Queue Layouts of Graphs
with Bounded Degree and Bounded Genus

Vida Dujmović ‡ Pat Morin ∗ David R. Wood ¶

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Abstract

Motivated by the question of whether planar graphs have bounded queue-number, we prove that planar graphs with maximum degree $\Delta$ have queue-number $O(\Delta^2)$, which improves upon the best previous bound of $O(\Delta^6)$. More generally, we prove that graphs with bounded degree and bounded Euler genus have bounded queue-number. In particular graphs with Euler genus $g$ and maximum degree $\Delta$ have queue-number $O(g + \Delta^2)$. As a byproduct we prove that if planar graphs have bounded queue-number, then graphs of Euler genus $g$ have queue-number $O(g)$.

1 Introduction

Bekos, Förster, Gronemann, Mchedlidze, Montecchiani, Raftopoulou, and Ueckerdt [1] recently proved that planar graphs with bounded (maximum) degree have bounded queue-number. We improve their bound and more generally show that graphs with bounded degree and bounded genus have bounded queue-number.

First we introduce queue layouts and give the background to the above results. For a graph $G$ and integer $k \geq 0$, a $k$-queue layout of $G$ consists of a linear ordering $\preceq$ of $V(G)$ and a partition $E_1, E_2, \ldots, E_k$ of $E(G)$, such that for $i \in \{1, \ldots, k\}$, no two edges in $E_i$ are nested with respect to $\preceq$. Here edges $vw$ and $xy$ are nested if $v < x < y < w$. The queue-number of a graph $G$, denoted by $qn(G)$, is the minimum integer $k$ such that $G$ has a $k$-queue layout. These definitions were introduced by Heath et al. [12, 13] as a dual to stack layouts (also called book embeddings). In a stack layout, no two edges in $E_i$ cross with respect to $\preceq$. Here edges $vw$ and $xy$ cross if $v < x < w < y$.

Heath et al. [12] conjectured that every planar graph has bounded queue number. This conjecture has remained open despite much research on queue layouts [3, 5–8, 10–14, 16, 18, 19]. Dujmović

‡School of Computer Science and Electrical Engineering, University of Ottawa, Ottawa, Canada (vida.dujmovic@uottawa.ca). Research supported by NSERC and the Ontario Ministry of Research and Innovation.

∗School of Computer Science, Carleton University, Ottawa, Canada (morin@scs.carleton.ca). Research supported by NSERC.

¶School of Mathematical Sciences, Monash University, Melbourne, Australia (david.wood@monash.edu). Research supported by the Australian Research Council.
and Wood [8] observed that every graph with \( m \) edges has a \( O(\sqrt{m}) \)-queue layout using a random vertex ordering. Thus every planar graph with \( n \) vertices has queue-number \( O(\sqrt{n}) \). Di Battista, Frati, and Pach [2] proved the first breakthrough on this topic, by showing that every planar graph with \( n \) vertices has queue-number \( O(\log^2 n) \). Dujmović [4] improved this bound to \( O(\log n) \) with a simpler proof.

Dujmović et al. [6] established (poly-)logarithmic bounds for more general classes of graphs.\(^1\) For example, they proved that every graph with \( n \) vertices and Euler genus \( g \) has queue-number \( O(g + \log n) \), and that every graph with \( n \) vertices excluding a fixed minor has queue-number \( \log^{O(1)} n \).

Recently, Bekos et al. [1] proved a second breakthrough result, by showing that planar graphs with bounded degree have bounded queue-number.

**Theorem 1** ([1]). *Every planar graph with maximum degree \( \Delta \) has queue-number at most \( 32(2\Delta - 1)^6 - 1 \).*

Note that bounded degree alone is not enough to ensure bounded queue-number. In particular, Wood [20] proved that for every integer \( \Delta \geq 3 \) and all sufficiently large \( n \), there are graphs with \( n \) vertices, maximum degree \( \Delta \), and queue-number \( \Omega(\sqrt{\Delta n^{1/2} - 1/\Delta}) \).

The first contribution of this paper is to improve the bound of Bekos et al. [1] from \( O(\Delta^6) \) to \( O(\Delta^2) \).

**Theorem 2.** *Every planar graph with maximum degree \( \Delta \) has queue-number at most \( 12\Delta^2 + 16\Delta + 3 \).*

We extend this result by showing that graphs with bounded Euler genus and bounded degree have bounded queue-number.

**Theorem 3.** *Every graph with Euler genus \( g \) and maximum degree \( \Delta \) has queue-number at most \( 4g + 36\Delta^2 + 48\Delta + 9 \).*

We remark that using well-known constructions [5, 7, 9], Theorem 3 implies that graphs with bounded Euler genus and bounded degree have bounded track-number, which in turn can be used to prove linear volume bounds for three-dimensional straight-line grid drawings of the same class of graphs. These results can also be extended for graphs with bounded degree that can be drawn in a surface of bounded Euler genus with a bounded number of crossings per edge (using [10, Theorem 6]). We omit all these details.

