1| \geq \max \{|i_3 - i_1|, |i_4 - i_1|\}.$$  

Thus $|S_0| \geq |S_1|$ since $|i_3 - i_1| \geq 1$ and $|i_4 - i_1| \geq 1$.

- Now we prove that $|S_0| \geq |S_2|$. That is,

$$|4c n'(i_3 - i_1)(i_4 - i_1)(Y_2 - Y_1)(Y_3 - Y_4)| \geq |p(i_4 - i_3)(k_2 Y_1 - k_1 Y_2)|.$$  

By inequality (1) and since $|Y_2 - Y_1| \geq 1$,

$$|S_0| \geq |S_2| \iff |4c n'(Y_3 - Y_4)| \geq |k_2 Y_1 - k_1 Y_2|,$$

$$\iff |2cn'(Y_3 - Y_4)| \geq \max \{|k_2 Y_1|, |k_1 Y_2|\},$$

which holds since $n' \geq k_1, k_2$ and $|2c(Y_3 - Y_4)| \geq \max \{Y_1, Y_2\}$ by Claim 1.

- Now we prove that $|S_0| \geq |S_3|$. That is,

$$|4c n'(i_3 - i_1)(i_4 - i_1)(Y_2 - Y_1)(Y_3 - Y_4)| \geq |p(k_2 - k_1)(Y_4(i_3 - i_1) - Y_3(i_4 - i_1))|.$$  

Since $n' \geq |k_2 - k_1|$ and since $|Y_2 - Y_1| \geq 1$,

$$|S_0| \geq |S_3| \iff |4c(i_3 - i_1)(i_4 - i_1)(Y_3 - Y_4)| \geq |Y_4(i_3 - i_1) - Y_3(i_4 - i_1)|,$$

$$\iff |2c(i_4 - i_1)(i_4 - i_1)(Y_3 - Y_4)| \geq \max \{|Y_4(i_3 - i_1)|, |Y_3(i_4 - i_1)|\},$$

which holds since $|2c(Y_3 - Y_4)| \geq \max \{Y_1, Y_2\}$ by Claim 1.

- Finally we prove that $|S_0| \geq |S_4|$. That is,

$$|4c n'(i_3 - i_1)(i_4 - i_1)(Y_2 - Y_1)(Y_3 - Y_4)| \geq$$

$$|(Y_2 - Y_1)((i_3 - i_4)(i_4^3 \mod p) - (i_3 - i_1)(i_1^3 \mod p) + (i_4 - i_1)(i_4^3 \mod p))|.$$  

Since $cn'|Y_3 - Y_4| \geq 1$,

$$|S_0| > |S_4| \iff |3p(i_3 - i_1)(i_4 - i_1)| \geq$$

$$|i_3 - i_1)(i_1^3 \mod p) - (i_3 - i_1)(i_1^3 \mod p) + (i_4 - i_1)(i_4^3 \mod p)|,$$

$$\iff |3p(i_3 - i_1)(i_4 - i_1)| \geq$$

$$3 \cdot \max \{|(i_3 - i_4)(i_4^3 \mod p)|, |(i_3 - i_1)(i_1^3 \mod p)|, |(i_4 - i_1)(i_4^3 \mod p)|\},$$

$$\iff |(i_3 - i_1)(i_4 - i_1)| \geq \max \{|i_3 - i_4|, |i_3 - i_1|, |i_4 - i_1|\},$$

which holds by inequality (1).

