On Linear Layouts of Graphs *

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Abstract

In a total order of the vertices of a graph, two edges with no endpoint in common can be crossing, nested, or disjoint. A $k$-stack (respectively, $k$-queue, $k$-arch) layout of a graph consists of a total order of the vertices, and a partition of the edges into $k$ sets of pairwise non-crossing (non-nested, non-disjoint) edges. Motivated by numerous applications, stack layouts (also called book embeddings) and queue layouts are widely studied in the literature, while this is the first paper to investigate arch layouts.

Our main result is a characterisation of $k$-arch graphs as the almost $(k+1)$-colourable graphs. That is, the graphs $G$ with a set $S$ of at most $k$ vertices, such that $G \setminus S$ is $(k+1)$-colourable. In addition, we survey the following fundamental questions regarding each type of layout, and in the case of queue layouts, provide simple proofs of a number of existing results. How does one partition the edges given a fixed ordering of the vertices? What is the maximum number of edges in each type of layout? What is the maximum chromatic number of a graph admitting each type of layout? What is the computational complexity of recognising the graphs that admit each type of layout?

A comprehensive bibliography of all known references on these topics is included.

Keywords: graph layout, graph drawing, stack layout, queue layout, arch layout, book embedding, queue-number, stack-number, page-number, book-thickness.

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1 Introduction

We consider undirected, finite, simple graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of $G$ are respectively denoted by $n = |V(G)|$ and $m = |E(G)|$. The subgraph of $G$ induced by a set of vertices $S \subseteq V(G)$ is denoted by $G[S]$, $G \setminus S$ denotes $G[V(G) \setminus S]$, and $G \setminus v$ denotes $G \setminus \{v\}$ for each vertex $v \in V(G)$.

A vertex ordering of an $n$-vertex graph $G$ is a bijection $\sigma : V(G) \to \{1, 2, \ldots, n\}$. We write $v <_\sigma w$ to mean that $\sigma(v) < \sigma(w)$. Thus $<_\sigma$ is a total order on $V(G)$. We say $G$ (or $V(G)$) is ordered by $<_\sigma$. At times, it will be convenient to express $\sigma$ by the list $(v_1, v_2, \ldots, v_n)$, where $\sigma(v_i) = i$. These notions extend to subsets of vertices in the natural way. Suppose that $V_1, V_2, \ldots, V_k$ are disjoint sets of vertices, such that each $V_i$ is ordered by $<_i$. Then $(V_1, V_2, \ldots, V_k)$ denotes the vertex ordering $\sigma$ such that $v <_\sigma w$ whenever $v \in V_i$ and $w \in V_j$ with $i < j$, or $v \in V_i$, $w \in V_i$, and $v <_i w$. We write $V_1 <_\sigma V_2 <_\sigma \cdots <_\sigma V_k$.

In a vertex ordering $\sigma$ of a graph $G$, let $L(e)$ and $R(e)$ denote the endpoints of each edge $e \in E(G)$ such that $L(e) <_\sigma R(e)$. Consider two edges $e, f \in E(G)$ with no common endpoint. There are the following three possibilities for the relative positions of the endpoints of $e$ and $f$ in $\sigma$.

Without loss of generality $L(e) <_\sigma L(f)$.

- $e$ and $f$ cross: $L(e) <_\sigma L(f) <_\sigma R(e) <_\sigma R(f)$.
- $e$ and $f$ nest and $f$ is nested inside $e$: $L(e) <_\sigma L(f) <_\sigma R(f) <_\sigma R(e)$
- $e$ and $f$ are disjoint: $L(e) <_\sigma R(e) <_\sigma L(f) <_\sigma R(f)$

A stack (respectively, queue, arch) in $\sigma$ is a set of edges $F \subseteq E(G)$ such that no two edges in $F$ are crossing (nested, disjoint) in $\sigma$. Observe that when traversing $\sigma$, edges in a stack (queue) appear in LIFO (FIFO) order — hence the names.

A linear layout of a graph $G$ is a pair $(\sigma, \{E_1, E_2, \ldots, E_k\})$ where $\sigma$ is a vertex ordering of $G$, and $\{E_1, E_2, \ldots, E_k\}$ is a partition of $E(G)$. A $k$-stack (queue, arch) layout of $G$ is a linear layout $(\sigma, \{E_1, E_2, \ldots, E_k\})$ such that each $E_i$ is a stack (queue, arch) in $\sigma$. At times, we write stack(e) = $\ell$ (or queue(e) = $\ell$, arch(e) = $\ell$) if $e \in E_i$. Layouts of $K_6$ of each type are illustrated in Figure 1.

Figure 1: Layouts of $K_6$: (a) 3-stack, (b) 3-queue, (c) 3-arch.