The proof of Theorem 3 uses Theorem 2 as a ‘black box’. Starting with a graph \( G \) of bounded Euler genus and bounded degree, we construct a planar subgraph \( G' \) of \( G \). We then apply Theorem 2 to obtain a queue layout of \( G' \), from which we construct a queue layout of \( G \). This approach suggests a direct connection between the queue-number of graphs with bounded Euler genus and planar graphs, regardless of degree considerations. The following theorem

\(^1\)The Euler genus of a graph \( G \) is the minimum integer \( k \) such that \( G \) embeds in the orientable surface with \( k/2 \) handles (and \( k \) is even) or the non-orientable surface with \( k \) cross-caps. Of course, a graph is planar if and only if it has Euler genus 0; see [15] for more about graph embeddings in surfaces. A graph \( H \) is a *minor* of a graph \( G \) if a graph isomorphic to \( H \) can be obtained from a subgraph of \( G \) by contracting edges.
establishes this connection. A class of graphs is hereditary if it is closed under taking induced subgraphs.

**Theorem 4.** Let $G$ be a hereditary class of graphs, such that every planar graph in $G$ has queue-number at most $k$. Then every graph in $G$ with Euler genus $g$ has queue-number at most $3k + 4g$.

Theorem 3 is an immediate corollary of Theorems 2 and 4, where $G$ is the class of graphs with maximum degree at most $\Delta$. Theorem 4, where $G$ is the class of all graphs, implies the following result of interest:

**Corollary 5.** If every planar graph has queue-number at most $k$, then every graph with Euler genus $g$ has queue-number at most $3k + 4g$.

For a graph $G$ and a set $A \subseteq V(G)$, let $G[A]$ be the subgraph of $G$ induced by $A$, which has vertex set $A$ and edge set $\{vw \in E(G) : v, w \in A\}$. For disjoint sets $A, B \subseteq V(G)$, let $G[A, B]$ be the bipartite graph with bipartition $\{A, B\}$ and edge set $\{vw \in E(G) : v \in A, w \in B\}$.

## 2 Planar Graphs of Bounded Degree

This section proves Theorem 2. The proof is inspired by the proof of Theorem 1 by Bekos et al. [1]. Here is high-level overview of their proof for a planar graph $G$ with maximum degree $\Delta$. First, Bekos et al. [1] construct a particular planar graph $G_1$ obtained from $G$ by subdividing each edge at most three times. Then they construct a planar graph $G_2$ from $G_1$ by replacing certain edges by pairs of trees and a perfect matching between their leaves. $G_2$ is called a `$\Delta$-matched' graph. The heart of the proof of Bekos et al. [1] is to construct an $O(\Delta)$-queue layout of any $\Delta$-matched graph, and thus of $G_2$. They then observe that the queue layout of $G_2$ also gives an $O(\Delta)$-queue layout of $G_1$. Finally, they use a generic lemma of Dujmović and Wood [8], which says that if some $(\leq c)$-subdivision of a graph has a $k$-queue layout, then the original graph has an $O(k^c)$-queue layout. Bekos et al. [1] apply this result with $k = O(\Delta)$ and $c = 3$, to obtain an $O(\Delta^3)$-queue layout of $G$.

It should be mentioned that there is a straightforward way to improve this $O(\Delta^6)$ bound. Lemma 11 in Appendix 1 shows that if some $(\leq c)$-subdivision of a graph has a $k$-queue layout for some fixed $c$, then the original graph has an $O(k^{c+1})$-queue layout. Moreover, in the proof of Bekos et al. [1], for every edge $e$ of $G$ that is subdivided three times, one of the edges in the subdivision of $e$ is assigned to a single queue ($Q_0$ in their notation). This observation, in conjunction with the proof of Lemma 11, leads to a $O(\Delta^5)$-queue layout of $G$.

Our proof of Theorem 2 initially follows a similar strategy. Starting with a planar graph $G$ with maximum degree $\Delta$, we consider the $(\leq 3)$-subdivision $G_1$ of $G$ constructed by Bekos et al. [1]. Note that Bekos et al. [1] explain in Section 3.3 of their paper that one can work directly with $G_1$ instead of the $\Delta$-matched graph $G_2$, and this is what we choose to do. The key properties of $G_1$ are summarised in the definition of ‘well-layered’ below. We then construct a partition of $V(G_1)$ with several desirable properties (see Lemma 7). This partition is implicit in the proof of Bekos et al. [1]—there is really nothing new in this part of our proof.
The main point of difference between our proof and that of Bekos et al. [1] is that we do not apply the generic `unsubdividing' lemma of Dujmović and Wood [8]. Instead we refine the partition of \( V(G_1) \) to obtain a similar partition of \( V(G) \) (see Lemma 8). From this partition one can determine an \( O(\Delta^2) \)-queue layout of \( G \). Note that in this \( O(\Delta^2) \)-queue layout, the vertex ordering is identical to that used by Bekos et al. [1], only the queue assignment is different. This fact shows the value in focusing on structural partitions rather than the final queue layout.

