Abstract. We prove that graphs excluding a fixed immersion have bounded nonrepetitive chromatic number. More generally, we prove that if $H$ is a fixed planar graph that has a planar embedding with all the vertices with degree at least 4 on a single face, then graphs excluding $H$ as a topological minor have bounded nonrepetitive chromatic number. This is the largest class of graphs known to have bounded nonrepetitive chromatic number.

1 Introduction

A vertex colouring of a graph is nonrepetitive if there is no path for which the first half of the path is assigned the same sequence of colours as the second half. More precisely, a $k$-colouring of a graph $G$ is a function $\psi$ that assigns one of $k$ colours to each vertex of $G$. A path $(v_1, v_2, \ldots, v_{2t})$ of even order in $G$ is repetitively coloured by $\psi$ if $\psi(v_i) = \psi(v_{t+i})$ for $i \in \{1, \ldots, t\}$. A colouring $\psi$ of $G$ is nonrepetitive if no path of $G$ of even order is repetitively coloured by $\psi$. Observe that a nonrepetitive colouring is proper, in the sense that adjacent vertices are coloured differently. The nonrepetitive chromatic number $\pi(G)$ is the minimum integer $k$ such that $G$ admits a nonrepetitive $k$-colouring. We only consider simple graphs without loops or parallel edges.

The seminal result in this area is by Thue [41], who in 1906 proved that every path is nonrepetitively 3-colourable. Thue expressed his result in terms of strings over an alphabet of three characters—Alon et al. [3] introduced the generalisation to graphs in 2002. Nonrepetitive graph colourings have since been widely studied [2–12, 21, 25–33, 35, 37–39]. The principle result of Alon et al. [3] was that graphs with maximum degree $\Delta$ are nonrepetitively $O(\Delta^2)$-colourable. Several subsequent papers improved the constant [16, 26, 30]. The best known bound is due to Dujmović et al. [16].

**Theorem 1** ([16]). Every graph with maximum degree $\Delta$ is nonrepetitively $(1 + o(1))\Delta^2$-colourable.

A number of other graph classes are known to have bounded nonrepetitive chromatic number. In
particular, trees are nonrepetitively 4-colourable [8, 33], outerplanar graphs are nonrepetitively 12-colourable [5, 33], and graphs with bounded treewidth have bounded nonrepetitive chromatic number [5, 33]. (See Section 2 for the definition of treewidth.) The best known bound is due to Kündgen and Pelsmajer [33].

**Theorem 2** ([33]). *Every graph with treewidth* $k$ *is nonrepetitively* $4^k$*-colourable.*

The primary contribution of this paper is to provide qualitative generalisations of Theorems 1 and 2 in terms of graph immersions and excluded topological minors.

A graph $G$ contains a graph $H$ as an immersion if the vertices of $H$ can be mapped to distinct vertices of $G$, and the edges of $H$ can be mapped to pairwise edge-disjoint paths in $G$, such that each edge $vw$ of $H$ is mapped to a path in $G$ whose endpoints are the images of $v$ and $w$. The image in $G$ of each vertex in $H$ is called a branch vertex. Structural and colouring properties of graphs excluding a fixed immersion have been widely studied [1, 13, 14, 18–20, 22–24, 34, 36, 40, 42]. We prove that graphs excluding a fixed immersion have bounded nonrepetitive chromatic number.

**Theorem 3.** *For every graph* $H$ *with* $t$ *vertices, every graph that does not contain* $H$ *as an immersion is nonrepetitively* $(4 + o(1))t^8$*-colourable.*

Since a graph with maximum degree $\Delta$ contains no star with $\Delta + 1$ leaves as an immersion, Theorem 3 implies that graphs with bounded degree have bounded nonrepetitive chromatic number (as in Theorem 1).

We strengthen Theorem 3 as follows (although without explicit bounds). A graph $G$ contains a graph $H$ as a strong immersion if $G$ contains $H$ as an immersion, such that for each edge $vw$ of $H$, no internal vertex of the path in $G$ corresponding to $vw$ is a branch vertex.