A graph admitting a $k$-stack (queue, arch) layout is called a $k$-stack (queue, arch) graph. The stack-number (queue-number, arch-number) of a graph $G$, denoted by $sn(G)$ ($qn(G)$, $an(G)$), is the minimum $k$ such that $G$ is a $k$-stack ($k$-queue, $k$-arch) graph.

Stack and queue layouts were respectively introduced by Ollmann [82] and Heath et al. [53, 57]. As far as we are aware, arch layouts have not previously been studied, although Dan Archdeacon\(^1\) suggests doing so.

Stack layouts are more commonly called book embeddings, and stack-number has been called book-thickness, fixed outer-thickness, and page-number. Applications of stack layouts include sorting permutations [34, 47, 83, 86, 99], fault tolerant VLSI design [17, 89–91], complexity theory [36, 37, 63], compact graph

\(^1\)http://www.emba.uvm.edu/~archdeac/problems/stackq.htm
encodings [61, 79], compact routing tables [43], and graph drawing [6, 24, 105, 106]. Numerous other aspects of stack layouts have been studied in the literature [7, 8, 10, 11, 14–16, 18, 20, 22, 32, 33, 35, 38, 40, 49–53, 57, 58, 60, 62, 65, 66, 70–74, 76–78, 81, 85, 88, 92–97, 100, 103, 108]. Stack layouts of directed graphs [23, 48, 55, 56] and posets [2, 3, 54, 69, 80, 98] have also been investigated.

Applications of queue layouts include sorting permutations [34, 59, 83, 86, 99], parallel process scheduling [5], matrix computations [85], and graph drawing [25, 27, 107]. Other aspects of queue layouts have been studied in the literature [29, 30, 39, 88, 92]. Queue layouts of directed graphs [55, 56] and posets [54] have also been investigated.

Table 1 summarises some of the known bounds on the stack-number and queue-number of various classes of graphs. A blank entry indicates that a more general result provides the best known bound.

<table>
<thead>
<tr>
<th>graph family</th>
<th>stack-number</th>
<th>reference</th>
<th>queue-number</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$ vertices</td>
<td>$\lceil \frac{n}{2} \rceil$</td>
<td>[4, 17, 46]</td>
<td>$\lceil \frac{n}{2} \rceil$</td>
<td>[57]</td>
</tr>
<tr>
<td>$m$ edges</td>
<td>$O(\sqrt{m})$</td>
<td>[73]</td>
<td>$e\sqrt{m}$</td>
<td>Theorem 4</td>
</tr>
<tr>
<td>proper minor-closed</td>
<td>bounded</td>
<td>[9, 11]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>genus $\gamma$</td>
<td>$O(\sqrt{\gamma})$</td>
<td>[72]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>tree-width $w$</td>
<td>$w$</td>
<td>[71]</td>
<td>$3^w \cdot 6^{(4^w-3w-1)/9} - 1$</td>
<td>[27, 31]</td>
</tr>
<tr>
<td>tree-width $w$, max. degree $\Delta$</td>
<td>$3^w \cdot 6^{(4^w-3w-1)/9} - 1$</td>
<td>[107]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>path-width $p$</td>
<td>$p$</td>
<td>[107]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>band-width $b$</td>
<td>$b - 1$</td>
<td>[97]</td>
<td>$\lceil \frac{b}{2} \rceil$</td>
<td>[57]</td>
</tr>
<tr>
<td>track-number $t$</td>
<td>$t - 1$</td>
<td>[27, 30, 107]</td>
<td></td>
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</tr>
<tr>
<td>toroidal</td>
<td>7</td>
<td>[32]</td>
<td></td>
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</tr>
<tr>
<td>planar</td>
<td>4</td>
<td>[108]</td>
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</tr>
<tr>
<td>bipartite planar</td>
<td>2</td>
<td>[21, 84]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2-trees</td>
<td>2</td>
<td>[88]</td>
<td>3</td>
<td>[28, 88]</td>
</tr>
<tr>
<td>Halin</td>
<td>2</td>
<td>[39]</td>
<td>3</td>
<td>[39]</td>
</tr>
<tr>
<td>X-trees</td>
<td>2</td>
<td>[17]</td>
<td>2</td>
<td>[57]</td>
</tr>
<tr>
<td>outerplanar</td>
<td>1</td>
<td>[4]</td>
<td>2</td>
<td>[53]</td>
</tr>
<tr>
<td>arched levelled planar</td>
<td>2</td>
<td>[53]</td>
<td>1</td>
<td>[53]</td>
</tr>
<tr>
<td>trees</td>
<td>1</td>
<td>[17]</td>
<td>1</td>
<td>[57]</td>
</tr>
</tbody>
</table>

Consider a vertex ordering $\sigma = (v_1, v_2, \ldots, v_n)$ of a graph $G$. For each edge $v_iv_j \in E(G)$, let the width of $v_iv_j$ in $\sigma$ be $|i - j|$, and let the midpoint of $v_iv_j$ be $\frac{1}{2}(i + j)$. The band-width of $\sigma$ is the maximum width of an edge of $G$ in $\sigma$. The band-width of $G$, denoted by $bw(G)$, is the minimum band-width of a vertex ordering of $G$. Consider the two fundamental observations:

**Observation 1 ([57]).** Edges whose widths differ by at most one are not nested.

**Observation 2.** Edges with the same midpoint are nested.

Observation 1 was made by Heath and Rosenberg [57]. Remarkably, Observation 2 seems to have gone unnoticed in the literature on queue layouts.