The following definitions are key concepts in our proofs (and that of several other papers on queue layouts [1, 5, 6, 8]). A **layering** of a graph \( G \) is a partition \((V_0, V_1, \ldots, V_t)\) of \( V(G) \) such that for every edge \( vw \in E(G) \), if \( v \in V_i \) and \( w \in V_j \), then \( |i − j| \leq 1 \). If \( r \) is a vertex in a connected graph \( G \) and \( V_i := \{v \in V(G) : \text{dist}(r, v) = i\} \) for all \( i \geq 0 \), then \((V_0, V_1, \ldots, V_t)\) is called a **BFS layering** of \( G \), where \( t := \max\{\text{dist}(r, v) : v \in V(G)\} \). Associated with a bfs layering is a **bfs spanning tree** \( T \) obtained by choosing, for each non-root vertex \( v \in V_i \) with \( i \geq 1 \), a neighbour \( w \) in \( V_{i-1} \), and adding the edge \( vw \) to \( T \). Thus \( \text{dist}_T(r, v) = \text{dist}_G(r, v) \) for each vertex \( v \) of \( G \). When the spanning tree \( T \) is obvious from the context, we call edges in \( T \) **tree edges** and edges not in \( T \) **non-tree edges**. An edge \( vw \in E(G) \) with \( v, w \in V_i \) for some \( i \geq 0 \) is called a **level edge**. An edge \( vw \in E(G) \) with \( v \in V_i \) and \( w \in V_{i+1} \) for some \( i \geq 0 \) is called a **binding edge**. Every tree edge is binding.

The following lemma of Pupyrev [17] shows that every planar graph has a drawing that highlights particular aspects of a BFS layering, as illustrated in Figure 1.

**Lemma 6** ([17]). For every connected planar graph \( G \) and every vertex \( r \) of \( G \), if \( T \) is the BFS tree and \((V_0, V_1, \ldots, V_t)\) is the BFS layering of \( G \) rooted at \( r \), then there is a drawing of \( G \) in \( \mathbb{R}^2 \) with the \( r \) at the origin and on the outer-face, such that for \( i \in \{1, 2, \ldots, t\} \),

- the vertices in \( V_i \) are drawn on a circle \( C_i \) of radius \( R_i \) centred at the origin, where \( 0 < R_1 < R_2 < \cdots < R_t \);
- each level edge \( vw \in E(G) \) with \( v, w \in V_i \) is drawn as an open curve between \( v \) and \( w \) strictly outside of \( C_i \); and
- each binding edge \( vw \) with \( v \in V_i \) and \( w \in V_{i+1} \) is drawn either:
  - as an open curve from \( v \) to \( w \) strictly between \( C_i \) and \( C_{i+1} \) (called a direct edge), or
  - as an open curve starting at \( v \) that crosses \( C_{i+1} \) once at a point distinct from \( w \), then stays outside of \( C_{i+1} \), and ends at \( w \) (called a hooked edge).

- each tree edge \( vw \in E(T) \) is direct and binding.

### 2.1 Well-Layered Planar Graphs

A planar graph \( G \) is **well-layered** if there is a BFS spanning tree \( T \) of \( G \) rooted at a vertex \( r \) such that every non-tree edge \( vw \in E(G) \setminus E(T) \) is a level edge in the corresponding BFS layering, and both \( v \) and \( w \) are leaves in \( T \) with degree 2 in \( G \). This implies that the set of non-tree edges are a matching in \( G \).
Figure 1: Drawing of planar graph on concentric circles: tree edges are blue, hooked edges are green, level edges are brown, direct non-tree edges are red.

Lemma 7. Let $G$ be well-layered planar graph with corresponding BFS spanning tree $T$ and BFS layering $(V_0, V_1, \ldots, V_t)$ rooted at a vertex $r$. Assume that every vertex in $G$ has at most $\Delta$ children in $T$. Then for $i \in \{1, 2, \ldots, t\}$, there is a partition $\{V_{i,a} : a \geq 0\}$ of $V_i$, and an ordering $\vec{V}_{i,a}$ of each set $V_{i,a}$ such that:

(a) for each non-tree edge $vw \in E(G) \setminus E(T)$, both $v$ and $w$ are in $V_{i,a}$ for some $i, a \geq 0$,

(b) for each tree edge $vw \in E(T)$, if $v \in V_{i,a}$ and $w \in V_{i+1,b}$ for some $a, b \geq 0$, then $b - \Delta a \in \{0, 1, \ldots, 2\Delta - 1\}$,

(c) for all $i, a \geq 0$, no two edges in $G[V_{i,a}]$ cross or nest with respect to the ordering $\vec{V}_{i,a}$, in particular, $\vec{V}_{i,a}$ defines a 1-queue layout of $G[V_{i,a}]$, and

(d) for all $i, a, b \geq 0$, the ordering $\vec{V}_{i,a} \vec{V}_{i+1,b}$ defines a 1-queue layout of $G[V_{i,a}, V_{i+1,b}]$.