**Theorem 4.** *For every fixed graph* $H$, *there exists a constant* $k$, *such that every graph* $G$ *that does not contain* $H$ *as a strong immersion is nonrepetitively* $k$*-colourable.*

Note that planar graphs with $n$ vertices are nonrepetitively $O(\log n)$-colourable [15], and the same is true for graphs excluding a fixed graph as a minor or topological minor [17]. It is unknown whether any of these classes have bounded nonrepetitive chromatic number. Our final result shows that excluding a special type of topological minor gives bounded nonrepetitive chromatic number.

**Theorem 5.** *Let* $H$ *be a fixed planar graph that has a planar embedding with all the vertices of $H$ with degree at least 4 on a single face. Then there exists a constant* $k$, *such that every graph* $G$ *that does not contain* $H$ *as a topological minor is nonrepetitively* $k$*-colourable.*

Graphs with bounded treewidth exclude fixed walls as topological minors. Since walls are planar graphs with maximum degree 3, Theorem 5 implies that graphs of bounded treewidth
have bounded nonrepetitive chromatic number (as in Theorem 2). Similarly, for every graph $H$ with $t$ vertices, the ‘fat star’ graph (which is the 1-subdivision of the $t$-leaf star with edge multiplicity $t$) contains $H$ as a strong immersion. Since fat stars embed in the plane with all vertices of degree at least 4 on a single face, Theorem 5 implies that graphs excluding a fixed graph as a strong immersion have bounded nonrepetitive chromatic number (as in Theorem 4). In this sense, Theorem 5 generalises all of Theorems 1 to 4.

The results of this paper, in relation to the best known bounds on the nonrepetitive chromatic number, are summarised in Figure 1.

2 Tree Decompositions

For a graph $G$ and tree $T$, a tree decomposition or $T$-decomposition of $G$ consists of a collection $\{T_x \subseteq V(G) : x \in V(T)\}$ of sets of vertices of $G$, called bags, indexed by the nodes of $T$, such that for each vertex $v \in V(G)$ the set $\{x \in V(T) : v \in T_x\}$ induces a connected subtree of $T$, and for each edge $vw$ of $G$ there is a node $x \in V(T)$ such that $v, w \in T_x$. The width of a $T$-decomposition is the maximum, taken over the nodes $x \in V(T)$, of $|T_x| - 1$. The treewidth of a graph $G$ is the minimum width of a tree decomposition of $G$. The adhesion of a tree decomposition $(T_x : x \in V(T))$ is $\max\{|T_x \cap T_y| : xy \in E(T)\}$. The torso of each node $x \in V(T)$ is the graph obtained from $G[T_x]$ by adding a clique on $T_x \cap T_y$ for each edge $xy \in E(T)$ incident to $x$.

Dujmović et al. [17] generalised Theorem 2 as follows:

**Lemma 6 ([17]).** If a graph $G$ has a tree decomposition with adhesion $k$ such that every torso is nonrepetitively $c$-colourable, then $G$ is nonrepetitively $c4^k$-colourable.

For integers $c, d \geq 0$ a graph $G$ has $(c,d)$-bounded degree if $G$ contains at most $c$ vertices with degree greater than $d$.

**Lemma 7.** Every graph with $(c,d)$-bounded degree is nonrepetitively $c + (1 + o(1))d^2$-colourable.

**Proof.** Assign a distinct colour to each vertex of degree at least $d$, and colour the remaining graph by Theorem 1. For each vertex $v$ of degree at least $d$, no other vertex is assigned the same colour as $v$. Thus $v$ is in no repetitively coloured path. The result then follows from Theorem 1.

Dvořák [18] proved the following structure theorem for graphs excluding a strong immersion.

**Theorem 8 ([18]).** For every fixed graph $H$, there exists a constant $k$, such that every graph $G$ that does not contain $H$ as a strong immersion has a tree decomposition such that each torso is $(k,k)$-bounded degree.