Our main result is a characterisation of $k$-arch graphs, given in Section 3. We also survey various fundamental questions regarding each type of layout, and in the case of queue layouts, provide new and simple proofs (based on Observation 2) of a number of existing results. In Section 2 we consider how to partition the edges given a fixed vertex ordering. In Section 4 we analyse the computational complexity of recognising the graphs that admit each type of layout. In Section 5 we consider the extremal questions: what is the
maximum number of edges in each type of layout, and what is the maximum chromatic number of a graph admitting each type of layout? Section 6 considers how to produce a queue layout of a graph \( G \) given queue layouts of the biconnected components of \( G \). In Section 7 we given a simple proof of the known result that queue-number is in \( O(\sqrt{m}) \).

2 Fixed Vertex Orderings

Consider the problem of assigning the edges of a graph \( G \) to the minimum number of stacks given a fixed vertex ordering \( \sigma \) of \( G \). This problem is equivalent to colouring a circle graph with the minimum number of colours. (A circle graph is the intersection graph of a set of chords of a circle.) As illustrated in Figure 2(a), a twist in \( \sigma \) is a matching \( \{v_iw_i \in E(G) : 1 \leq i \leq k\} \) such that

\[
v_1 <_\sigma v_2 <_\sigma \cdots <_\sigma v_k <_\sigma w_1 <_\sigma w_2 <_\sigma \cdots <_\sigma w_k .
\]

A vertex ordering with a \( k \)-edge twist needs at least \( k \) stacks, since each edge of a twist must be in a distinct stack. Unfortunately the converse is not true. There exists vertex orderings with no \((k+1)\)-edge twist that require \( \Omega(k \log k) \) stacks [67]. Moreover, it is \( \mathcal{NP} \)-complete to test if a fixed vertex ordering of a graph admits a \( k \)-stack layout [41]². On the other hand, Kostochka [68] proved that a vertex ordering with no 3-edge twist admits a 5-stack layout, and Ageev [1] proved that 5-stacks are sometimes necessary in this case. In general, Kostochka and Kratochvíl [67] proved that a vertex ordering with no \((k+1)\)-edge twist admits a \( 2^{k+6} \)-stack layout³, thus improving on previous bounds by Gyárfás [44, 45]. Hence the stack-number of a graph \( G \) is bounded by the minimum, taken over all vertex orderings \( \sigma \) of \( G \), of the maximum number of edges in a twist in \( \sigma \).

![Figure 2: (a) 5-edge twist, (b) 5-edge rainbow, (c) 5-edge necklace.](image)

Now consider the analogous problem for queue layouts: assign the edges of a graph \( G \) to the minimum number of queues given a fixed vertex ordering \( \sigma \) of \( G \). As illustrated in Figure 2(b), a rainbow in \( \sigma \) is a matching \( \{v_iw_i \in E(G) : 1 \leq i \leq k\} \) such that

\[
v_1 <_\sigma v_2 <_\sigma \cdots <_\sigma v_k <_\sigma w_k <_\sigma w_{k-1} <_\sigma \cdots <_\sigma w_1 .
\]

The rainbow \( \{v_iw_i : 2 \leq i \leq k\} \) is said to be inside \( v_1w_1 \). We now give a new and simple proof of a result by Heath and Rosenberg [57].

**Lemma 1.** [57] A vertex ordering of a graph \( G \) admits a \( k \)-queue layout of \( G \) if and only if it has no \((k+1)\)-edge rainbow.

**Proof.** A \( k \)-queue layout has no \((k+1)\)-edge rainbow since each edge of a rainbow must be in a distinct queue. Conversely, suppose we have a vertex ordering with no \((k+1)\)-edge rainbow. For every edge \( vw \in E(G) \), let

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²Unger [101, 102] claimed that it is \( \mathcal{NP} \)-complete to determine whether a given vertex ordering of a graph \( G \) admits a 4-stack layout, and that there is a \( O(n \log n) \) time algorithm in the case of 3-stack layouts. Crucial details are missing from these papers.

³Unger [101] claimed without proof that a vertex ordering with no \((k+1)\)-edge twist admits a \( 2k \)-stack layout. This claim is refuted by Ageev [1] in the case of \( k = 2 \).
queue\((vw)\) be the maximum number of edges in a rainbow inside \(vw\) plus one. If \(vw\) is nested inside \(xy\) then queue\((vw)\) < queue\((xy)\). Hence we have a valid queue assignment. The number of queues is at most \(k\).