Proof. Apply Lemma 6 to obtain a drawing of $G$ on concentric circles $C_1, C_2, \ldots, C_t$. For each vertex $v \in V(G)$, let $\ell(v) := \text{dist}_G(v, r) - t$. Thus $\ell(v) \in \{0, 1, \ldots, t\}$ for every vertex $v$ of $G$, and $\ell(r) = t$. For each vertex $v$ of $G$, let $T_v$ be the subtree of $T$ rooted at $v$.

For each non-tree edge $vw \in E(G) \setminus E(T)$, let $D_{vw}$ be the cycle obtained from the $vw$-path in $T$ by adding the edge $vw$. Note that if $v, w \in V_i$ then the $vw$-path in $T$ is drawn within the interior of $C_i$ (since every tree edge is binding and direct) and $vw$ is drawn outside of $C_i$.

Let $G^+$ be the multigraph with vertex set $V(G)$, where each tree edge $vw \in E(T)$ has multiplicity 1 in $G^+$, and each non-tree edge $vw \in E(G) \setminus E(T)$ has multiplicity $\Delta^{\ell(v)}$ (which equals $\Delta^{\ell(w)}$ since every non-tree edge is a level edge). Note that $T$ is a spanning tree of $G^+$.

A key property of this construction is that for each vertex $v$ of $G$, the number of non-tree edges in $G^+$ with one endpoint in $T_v$ is at most $\Delta^{\ell(v)}$. We prove this claim by induction on $\ell(v)$. First note that if $v$ is a leaf in $T$, then $T_v$ is simply the vertex $v$, and $v$ is incident to at most one non-tree edge in $G$, and thus is incident to at most $\Delta^{\ell(v)}$ edges in $G^+$. So the claim holds
We now prove (a). Consider a non-tree edge $vw$. Let $v_1, \ldots, v_d$ be the children of $v$, where $v_1, \ldots, v_p$ are not leaves of $T$, and $v_{p+1}, \ldots, v_d$ are leaves of $T$. Note that $\ell(v_i) = \ell(v) - 1$ for $i \in \{1, \ldots, d\}$. By induction, for each $i \in \{1, \ldots, p\}$, the number of non-tree edges in $G^+$ with one endpoint in $T_{v_i}$ is at most $\Delta^{\ell(v_i)} = \Delta^{\ell(v) - 1}$. For $i \in \{p + 1, \ldots, d\}$, we have already shown that the number of non-tree edges in $G^+$ incident to $v_i$ is at most $\Delta^{\ell(v_i)} = \Delta^{\ell(v) - 1}$ edges. In total, there are at most $d\Delta^{\ell(v) - 1} \leq \Delta^{\ell(v)}$ non-tree edges in $G^+$ incident to $T_v$, as claimed.

Let $G^*$ be the dual of $G^+$. Let $T^*$ be the spanning subgraph of $G^*$ consisting of those edges of $G^*$ dual to edges of $E(G^+) \setminus E(T)$. It is well known (and easily follows from Euler’s formula) that $T^*$ is a spanning tree of $G^*$. ($T^*$ is sometimes called a co-tree; note that $T$ and $T^*$ can be simultaneously drawn without crossing each other.) Let $r^*$ be the vertex of $T^*$ dual to the outer-face of $G^+$. Consider $T^*$ to be rooted at $r^*$.

For each face $f$ of $G^+$, let $d(f)$ be the distance in $T^*$ between $r^*$ and the vertex of $T^*$ dual to $f$. For each vertex $v$ of $G^+$, let $m(v)$ be the minimum of $d(f)$ taken over all faces $f$ of $G^+$ incident with the subtree of $T$ rooted at $v$, and let

$$g(v) := \left\lfloor \frac{m(v)}{\Delta^{\ell(v)}} \right\rfloor.$$

For $i, a \geq 0$, let

$$V_{i,a} := \{v \in V_i : g(v) = a\}.$$

Let $\overrightarrow{V_{i,a}}$ be the ordering of $V_{i,a}$ along circle $C_v$ where the outer-face defines the start and endpoint. (In the language of Bekos et al. [1], $m(v)$ is analogous to the ‘matching-value’ of $v$, and $g(v)$ is the ‘layer-group’ of $v$.)

This concludes the description of the partition $\{V_{i,a} : i, a \geq 0\}$ and the orderings $\overrightarrow{V_{i,a}}$. We now show these satisfy the claims of the lemma.

We now prove (a). Consider a non-tree edge $vw \in E(G) \setminus E(T)$. By assumption, both $v$ and $w$ are in $V_i$ for some $i \geq 0$, and both $v$ and $w$ are leaves in $T$. Thus $\ell(v) = \ell(w)$, and $v$ is the only vertex in the subtree rooted at $v$, and similarly for $w$. Since $\deg_G(v) = \deg_G(w) = 2$, the faces incident to $v$ are exactly the same faces incident to $w$. Thus $m(v) = m(w)$, implying $v$ and $w$ are in $V_{i,a}$ where $a = [m(v)/\Delta^{\ell(v)}]$. This proves (a).