Lemmas 6 and 7 and Theorem 8 imply Theorem 4.
3 Weak Immersions

The above proof of Theorem 4 gives no explicit bound on the constant $k$. In this section we prove explicit bounds on the nonrepetitive chromatic number of graphs excluding a weak immersion. The starting point is the following structure theorem of Wollan [42]. For a tree $T$ and graph $G$, a $T$-partition of $G$ is a partition $(T_x \subseteq V(G) : x \in V(T))$ of $V(G)$ indexed by the nodes of $T$. Each set $T_x$ is called a bag. Note that a bag may be empty. For each edge $xy$ of a tree $T$, let $T(xy)$
and \( T(yx) \) be the components of \( T - xy \) where \( x \) is in \( T(xy) \) and \( y \) is in \( T(yx) \). For each edge \( xy \in E(T) \), let \( G(T, xy) := \bigcup \{ T_z : z \in V(T(xy)) \} \) and \( G(T, yx) := \bigcup \{ T_z : z \in V(T(yx)) \} \). Let \( E(T, xy) \) be the set of edges in \( G \) between \( G(T, xy) \) and \( G(T, yx) \). The adhesion of a \( T \)-partition \( (T_x : x \in V(T)) \) is the maximum, taken over all edges \( xy \) of \( T \), of \( |E(T, xy)| \). For each node \( x \) of \( T \), the torso of \( x \) (with respect to a \( T \)-partition) is the graph obtained from \( G \) by identifying \( G(T, yx) \) into a single vertex for each edge \( xy \) incident to \( x \) (deleting resulting parallel edges and loops).

**Theorem 9** ([42]). For every graph \( H \) with \( t \) vertices, for every graph \( G \) that does not contain \( H \) as a weak immersion, there is a \( T \)-partition of \( G \) with adhesion at most \( t^2 \) such that each torso has \((t, t^2)\)-bounded degree.

Theorem 9 leads to the following new structure theorem of independent interest.

**Theorem 10.** For every graph \( H \) with \( t \) vertices, for every graph \( G \) that does not contain \( H \) as a weak immersion has a tree decomposition with adhesion at most \( t^2 \) such that every torso has \((t, t^4 + 2t^2)\)-bounded degree.

**Proof.** Consider the \( T \)-partition \( (T_x : x \in V(T)) \) of \( G \) from Theorem 9. Let \( T' \) be obtained from \( T \) by orienting each edge towards some root vertex. We now define a tree decomposition \( (T_x^* : x \in V(T)) \) of \( G \). Initialise \( T_x^* := T_x \) for each node \( x \in V(T) \). For each edge \( vw \) of \( G \), if \( v \in T_x \) and \( w \in T_y \) and \( z \) is the least common ancestor of \( x \) and \( y \) in \( T' \), then add \( v \) to \( T^*_\alpha \) for each node \( \alpha \) on the \( \overline{xz} \) path in \( T' \), and add \( w \) to \( T^*_\alpha \) for each node \( \alpha \) on the \( \overline{yz} \) path in \( T' \). Thus each vertex \( v \in T_x \) is in a sequence of bags that correspond to a directed path from \( x \) to some ancestor of \( x \) in \( T' \). By construction, the endpoints of each edge are in a common bag. Thus \( (T_x^* : x \in V(T)) \) is a tree decomposition of \( G \).

Consider a vertex \( v \in T_x^* \cap T_y^* \) for some edge \( \overline{xy} \) of \( T' \). Then \( v \) has a neighbour \( w \) in \( G(T, yx) \), and \( vw \in E(T, xy) \). Thus \( |T_x^* \cap T_y^*| \leq |E(T, xy)| \leq t^2 \). That is, the tree decomposition \( (T_x^* : x \in V(T)) \) has adhesion at most \( t^2 \).

Let \( G^x_\infty \) be the torso of each node \( x \in V(T) \) with respect to the tree decomposition \( (T_x^* : x \in V(T)) \). That is, \( G^x_\infty \) is obtained from \( G[T_x^*] \) by adding a clique on \( T_x^* \cap T_y^* \) for each edge \( xy \) of \( T \). Our goal is to prove that \( G^x_\infty \) has \((t, t^4 + 2t^2)\)-bounded degree.