This result implies that the queue-number of a graph \(G\) is the minimum, taken over all vertex orderings \(\sigma\) of \(G\), of the maximum number of edges in a rainbow in \(\sigma\). Hence determining the queue-number of a graph is merely a question of finding the right vertex ordering. Heath and Rosenberg [57] presented a \(O(m \log \log n)\) time algorithm for computing the \(k\)-queue layout in Lemma 1.

Now consider the problem of assigning the edges of a graph \(G\) to the minimum number of arches given a fixed vertex ordering \(\sigma\) of \(G\). As illustrated in Figure 2(c), a necklace in \(\sigma\) is a matching \(\{v_iw_i : 1 \leq i \leq k\}\) such that

\[
v_1 <_{\sigma} w_1 <_{\sigma} v_2 <_{\sigma} w_2 <_{\sigma} \cdots <_{\sigma} v_k <_{\sigma} w_k.
\]

The necklace \(\{v_iw_i : 1 \leq i \leq k-1\}\) is said to precede the edge \(v_kw_k\).

**Lemma 2.** A vertex ordering of a graph \(G\) admits a \(k\)-arch layout of \(G\) if and only if it has no \((k+1)\)-edge necklace.

**Proof.** A \(k\)-arch layout has no \((k+1)\)-edge necklace, since each edge of a necklace must be assigned to a distinct arch. Conversely, suppose we have a vertex ordering with no \((k+1)\)-edge necklace. For every edge \(vw \in E(G)\), let arch\((vw)\) be the maximum number of edges in a necklace that precedes \(vw\) plus one. If \(vw\) and \(xy\) are disjoint then, without loss of generality, \(vw\) is in a necklace that precedes \(xy\), and thus arch\((vw)\) < arch\((xy)\). Hence we have a valid arch assignment. The number of arches is at most \(k\).

Lemma 2 implies that the arch-number of a graph \(G\) is the minimum, taken over all vertex orderings \(\sigma\) of \(G\), of the maximum number of edges in a necklace in \(\sigma\). For example, \(an(K_n) = \lceil \frac{n}{2}\rceil\). Now consider the following algorithm.

Algorithm **ASSIGNARCHES**(\(G, \sigma\))
1. let \(k_0 = 0\)
2. let \((v_1, v_2, \ldots, v_n) = \sigma\)
3. for \(i = 1, 2, \ldots, n\) do
4. for each edge \(v_iv_j \in E(G)\) with \(i < j\), let arch\((v_iv_j)\) = 1 + \(k_{i-1}\)
5. let \(k_i = k_{i-1}\)
6. for each edge \(v_iv_j \in E(G)\) with \(j < i\), let \(k_i = \max\{k_i, 1 + k_{j-1}\}\)

**Lemma 3.** Given a vertex ordering \(\sigma\) of an \(n\)-vertex \(m\)-edge graph \(G\), the algorithm **ASSIGNARCHES**\((G, \sigma)\) assigns the edges of \(G\) to the minimum number of arches with respect to \(\sigma\) in \(O(n + m)\) time.

**Proof.** It is easily verified that the algorithm maintains the invariant that \(k_i\) is the maximum number of edges in a necklace in the vertex ordering \((v_1, v_2, \ldots, v_i)\). Hence, for every edge \(vw \in E(G)\), arch\((vw)\) is the maximum number of edges in a necklace that precedes \(vw\) plus one. Thus, as in Lemma 2, we have an assignment of the edges to the minimum number of arches.

The proofs of Lemmata 1 and 2 hide an application of the easy half of Dilworth’s Theorem [26] for partitioning a poset into \(k\) antichains, where \(k\) is the maximum size of a chain. In Lemma 1, \(e \prec f\) if \(e\) is nested inside \(f\), and in Lemma 2, \(e \prec f\) if \(R(e) <_{\sigma} L(f)\).
3 Arch Layout Characterisation

A graph $G$ is almost $k$-colourable if there is a set $S \subseteq V(G)$ of at most $k-1$ vertices such that $G \setminus S$ is $k$-colourable.

**Theorem 1.** A graph $G$ has arch-number $\text{an}(G) \leq k$ if and only if $G$ is almost $(k+1)$-colourable.

**Proof.** ($\Longleftarrow$) First suppose that $G$ is almost $(k+1)$-colourable. Then there is a set of vertices $S = \{x_1, x_2, \ldots, x_k\} \subseteq V(G)$ such that $G \setminus S$ is $(k+1)$-colourable. Let $V_1, V_2, \ldots, V_{k+1}$ be the colour classes in such a colouring. Let $\sigma$ be a vertex ordering such that

$$V_1 <_{\sigma} x_1 <_{\sigma} V_2 <_{\sigma} x_2 <_{\sigma} \cdots <_{\sigma} V_k <_{\sigma} x_k <_{\sigma} V_{k+1}.$$ 

Clearly every necklace in $\sigma$ has at most $k$ edges. By Lemma 2, $\sigma$ admits a $k$-arch layout of $G$.