We now prove (b). Consider a non-leaf vertex $v$ with $\ell(v) = \ell$. Then all the edges incident to $v$ are in $T$. Let $x$ and $y$ be two children of $v$ consecutive in the embedding of $G$. Observe that $m(y) - m(x)$ is maximised when all the non-tree edges incident to $T_x$ go ‘under’ $T_y$. The number of such edges is at most $\Delta^{\ell(x)} = \Delta^{\ell - 1}$. Thus $m(y) \leq m(x) + \Delta^{\ell - 1}$. Since $v$ has at most $\Delta$ children, $m(y) \leq m(x) + \Delta^\ell$ for all children $x$ and $y$ of $v$. Every face incident with $T_v$ is incident to $T_x$ for some child $x$ of $v$. Thus $m(v)$ equals the minimum of $m(x)$ taken over all children $x$ of $v$. Hence $m(y) \leq m(v) + \Delta^\ell$ for all children $y$ of $v$, implying

$$g(y) = \left\lfloor \frac{m(y)}{\Delta^{\ell - 1}} \right\rfloor \leq \frac{m(y)}{\Delta^{\ell - 1}} \leq \Delta \frac{m(v)}{\Delta^\ell} + \Delta < \Delta \left( \left\lfloor \frac{m(v)}{\Delta^\ell} \right\rfloor + 1 \right) + \Delta = \Delta g(v) + 2\Delta.$$

Since $g(v)$ and $g(y)$ are integers, $g(y) \leq \Delta g(v) + 2\Delta - 1$. Moreover, $m(v) \leq m(y)$, implying

$$\Delta g(v) = \Delta \left\lfloor \frac{m(v)}{\Delta^\ell} \right\rfloor \leq \Delta \frac{m(v)}{\Delta^\ell} \leq \frac{m(y)}{\Delta^{\ell - 1}} < \left\lfloor \frac{m(y)}{\Delta^{\ell - 1}} \right\rfloor + 1 = g(y) + 1.$$
Since \( g(v) \) and \( g(y) \) are integers, \( g(y) \geq \Delta g(v) \). Summarising, if \( y \) is a child of \( v \) in \( T \) then
\[
g(y) - \Delta g(v) \in \{0, 1, \ldots, 2\Delta - 1\}.
\]

This says that for each tree edge \( vw \in E(T) \) where \( v \in V_{i,a} \) and \( w \in V_{i+1,b} \) we have \( b - \Delta a \in \{0, 1, \ldots, 2\Delta - 1\} \), which proves (b).

We now prove (c), which claims that no two edges in \( G[V_{i,a}] \) cross or nest with respect to the ordering \( \overrightarrow{V_{i,a}} \). Consider edges \( vw, pq \in E(G) \) with \( v, w, p, q \in V_{i,a} \). Let \( \ell := \ell(v) = \ell(w) \). Neither \( vw \) nor \( pq \) are tree edges. Suppose on the contrary that \( vw \) and \( pq \) cross with respect to \( \overrightarrow{V_{i,a}} \). Without loss of generality, \( v < p < w < q \) in \( \overrightarrow{V_{i,a}} \). Then \( v, p, w, q \) appear in this order on the circle \( C_i \). Since \( vw \) and \( pq \) are drawn outside \( C_i \), these edges cross in the drawing of \( G \), which is a contradiction. Thus no two edges in \( G[V_{i,a}] \) cross with respect to \( \overrightarrow{V_{i,a}} \). Now suppose that \( vw \) and \( pq \) nest with respect to \( \overrightarrow{V_{i,a}} \). Without loss of generality, \( v < p < q < w \) in \( \overrightarrow{V_{i,a}} \). Thus \( v, p, q, w \) appear in this order on \( C_i \). Hence both \( T_p \) and \( T_q \) are inside \( D_{vw} \) and the outer-face of \( G^+ \) is outside \( D_{vw} \). Since \( v \) is the only edge of \( D_{vw} \) not in \( T \), every path in \( T^* \) from \( r^* \) to a vertex dual to a face incident with \( pw \) or \( q \) must include the edges of \( T^* \) dual to \( vw \). Let \( f \) be the face of \( G \) immediately below \( vw \). Since \( v \) has multiplicity \( \Delta^f \) in \( G^+ \), for every face \( f' \) incident with \( T_p \) or \( T_q \), we have \( d(f') \geq d(f) + \Delta^f \). Since \( w \) is incident with \( f \), we have \( m(w) \leq d(f) \). Thus \( m(p) \geq d(f) + \Delta^f \geq m(w) + \Delta^f \). Hence
\[
\frac{m(p)}{\Delta^f} \geq \frac{m(w)}{\Delta^f} + 1.
\]

Therefore
\[
g(p) = \left\lceil \frac{m(p)}{\Delta^f} \right\rceil > \left\lceil \frac{m(w)}{\Delta^f} \right\rceil = g(w),
\]
which implies that \( p \) and \( w \) are not both in \( V_{i,a} \). This contradiction shows that \( \overrightarrow{V_{i,a}} \) defines a 1-queue layout of \( G[V_{i,a}] \) for all \( i, a \geq 0 \). This proves (c).