Consider a vertex \( v \) of \( G^x_\infty \). Then \( v \) is in at most one child bag \( y \) of \( x \), as otherwise \( v \) would belong to a set of bags that do not correspond to a directed path in \( T' \). Since \( (T_x^* : x \in V(T)) \) has adhesion at most \( t^2 \), \( v \) has at most \( t^2 \) neighbours in \( T_x^* \cap T_p^* \), where \( p \) is the parent of \( x \) and \( v \) has at most \( t^2 \) neighbours in \( T_x^* \cap T_y^* \). Thus the degree of \( v \) in \( G^x_\infty \) is at most the degree of \( v \) in \( G[T_x^*] \) plus \( 2t^2 \). Call this property \((\star)\).

First consider the case that \( v \not\in T_x \). Let \( z \) be the node of \( T \) for which \( v \in T_z \). Since \( v \in T_x^* \), by construction, \( x \) is an ancestor of \( z \). Let \( y \) be the node immediately before \( x \) on the \( \overline{zx} \) path in \( T' \).

We now bound the number of neighbours of \( v \) in \( T_x^* \). Say \( w \in N_G(v) \cap T_x^* \). If \( w \) is in \( G(T, xy) \) then
Let $e_w$ be the edge $vw$. Otherwise, $w$ is in $G(T,yx)$ and thus $w$ has a neighbour $u$ in $G(T,xy)$ since $w \in T_x^+$, let $e_w$ be the edge $wu$. Observe that $\{e_w : w \in N_G(v) \cap T_x^+ \} \subseteq E(T,xy)$, and thus $|\{e_w : w \in N_G(v) \cap T_x^+ \}| \leq t^2$. Since $e_u \neq e_w$ for distinct $u,w \in N_G(v) \cap T_x^+$, we have $|N_G(v) \cap T_x^+| \leq t^2$. By ($\ast$), the degree of $v$ in $G^+_x$ is at most $3t^2$.

Now consider the case that $v \in T_x$. Suppose further that $v$ is not one of the at most $t$ vertices of degree greater than $t^2$ in the torso $Q$ of $x$ with respect to the given $T$-partition. Suppose that in $Q$, $v$ has $d_1$ neighbours in $T_x$ and $d_2$ neighbours not in $T_x$ (the identified vertices). So $d_1 + d_2 \leq t^2$. Consider a neighbour $w$ of $v$ in $G[T_x^+]$ with $w \notin T_x$. Then $w \in G(T,xy)$ for some child $y$ of $x$. For at most $d_2$ children $y$ of $x$, there is a neighbour of $v$ in $G(T,xy)$. Furthermore, for each child $y$ of $x$, $v$ has at most $t^2$ neighbours in $G(T,xy)$ since the $T$-partition has adhesion at most $t^2$. Thus $v$ has degree at most $d_1 + d_2 t^2 \leq t^4$ in $G[T_x^+]$. By ($\ast$), $v$ has degree at most $2t^2 + t^4$ in $G^+_x$.

Since $3t^2 \leq t^4 + 2t^2$, the torso $G^+_x$ has $(t,t^4 + 2t^2)$-bounded degree. □

Lemma 6 and Theorem 10 imply that for every graph $H$ with $t$ vertices, every graph that does not contain $H$ as an immersion is nonrepetitively $4t^4+O(t^2)$-colourable, which is Theorem 3 with a weaker bound. A key ingredient in the polynomial bound in Theorem 3, is the following definitions and lemma due to Kündgen and Pelsmajer [33], which were introduced as steps towards proving Theorem 2. A layering of a graph $G$ is a partition $(V_1, \ldots, V_n)$ of $V(G)$ such that for each edge $vw$ of $G$, if $v \in V_i$ and $w \in V_j$ then $|i-j| \leq 1$. A layering $(V_1, \ldots, V_n)$ of a graph $G$ is shadow-complete if for $2 \leq i \leq n$, for each component $H$ of $G[V_i \cup \cdots \cup V_n]$ the set of neighbours in $V_{i-1}$ of vertices in $H$ is a clique.