**Case** $N = 2$: Without loss of generality $i_1 = i_2 \neq i_3 = i_4$. If $Y_1 = Y_2$ and $Y_3 = Y_4$ then $e$ and $e'$ do not cross as otherwise there would be an X-crossing in the track layout. If $Y_1 = Y_2$ and $Y_3 \neq Y_4$ (or $Y_1 \neq Y_2$ and $Y_3 = Y_4$) then $e$ and $e'$ do not cross, by considering the projection in the $XY$-plane. Thus we can assume that $Y_1 \neq Y_2$ and $Y_3 \neq Y_4$. It follows that

$$D = p(i_1 - i_3) \left(5 \cdot 4c n'(Y_2 - Y_1)(Y_4 - Y_3)(i_3 - i_1) + (k_1 - k_2)(Y_4 - Y_3) + (k_4 - k_3)(Y_2 - Y_1)\right).$$  

As in Claim 2, to show that $D \neq 0$ it suffices to show that

$$|4c n'(Y_2 - Y_1)(Y_4 - Y_3)(i_3 - i_1)| \geq |(k_1 - k_2)(Y_4 - Y_3)|,$$  

(2)
and

\[ |4cn'(Y_2 - Y_1)(Y_4 - Y_3)(i_3 - i_1)| \geq |(k_4 - k_3)(Y_2 - Y_1)|. \]  

(3)

Inequalities (2) and (3) hold since \( n' > |k_1 - k_2| \) and \( n' > |k_4 - k_3| \).

Note that the constant 20 in the definition of \( Z(i, j, k) \) in the proof of Lemma 5 is chosen to enable a relatively simple proof. It is easily seen that it can be reduced. The proof of the next lemma is based on an idea of Pach et al. [34] for balancing the size of the colour classes in a vertex colouring.

**Lemma 6.** Let \( G \) be an \( n \)-vertex graph with a \( c \)-colouring \( \{V_i : 0 \leq i \leq c - 1\} \), and a track layout \( \{T_{i,j} : 0 \leq i \leq c - 1, 1 \leq j \leq t_i\} \), such that each \( T_{i,j} \subseteq V_{i} \). Let \( k = \sum_i t_i \) be the total number of tracks. Then \( G \) has a \( O(c) \times O(ck) \times O(c^4 n) \) drawing.

**Proof.** Replace each track by tracks of size exactly \( \lceil \frac{n}{k} \rceil \), except for at most one track of size at most \( \lceil \frac{n}{c} \rceil \). Order the vertices within each track according to the original track, and consider the new tracks to belong to the same colour class as the original. Clearly no X-crossing is created. Within \( V_i \) there are now at most \( t_i + |V_i|/\lceil \frac{n}{c} \rceil \) tracks. The total number of tracks is \( \sum_i (t_i + |V_i|/\lceil \frac{n}{c} \rceil) \leq 2k \).

For each colour class \( V_i \), partition the set of tracks in \( V_i \) into sets of size exactly \( \lceil \frac{2n}{k} \rceil \), except for one set of size at most \( \lceil \frac{2n}{ck} \rceil \). Consider each set to correspond to a colour. The number of colours is now at most \( c + 2k/\lceil \frac{2n}{ck} \rceil \leq 2c \). Applying Lemma 5 with \( 2c \) colours, \( n' = \lceil \frac{n}{k} \rceil \), and \( t = \lceil \frac{2n}{ck} \rceil \), we obtain the desired drawing.

**Theorem 6.** Let \( G \) be a \( c \)-colourable graph with \( n \) vertices and track-number \( tn(G) \leq t \). Then \( G \) has a \( O(c) \times O(c^2 t) \times O(c^4 n) \) drawing.

**Proof.** Let \( \{V_i : 0 \leq i \leq c - 1\} \) be a \( c \)-colouring of \( G \). Let \( \{T_{i,j} : 1 \leq i \leq j \leq t\} \) be a \( t \)-track layout of \( G \). For all \( 0 \leq i \leq c - 1 \) and \( 1 \leq j \leq t \), let \( T_{i,j} = V_i \cap T_j \). Then \( \{V_i : 0 \leq i \leq c - 1\} \) and \( \{T_{i,j} : 0 \leq i \leq c - 1, 1 \leq j \leq t\} \) satisfy Lemma 6 with \( k = ct \). Thus \( G \) has the desired drawing.

In the case of bipartite graphs we have a simple proof of Theorem 6 with improved constants.