($\Longrightarrow$) The proof is by induction on $k$. For $k=0$, the result is trivial. Suppose that $\text{an}(G) \leq k-1$ implies $G$ is almost $k$-colourable. Let $G$ be a $k$-arch graph with vertex ordering $\sigma = (v_1, v_2, \ldots, v_n)$.

Let $V_{\leq p} = (v_1, v_2, \ldots, v_p)$ and $V_{>p} = (v_{p+1}, v_{p+2}, \ldots, v_n)$. It is simple to verify that the maximum number of edges in a necklace in $V_{\leq p}$ is equal to, or one less than, the maximum number of edges in a necklace in $V_{\leq p+1}$, for all $1 \leq p \leq n-1$. Consequently, there is maximum number $i$ such that $V_{\leq i}$ admits a $(k-1)$-arch layout. By the maximality of $i$, $V_{\leq i+1}$ contains a $k$-edge necklace. Therefore $V_{>i+1}$ is an independent set of $G$, otherwise an edge of $G[V_{>i+1}]$ together with the $k$-edge necklace of $V_{\leq i+1}$ would comprise a $(k+1)$-edge necklace. Therefore, $G[V_{>i}]$ is a forest, at most one component of which is a star centred at $v_{i+1}$, and the remaining components are isolated vertices.

By the induction hypothesis there is a set $S_{k-1}$ of $k-1$ vertices such that $G[V_{\leq i} \setminus S_{k-1}]$ is $k$-colourable. Since $G[V_{>i}]$ is a star centred at $v_{i+1}$ with some isolated vertices, it follows that for $S_k = S_{k-1} \cup \{v_{i+1}\}$, the induced subgraph $G[V \setminus S_k]$ is $(k+1)$-colourable. Therefore, $G$ is almost $(k+1)$-colourable.

Arch-number and chromatic number are tied in the following strong sense.

**Theorem 2.** The arch-number of every graph $G$ satisfies:

$$\text{an}(G) + 1 \leq \chi(G) \leq 2 \text{an}(G) + 1.$$ 

**Proof.** By Theorem 1, $G$ is almost $(\text{an}(G) + 1)$-colourable. Thus it is $(2 \text{an}(G) + 1)$-colourable. Conversely, $G$ is almost $\chi(G)$-colourable, and $\text{an}(G) \leq \chi(G) - 1$ by Theorem 1.

Theorem 2 implies that any graph family that has bounded chromatic-number also has bounded arch-number. Examples include graphs with bounded maximum degree, graphs with bounded tree-width, and graphs with an excluded clique minor, and so on.

**Lemma 4.** Planar graphs have arch-number at most three and this bound is tight.

**Proof.** The Four Colour Theorem and Theorem 1 imply that all planar graphs have arch-number at most three. Any planar graph $G$ containing three vertex-disjoint $K_4$ subgraphs is not almost 3-colourable. By Theorem 1, $\text{an}(G) = 3$.

4 Computational Complexity

The 1-stack graphs are precisely the outerplanar graphs [4], and thus can be recognised in $O(n)$ time [75]. 2-stack graphs are characterised as the subgraphs of planar Hamiltonian graphs [4], which implies that it is $NP$-complete to test if $\text{sn}(G) \leq 2$ [104]. Heath and Rosenberg [57] characterised 1-queue graphs as the ‘arched levelled’ planar graphs, and proved that it is $NP$-complete to recognise such graphs.
Lemma 5. There is a $O(n(n + m))$ time algorithm to determine if a given $n$-vertex $m$-edge graph $G$ has arch-number $an(G) \leq 1$.

Proof. By Theorem 1, $an(G) \leq 1$ if and only if there is a vertex $v$ such that $G \setminus v$ is bipartite. The result follows since bipartiteness can be tested in $O(n + m)$ time by breadth-first search. \qed

Note that almost bipartite graphs have been studied by Prömel et al. [87].

Theorem 3. Given a graph $G$ and an integer $k$, it is N\P-complete to determine if $G$ has arch-number $an(G) \leq k$.

Proof. The problem is clearly in N\P. The remainder of the proof is a reduction from the graph $k$-colourability problem: given a graph $G$ and an integer $k$, is $\chi(G) \leq k$? Let $G'$ be the graph comprised of $k$ components each isomorphic to $G$. We claim that $\chi(G) \leq k$ if and only if $G'$ is almost $k$-colourable. The result will follow from Theorem 1 and since graph $k$-colourability is N\P-complete [64].