We now prove (d). Suppose on the contrary that \( v < x \) in \( \overrightarrow{V_{i,a}} \) and \( y < w \) in \( \overrightarrow{V_{i+1,b}} \) for some edges \( vw, xy \in E(G) \) for some \( i, a, b \geq 0 \). Thus \( v \) is to the left of \( x \) in \( C_i \) and \( y \) is to the left of \( w \) in \( C_{i+1} \). Since every non-tree edge is a level edge, both \( vw \) and \( xy \) are tree edges, which are drawn direct between \( C_i \) and \( C_{i+1} \). Thus \( vw \) and \( xy \) cross. This contradiction shows that no two edges of \( G[V_{i,a}, V_{j,b}] \) are nested in the ordering \( \overrightarrow{V_{i,a}V_{i+1,b}} \) which thus defines a 1-queue layout of \( G[V_{i,a}, V_{i+1,b}] \). This proves (d).

Note that Lemma 7 implies that every well-layered graph has a 2\( \Delta \)-queue layout, as proved by Bekos et al. [1]. To see this, let \( \overrightarrow{V_i} \) be the ordering \( \overrightarrow{V_i,0V_i,1} \ldots \) of \( V_i \). Then take the ordering \( \overrightarrow{V_0V_1V_2} \ldots \) of \( V(G) \). By Lemma 7(c), every level edge can be assigned to a single queue \( Q^* \).

Assign each tree edge \( vw \in E(T) \) where \( v \in V_{i,a} \) and \( w \in V_{i+1,b} \) to \( Q_{b-\Delta a} \). By Lemma 7(b) this introduces 2\( \Delta \) queues. Suppose that tree edges \( vw \) and \( pq \) in \( Q_j \) are nested for some \( j \in \{0, 1, \ldots, 2\Delta - 1\} \), with \( v < p < q < w \) in the ordering. Then \( v \in V_{i,a} \), \( p \in V_{b,0} \), \( q \in V_{i+1,b} \) and \( w \in V_{i+1,d} \) for some \( i, a, b, c, d \geq 0 \) with \( j = d - \Delta a = c - \Delta b \). Thus \( d - c = \Delta(a - b) \). Since \( v < p < q < w \) in the ordering, \( d - c \geq 0 \) and \( a - b \leq 0 \). Thus \( a = b \) and \( c = d \), which contradicts Lemma 7(d).\[\square\]
2.2 General Planar Graphs

We now extend Lemma 7 for all planar graphs.

**Lemma 8.** Let $G$ be a planar graph with a BFS spanning tree $T$ and BFS layering $(V_0, V_1, \ldots, V_t)$ rooted at a vertex $r$. Assume that every vertex in $G$ has degree at most $\Delta + 1$ and has most $\Delta$ children in $T$. Then for $i \in \{1, 2, \ldots, t\}$, there is a partition $\{V_{i,a} : a \geq 0\}$ of $V_i$, and an ordering $\overrightarrow{V_{i,a}}$ of each set $V_{i,a}$, such that:

(a) for each level edge $vw \in E(G)$, if $v \in V_{i,a}$ and $w \in V_{i,b}$ then $|a - b| \leq 1$;

(b) for each tree edge $vw \in E(T)$, if $v \in V_{i,a}$ and $w \in V_{i+1,b}$ then $b - a\Delta \in \{0, 1, \ldots, 2\Delta - 1\}$;

(c) for each non-tree binding edge $vw \in E(G) \setminus E(T)$, if $v \in V_{i,a}$ and $w \in V_{i+1,b}$ then $b - a\Delta \in \{-1, 0, \ldots, 2\Delta\}$;

(d) the ordering $\overrightarrow{V_{i,a}V_{i+1,a}}$ defines a 1-queue layout of $G[V_{i,a}, V_{i+1,a+1}]$ for all $i, a \geq 0$.

(e) the ordering $\overrightarrow{V_{i,a}V_{i+1,b}}$ defines a $(6\Delta + 1)$-queue layout of $G[V_{i,a}, V_{i+1,b}]$ for all $i, a, b \geq 0$.

(f) the ordering $\overrightarrow{V_{i,a}}$ defines a $2\Delta$-queue layout of $G[V_{i,a}]$ for all $i, a \geq 0$.