**Lemma 11** ([33]). If a graph $G$ has a shadow-complete layering such that the graph induced by each layer is nonrepetitively $c$-colourable, then $G$ is nonrepetitively $4c$-colourable.

**Proof of Theorem 3.** By Theorem 9, for some tree $T$, there is a $T$-partition of $G$ with adhesion at most $t^2$ such that each torso has $(t,t^2)$-bounded degree. Root $T$ at some node $r$. For $i \geq 0$, let $V_i$ be the set of vertices of $G$ in a bag of $T$ whose corresponding node in $T$ is at distance $i$ from $r$. Then $(V_0, \ldots, V_n)$ is a layering of $G$, for some integer $n$.

For each edge $xy$ of $T$, with $y$ the child of $x$, let $C(xy)$ be the set of vertices in $T_x$ adjacent to some vertex in $T_y$. Then $|C(xy)| \leq t^2$ by the adhesion property. Add a clique on $C(xy)$ for each edge $xy$ to obtain a graph $G'$. Then $(V_0, \ldots, V_n)$ defines a shadow complete layering of $G'$.

Let $H$ be the subgraph of $G$ induced by some bag $T_x$. Let $H'$ be the subgraph of $G'$ induced by $T_x$. Let $Q$ be the torso of $T_x$ in the original graph $G$. Consider a vertex $v$ in $T_x$. Say $v$ is in $k$ of the cliques $C(xy)$ where $y$ is a child of $x$. Then $\text{deg}_Q(v) \geq k$ and $\text{deg}_{H'}(v) \leq \text{deg}_H(v) + kt^2$. At most $t$ vertices $v$ in $H$ have $\text{deg}_Q(v) > t^2$. Say $v$ is not one of these vertices. Then $k \leq \text{deg}_Q(v) \leq t^2$ and $\text{deg}_{H'}(v) \leq \text{deg}_H(v) + kt^2 \leq \text{deg}_Q(v) + t^4 \leq t^2 + t^4$. Thus $H'$ is a $(t,t^4 + t^2)$-bounded degree graph.
Hence each component of the subgraph of $G'$ induced by each layer has $(t, t^2 + t^4)$-bounded degree. Thus each layer of $G'$ is nonrepetitively $(1 + o(1))t^8$ colourable by Lemma 7. By Lemma 11, $G'$ and hence $G$ is nonrepetitively $(4 + o(1))t^8$ colourable.

4 Excluding a Topological Minor

Theorem 5 is an immediate corollary of Lemma 6 and the following structure theorem of Dvořák [18] that extends Theorem 8.

**Theorem 12** ([18]). Let $H$ be a fixed planar graph that has a planar embedding with all the vertices of $H$ with degree at least 4 on a single face. Then there exists a constant $k$, such that every graph $G$ that does not contain $H$ as a topological minor has a tree decomposition such that each torso has $(k, k)$-bounded degree.

While Theorem 12 is not explicitly stated in [18], we now explain that it is in fact a special case of Theorem 3 in [18]. This result provides a structural description of graphs excluding a given topological minor in terms of the following definition. For a graph $H$ and surface $\Sigma$, let $mf(H, \Sigma)$ be the minimum, over all possible embeddings of $H$ in $\Sigma$, of the minimum number of faces such that every vertex of degree at least 4 is incident with one of these faces. By assumption, for our graph $H$ and for every surface $\Sigma$, we have $mf(H, \Sigma) = 1$. In this case, Theorem 3 of Dvořák [18] says that for some integer $k = k(H)$, every graph $G$ that does not contain $H$ as a topological minor is a clique sum of $(k, k)$-bounded degree graphs. It immediately follows that $G$ has the desired tree decomposition. See Corollary 1.4 in [34] for a closely related structure theorem.

The following natural open problem arises from this research: Do graphs excluding a fixed planar graph as a topological minor have bounded nonrepetitive chromatic number? And what is the structure of such graphs?

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References