If $G$ is $k$-colourable then so is $G'$, and thus $G'$ is almost $k$-colourable. Conversely, if $G'$ is almost $k$-colourable then there is a set $S$ of at most $k - 1$ vertices such that $\chi(G' \setminus S) \leq k$. Since $|S| \leq k - 1$, $G' \setminus S$ contains a component isomorphic to $G$, and thus $G$ is $k$-colourable. \qed

The next result follows from the reduction in Theorem 3 and since it is N\P-complete to determine if a 4-regular planar graph is 3-colourable [19, 42].

Corollary 1. It is N\P-complete to determine if a given 4-regular planar graph $G$ has arch-number $an(G) \leq 2$. \qed

5 Extremal Questions

In this section we consider the extremal questions:

- what is the maximum number of edges in a particular type of layout?
- what is the maximum chromatic number of a graph admitting a particular type of layout?

The answer to the first question for stack layouts has been observed by many authors.

Lemma 6. [4, 18, 65] Every $s$-stack $n$-vertex graph has at most $(s + 1)n - 3s$ edges, and this bound is tight for all even $n \geq 4$ and all $1 \leq s \leq \frac{n}{2}$.

Proof. It will be beneficial to view the vertex ordering $(v_0, v_1, \ldots, v_{n-1})$ as circular. Each edge $v_i v_{(i+1) \mod n}$ is said to be a boundary edge. Each stack has at most $2n - 3$ edges, since a 1-stack graph is outerplanar. Every boundary edge can be assigned to any stack. Thus there are at most $n - 3$ non-boundary edges in each stack, and at most $n$ boundary edges, giving a total of at most $n(n - 3) + n = (s + 1)n - 3s$ edges.

Now for the lower bound. As illustrated in Figure 3(a), for each $0 \leq i \leq s - 1$, let

$$E_i = \{v_j v_k : \left[\frac{1}{2}(j + k)\right] \equiv i \pmod{\frac{n}{2}}\}.$$ 

Then $E_0, E_1, \ldots, E_{s-1}$ are edge-disjoint paths, each of which is a stack of $n - 3$ non-boundary edges. Adding the boundary edges to any stack, we obtain an $s$-stack graph with the desired number of edges. Note that with $s = \frac{n}{2}$, we obtain an $\frac{n}{2}$-stack layout of $K_n$. \qed

As observed by Bernhart and Kainen [4], Lemma 6 implies that (every induced subgraph of) an $s$-stack graph has a vertex of degree less than $2s + 2$, and is therefore vertex $(2s + 2)$-colourable by the minimum-degree-greedy algorithm. This result can be improved for small $s$. 1-stack graphs are outerplanar, which are 3-colourable, and 2-stack graphs are planar, which are 4-colourable.
Open Problem 1. What is the maximum chromatic number $\chi$ of the $s$-stack graphs? In general, $\chi \in \{2s, 2s + 1, 2s + 2\}$ since $\chi(K_n) = 2sn(K_n)$ for even $n$.

Now consider the maximum number of edges in a $k$-queue graph. The answer for $k = 1$ was given by Heath and Rosenberg [57] and Pemmaraju [85]. We now give a simple proof for this case. The proof by Heath and Rosenberg [57] is based on the characterisation of 1-queue graphs as the arched levelled planar graphs. The proof by Pemmaraju [85] is based on a relationship between queue layouts and 'staircase covers of matrices'. The observant reader will notice parallels between the following proof and that of the lower bound on the volume of three-dimensional drawings due to Bose et al. [13].

**Lemma 7.** A queue in a graph with $n$ vertices has at most $2n - 3$ edges.

**Proof.** By Observation 2, distinct edges with the same midpoint are nested. Since every midpoint is in $\{\frac{3}{2}, 2, 2\frac{1}{2}, \ldots, n - \frac{1}{2}\}$, there are at most $2n - 3$ midpoints. The result follows since no two edges in a queue are nested.

An immediate generalisation of Lemma 7 is that every $k$-queue graph has at most $k(2n - 3)$ edges [57]. The following improved upper bound was first discovered by Pemmaraju [85] with a longer proof. That this bound is tight for all values of $n$ and $k$ is new.

**Lemma 8.** Every $k$-queue graph with $n$ vertices has at most $2kn - k(2k + 1)$ edges. For every $k$ and $n \geq 2k + 1$, there exists an $n$-vertex graph with queue-number $k$ and $2kn - k(2k + 1)$ edges.