**Proof.** Apply Lemma 6 to obtain a drawing of $G$ on concentric circles $C_1, C_2, \ldots, C_t$ rooted at $r$.

Let $G'$ be obtained by subdividing edges of $G$ as follows, as illustrated in Figure 2. Initialise $T' := T$ and $V'_i := V_i$ for each $i \geq 0$. For each level edge $vw \in E(G)$ with $v, w \in V_i$ for some $i \geq 0$:

- replace $vw$ by a path $vxyw$ in $G'$ (where $x$ and $y$ are new vertices);
- add the edges $vx$ and $wy$ to $T'$ (so $x$ and $y$ are leaves in $T'$ with degree 2 in $G'$); and
- add $x$ and $y$ to $V'_{i+1}$.

For each non-tree binding edge $vw \in E(G) \setminus E(T)$ with $v \in V_i$ and $w \in V_{i+1}$ for some $i \geq 0$:

- replace $vw$ by a path $vxyzw$ in $G'$ (where $x, y, z$ are new vertices);
- add the edges $vx, xy$ and $wz$ to $T'$ (so $y$ and $z$ are leaves in $T'$ with degree 2 in $G'$); and
- add $x$ to $V'_{i+1}$, and add $y$ and $z$ to $V'_{i+2}$.

Observe that $T'$ is a bfs spanning tree of $G'$, and $G'$ is well-layered with respect to the layering $V'_0, V'_1, \ldots, V'_t$. For $i \geq 0$, let $\{V'_{i,a} : a \geq 0\}$ be the partition of $V'_i$ from Lemma 7 applied to $G'$. Let $V_{i,a} := V'_{i,a} \cap V(G)$ for $i, a \geq 0$, where $\overrightarrow{V_{i,a}}$ inherits its order from $\overrightarrow{V'_{i,a}}$.

We now prove (a). Consider a level edge $vw \in E(G)$ with $v \in V_{i,a}$ and $w \in V_{i,b}$. Let $vxyw$ be the corresponding path in $G'$. Then $xy$ is a level non-tree edge of $G'$ with $v, w \in V'_{i+1}$. By
Lemma 7(a), both $x$ and $y$ are in $V_{i+1,a}$ for some $c \geq 0$. Since $vx$ and $wy$ are tree edges in $G'$ by Lemma 7(b), we have $c - \Delta a = \alpha$ and $c - \Delta b = \beta$ for some $\alpha, \beta \in \{0, 1, \ldots, 2\Delta - 1\}$. Thus $c = \alpha + \Delta a = \beta + \Delta b$, implying $\Delta(a - b) = \beta - \alpha \leq 2\Delta - 1$ and $a - b \leq 1$. Similarly $b - a \leq 1$. Thus $|a - b| \leq 1$. This proves (a).

Property (b) follows immediately from Lemma 7(b) since a tree edge $vw$ is a tree edge in $G'$ with $v \in V_{i,a}'$ and $w \in V_{i+1,b}'$, in which case $b - \Delta a \in \{0, 1, \ldots, 2\Delta - 1\}$.

We now prove (c). Consider a binding non-tree edge $vw \in E(G) \setminus E(T)$ with $v \in V_{i,a}$ and $w \in V_{i+1,b}$. Let $vxyzw$ be the corresponding path in $G'$. Then $vx, xy$ and $wz$ are tree edges in $G'$, and $yz$ is a level edge in $G'$. Moreover, $x \in V_{i+1}'$ and $y, z \in V_{i+2}'$. Then $x \in V_{i+1,c}$ for some $c \geq 0$, and $y, z \in V_{i+2,d}$ for some $d \geq 0$ by Lemma 7(a). Since $vx, xy$ and $wz$ are tree edges, by Lemma 7(b), $c - \Delta a = \alpha$ and $d - \Delta b = \gamma$ for some $\alpha, \beta, \gamma \in \{0, 1, \ldots, 2\Delta - 1\}$. Thus $d = \beta + \Delta c = \gamma + \Delta b$, implying $\Delta(c - b) = \beta - \gamma \leq 2\Delta - 1$ and $c - b \leq 1$. Similarly, $\Delta(b - c) = \beta - \gamma \leq 2\Delta - 1$ and $b - c \leq 1$. Now $b \leq c + 1 = \alpha + \Delta a + 1$, implying $b - a \Delta \leq \alpha + 1 \leq 2\Delta$. Similarly, $\Delta(c - b) = \gamma - \beta \leq 2\Delta - 1$ and $c - b \leq 1$. Thus $b \geq c - 1 = \Delta a + \alpha - 1 \geq \Delta a - 1$, implying $b - a \Delta \geq -1$. In summary, $b - a \Delta \in \{-1, 0, \ldots, 2\Delta\}$. This proves (c).