**Lemma 7.** Every \( n \)-vertex bipartite graph \( G \) with track-number \( tn(G) \leq t \) has a \( 2 \times t \times n \) drawing.

**Proof.** Let \( \{A, B\} \) be the bipartition of \( V(G) \). Let \( \{T_i : 1 \leq i \leq t\} \) be a \( t \)-track layout of \( G \). For each \( 1 \leq i \leq t \), let \( A_i = T_i \cap A \) and \( B_i = T_i \cap B \). Order each \( A_i \) and \( B_i \) as in \( T_i \). Place the \( j \)-th vertex in \( A_i \) at \( (0, i, j + \sum_{k=1}^{i-1} |A_k|) \). Place the \( j \)-th vertex in \( B_i \) at \( (1, t - i + 1, j + \sum_{k=1}^{i-1} |B_k|) \). The drawing is thus \( 2 \times t \times n \). Let \( A_i B_j \) be the set of edges with one endpoint in \( A_i \) and the other in \( B_j \). There is no crossing between edges in \( A_i B_j \) and \( A_i B_j \) as otherwise there would be an X-crossing in the track layout. Clearly there is no crossing between edges in \( A_i B_j \) and \( A_i B_k \) for \( j \neq k \). Suppose there is a crossing between edges in \( A_i B_j \) and \( A_k B_{\ell} \) with \( i \neq k \) and \( j \neq \ell \), and without loss of generality \( i < k \). Then the projections of the edges in the \( XY \)-plane also cross, and thus \( j < \ell \). This implies that the projections of the edges in the \( XZ \)-plane do not cross, and thus the edges do not cross.

The next result is one of the main contributions of this paper.

**Theorem 7.** Every planar graph with \( n \) vertices has a \( O(1) \times O(n^{1/2}) \times O(n) \) drawing with \( O(n^{3/2}) \) volume.
Proof. Planar graphs have $O(n^{1/2})$ path-width (see [2]), and thus have $O(n^{1/2})$ track-number by Lemma 1(a). The result follows from Theorem 6 since planar graphs are 4-colourable.

The following generalisation of Theorem 7 for graphs $G$ with genus $\gamma$ follows from Lemma 1(c), Theorem 6, and the classical result of Heawood [23] that $\chi(G) \in O(\gamma^{1/2})$.

**Theorem 8.** Every $n$-vertex graph with genus $\gamma$ has a $O(\gamma^{1/2}) \times O(\gamma^{3/2}n^{1/2}) \times O(\gamma^2 n)$ drawing with $O(\gamma^4 n^{3/2})$ volume.

The next generalisation of Theorem 7 for graphs with no $K_h$-minor follows from Lemma 1(d), Theorem 6, and the result independently due to Kostochka [28] and Thomason [36] that $\chi(G) \in O(h \log^{1/2} h)$ (see [12]).

**Theorem 9.** Every $n$-vertex graph with no $K_h$-minor has a $O(h \log^{1/2} h) \times O(h^{7/2} \log h \cdot n^{1/2}) \times O(h^4 \log^2 h \cdot n)$ drawing with volume $O(h^{17/2} \log^2 h \cdot n^{3/2})$.

Finally we consider the maximum degree as a parameter. By the sequential greedy algorithm, $G$ is $(\Delta + 1)$-colourable. Thus by Theorems 1 and 6 we have:

**Theorem 10.** Every graph with $n$ vertices, $m$ edges, and maximum degree $\Delta$ has a $O(\Delta) \times O(\Delta^{5/2} m^{1/2}) \times O(\Delta^4 n)$ drawing with $O(\Delta^{15/2} m^{1/2} n^{1/2})$ volume.

By Theorems 8, 9 and 10 and since graphs with constant maximum degree have $O(n)$ edges we have:

**Corollary 1.** Every $n$-vertex graph with constant genus, or with no $K_h$-minor for some constant $h$, or with constant maximum degree has a $O(1) \times O(n^{1/2}) \times O(n)$ drawing with $O(n^{3/2})$ volume.

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References