**Proof.** By Observation 2, distinct edges with the same midpoint are nested. Since at most $k$ edges are pairwise nested in a $k$-queue layout, at most $k$ edges have the same midpoint. Moreover, for all integers $1 \leq i \leq k$, at most $i - 1$ edges have a midpoint of $i$, and at most $i - 1$ edges have a midpoint of $i - \frac{1}{2}$. At the other end of the vertex ordering, for all integers $1 \leq i \leq k - 1$, at most $i$ edges have a midpoint of $n - i$, and at most $i$ edges have a midpoint of $n - i + \frac{1}{2}$. It follows that the number of edges is at most

$$2 \sum_{i=1}^{k}(i - 1) + (2n - 4k + 1)k + 2 \sum_{i=1}^{k-1}i = 2kn - k(2k + 1).$$

We now prove the lower bound. As illustrated in Figure 3(b), let $P_n^s$ denote the $s$th power of the $n$-vertex path $P_n$. That is, $P_n^s$ has $V(P_n^s) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n^s) = \{v_iv_j : |i - j| \leq s\}$. Heath and Rosenberg [57] proved that $qn(P_n^{2k}) = k$, where for each $1 \leq \ell \leq k$, the set of edges $\{v_iv_j : 2\ell - 1 \leq |i - j| \leq 2\ell\}$ is a queue in the vertex ordering $(v_1, v_2, \ldots, v_n)$. (This is Observation 1.) Swaminathan et al. [97] proved that $P_n^{2k}$ has $2kn - k(2k + 1)$ edges. ($P_n^s$ appears in [57, 97] with regard to the relationship between band-width and queue- and stack-number, respectively.)

![Figure 3: Edge-maximal layouts: (a) 2-stack, (b) 2-queue.](image)
Lemma 8 implies that (every induced subgraph of) a \( k \)-queue graph has a vertex of degree less than \( 4k \), and is therefore \( 4k \)-colourable by the minimum-degree-greedy algorithm.

**Open Problem 2.** What is the maximum chromatic number \( \chi \) of a \( k \)-queue graph. We know that \( \chi \in \{2k+1, 2k+2, \ldots, 4k\} \) since \( \chi(K_n) = 2q_n(K_n) + 1 \) for odd \( n \) (by Lemma 1). Note that the extremal example \( P_n^{2k} \) in Lemma 8 also has chromatic number \( 2k+1 \).

We now prove that the lower bound in Open Problem 2 is attainable in the case of \( k = 1 \).

**Lemma 9.** Every 1-queue graph \( G \) is 3-colourable.

**Proof.** Let \( \sigma \) be the vertex ordering in a 1-queue layout of a graph \( G \). Partition the vertices into independent sets \( V_1, V_2, \ldots, V_k \) such that \( \sigma = (V_1, V_2, \ldots, V_k) \), and for all \( 1 \leq i \leq k - 1 \), there exists an edge in \( G[V_i \cup V_{i+1}] \). Such a partition can be computed by starting with each vertex in its own set, and repeatedly merging consecutive sets that have no edge between them. For all \( s \geq 3 \), there is no edge in any \( G[V_i \cup V_{i+s}] \), as otherwise it would be nested with an edge in \( G[V_{i+1} \cup V_{i+2}] \). Thus for each \( 0 \leq j \leq 2 \), \( W_j = \bigcup \{ V_i : i \equiv j \pmod{3} \} \) is an independent set, and \( \{ W_0, W_1, W_2 \} \) is a 3-colouring of \( G \). \( \square \)

The next lemma shows that, in terms of the maximum number of edges, arch layouts behave very differently from stack and queue layouts. Even 1-arch layouts may have a quadratic number of edges.

**Lemma 10.** The maximum number of edges in a \( k \)-arch layout with \( n \) vertices is at most

\[
\frac{kn(n+2) - k(2k+1)}{2(k+1)},
\]

which is attainable whenever \( k+1 \) divides \( n-k \).

**Proof.** Let \( G \) be a \( k \)-arch graph with \( n \) vertices and the maximum number of edges. By Theorem 1, \( G \) is almost \((k+1)\)-colourable. That is, there is a set \( S \subseteq V(G) \) of at most \( k \) vertices such that \( G \setminus S \) is \((k+1)\)-colourable. Each vertex \( v \in S \) is adjacent to every other vertex in \( G \), as otherwise \( G \) is not edge-maximal. The \((k+1)\)-colourable graph with the maximum number of edges is the complete \((k+1)\)-partite graph with partitions whose sizes are as equal as possible. Thus \( G \setminus S \) is this graph. When the partitions have the same size we obtain the most edges. Here, \( G \setminus S \) is obtained from \( K_{n-k} \) by removing \( k+1 \) vertex-disjoint copies of the complete graph on \( (n-k)/(k+1) \) vertices. Thus the number of edges is

\[
\binom{n}{2} - (k+1) \binom{(n-k)/(k+1)}{2},
\]

which is easily seen to reduce to (1). \( \square \)

## 6 Biconnected Components

Clearly the stack-number (respectively, queue-number) of a graph is at most the maximum stack-number (queue-number) of its connected components. Bernhart and Kainen [4] proved that the stack-number of a graph is at most the maximum stack-number of its maximal biconnected components (blocks). We now prove an analogous result for queue layouts.

**Lemma 11.** Every graph \( G \) has queue-number \( q_n(G) \leq 1 + \max\{ q_n(B) : B \text{ is a block of } G \} \).
Proof. Clearly we can assume that $G$ is connected. Let $T$ be the block-tree of $G$. That is, there is one node in $T$ for each block of $G$, and two vertices of $T$ are adjacent if the corresponding blocks share a cut-vertex. $T$ is acyclic, as otherwise a cycle in $T$ would correspond to a single block in $G$. Root $T$ at an arbitrary node $r$. For all $i \geq 0$, let $D_i$ be the set of blocks of $G$ whose corresponding node in $T$ is at distance $i$ from $r$. For all $i \geq 1$, each block $B \in D_i$ has exactly one cut-vertex $v$ that is also in some block $B' \in D_{i-1}$. We call $v$ the parent cut-vertex of $B$, $B'$ is the parent block of $B$, and $B$ is a child block of $B'$. Observe that each cut-vertex $v$ is the parent cut-vertex of all but one block containing $v$. If a vertex $v$ is in only one block $B$ then we say $v$ is grouped with $B$. Otherwise $v$ is a cut-vertex and we say $v$ is grouped with the block for which it is not the parent cut-vertex.

Let $d(v)$ denote the distance from $r$ to each node $v \in V(T)$. Let $\sigma$ be a breadth-first vertex ordering of $T$ satisfying the following properties for all nodes $x, y \in V(T)$. If $x <_{\sigma} y$ then $d(x) \leq d(y)$. If $d(x) = d(y)$, the parent nodes of $x$ and $y$ are $p$ and $q$ respectively, and $p <_{\sigma} q$ then $x <_{\sigma} y$. The child nodes $\{y_1, \ldots, y_d\}$ of a node $x$ are ordered as follows. Let $B_x$ be the block of $G$ corresponding to $x$. Let $v_i$, $1 \leq i \leq d$, be the parent cut-vertex of the block of $G$ corresponding to $y_i$. Then each $v_i$ is in $B_x$. Suppose that $v_1 \leq v_2 \leq \cdots \leq v_d$ in the given queue layout of $B_x$. Then order the child nodes $y_1, y_2, \ldots, y_d$ in $\sigma$. Now replace each node $x$ in $\sigma$ by the vertex ordering in the given queue layout of $B_x$, where each cut-vertex is placed with the block it is grouped with. Let the edges of $G$ inherit their queue assignment from the given queue layout of the blocks, except for edges from a cut-vertex $v$ to neighbours of $v$ that are grouped with the blocks for which $v$ is the parent cut-vertex. By the choice of ordering for child nodes in $\sigma$, these edges can be placed in a single queue. Thus the total number of queues is as claimed. 

\[ \Box \]

7 A Bound on the Queue-Number

Malitz [73] proved that the stack-number of an $m$-edge graph is $O(\sqrt{m})$. The exact bound is in fact $72\sqrt{m}$. Heath and Rosenberg [57] (see also Shahrokhi and Shi [92]) observed that an analogous method proves that queue-number is $O(\sqrt{m})$. We now establish this result using a simplified version of the argument of Malitz [73] and with an improved constant.

**Theorem 4.** Every graph $G$ with $m$ edges has queue-number $qn(G) < e\sqrt{m}$, where $e$ is the base of the natural logarithm.

**Proof.** Let $n = |V(G)|$. Let $\sigma$ be a random vertex ordering of $G$. For all positive integers $k$, let $A_k$ be the event that there exists a $k$-edge rainbow in $\sigma$. Then the probability

\[ \Pr\{A_k\} \leq \frac{m^k}{k!} \cdot \frac{n!}{(2k)!} \cdot \frac{2^k k!(n-2k)!}{n!} \cdot \frac{2^k k!(n-2k)!}{n!} = \frac{(2m)^k}{(2k)!} \cdot \frac{2^{2k} k!^2}{n!^2} \]

where:

(1) is an upper bound on the number of $k$-edge matchings $M$,

(2) is the number of vertex positions in $\sigma$ for $M$, and

(3) is the probability that $M$ with fixed vertex positions is a rainbow.

Thus

\[ \Pr\{A_k\} \leq \frac{m^k}{k!} \cdot \frac{n!}{(2k)!} \cdot \frac{2^k k!(n-2k)!}{n!} = \frac{(2m)^k}{(2k)!} \cdot \frac{2^{2k} k!^2}{n!^2} \]

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By Stirling’s formula, \( P\{A_k\} < \left( \frac{e^2 m}{2k}\right)^k \). Let \( k_0 = \lfloor e \sqrt{m} \rfloor \). Thus, \( P\{A_{k_0}\} > 1 - \left( \frac{1}{2} \right)^{\lfloor e \sqrt{m} \rfloor} > 0 \). That is, with positive probability a random vertex ordering has no \( k_0 \)-edge rainbow. Hence there exists a vertex ordering with no \( k_0 \)-edge rainbow. By Lemma 1, \( G \) has a queue layout with \( k_0 - 1 < e \sqrt{m} \) queues.

References


J. L. GANLEY, Stack and queue layouts of Halin graphs, 1995, manuscript.