We now prove (d). Suppose on the contrary that edges $vw$ and $pq$ in $G[V_{i,a}, V_{i,q+1}]$ are nested in $V_{i,a}V_{i,a+1}$. Without loss of generality, $v < p$ in $V_{i,a}'$ and $q < w$ in $V_{i,a+1}'$. Thus $v < p$ and $q < w$ in $C_i$. If $v < p < w$ in $C_i$, then $m(p) \geq m(w)$, which contradicts the fact that $g(w) > g(p)$. Thus $v < w < p$. If $v < q < w$ in $C_i$, then $vw$ crosses $pq$ in the drawing of $G$. Thus $q < v < w < p$ in $C_i$. Since $pq \in E(G)$, we have $m(v) \geq m(q)$, which contradicts the fact that $g(q) > g(v)$. Therefore the ordering $V_{i,a}V_{i,a+1}$ defines a 1-queue layout of $G[V_{i,a}, V_{i,a+1}]$. 

Figure 2: Creating the subdivision $G'$: (a) level edge, (b) non-tree binding edge.
This proves (d).

We now prove (e). That is, for \( i, a, b \geq 0 \), we show that the ordering \( \overrightarrow{V_i,a}V_{i+1,b} \) defines a \((6\Delta + 1)\)-queue layout of \( G[V_i,a, V_{i+1,b}] \). Each edge in \( G[V_i,a, V_{i+1,b}] \) is either direct or hooked. We first show that one queue suffices for direct edges in \( G[V_i,a, V_{i+1,b}] \). Suppose on the contrary that there are two direct edges \( uv \) and \( xy \) in \( G[V_i,a, V_{i+1,b}] \) with \( u < v \) in \( \overrightarrow{V_i,a} \) and \( y < w \) in \( \overrightarrow{V_{i+1,b}} \). Then \( uv \) and \( xy \) are drawn between \( C_i \) and \( C_{i+1} \) with \( u < w \) in \( C_i \) and \( x < y \) in \( C_{i+1} \). Thus \( uv \) and \( xy \) cross. This contradiction shows that one queue suffices for direct edges in \( G[V_i,a, V_{i+1,b}] \).

Now consider a hooked edge \( vw \) in \( G[V_i,a, V_{i+1,b}] \) with \( v \in V_i,a \) and \( w \in V_{i+1,b} \). Let \( vxyzw \) be the corresponding path in \( G' \). Then \( vx, xy \) and \( wz \) are tree edges in \( G' \), and \( yz \) is a level edge in \( G' \). Moreover, \( x \in V_{i+1,c} \) for some \( c \geq 0 \), and \( y, z \in V_{i+2,d} \) for some \( d \geq 0 \) by Lemma 7(a). Since \( vx, xy \) and \( wz \) are tree edges, by Lemma 7(b), \( c - \Delta a = \alpha \) and \( d - \Delta b = \beta \) and \( d - \Delta b = \gamma \) for some \( \alpha, \beta, \gamma \in \{0, 1, \ldots, 2\Delta - 1\} \). Thus \( d = \beta + \Delta c = \gamma + \Delta b \) implies \( \Delta(c - b) = \gamma - \beta \leq 2\Delta - 1 \) and \( c - b \leq 1 \). Similarly, \( \Delta(b - c) = \beta - \gamma \leq 2\Delta - 1 \) and \( b - c \leq 1 \). Thus \( c \in \{b - 1, b, b + 1\} \). Assign \( vw \) to queue \( Q^j_2 \) where \( \eta = c - b \). This introduces \( 6\Delta \) queues.

We claim that this is a valid queue assignment. Suppose on the contrary that there are hooked edges \( vw \) and \( pq \) in \( Q^j_2 \) with \( v < p \) in \( \overrightarrow{V_i,a} \) and \( q < w \) in \( \overrightarrow{V_{i+1,b}} \). Let \( vxyzw \) be the path corresponding to \( vw \) in \( G' \). Let \( prstq \) be the path corresponding to \( pq \) in \( G' \). Then \( x, y, z, r, s, t \) are distinct vertices, and \( x, r \in V_{i+1,b+y} \) and \( y, z, t \in V_{i+2,d} \) where \( d = \Delta b + \gamma \). By Lemma 7(d) and since \( v < p \) in \( \overrightarrow{V_i,a} \), we have \( x < r \) in \( \overrightarrow{V_{i+1,c}} \). This in turn implies that \( y < s \) in \( \overrightarrow{V_{i+2,d}} \) by Lemma 7(d). Similarly, by Lemma 7(d) and since \( q < w \) in \( \overrightarrow{V_{i+1,b}} \), we have \( t < z \) in \( \overrightarrow{V_{i+2,d}} \). This implies that \( y < t < s < z \) or \( y < t < z < s \) or \( y < s < t < z \) or \( t < z < y < s \) or \( t < y < z < s \) or \( t < y < s < z \) or \( t < y < s < z \) in \( \overrightarrow{V_{i+2,d}} \). Thus \( yz \) and \( st \) are either nest or cross in \( \overrightarrow{V_{i+2,d}} \), which contradicts Lemma 7(c). Hence no two edges in \( Q^j_2 \) nest. Therefore \((Q^j_2 : \eta \in \{-1,0,1\}, j \in \{0,1,\ldots,2\Delta - 1\}\) is a \(6\Delta\)-queue layout of the hooked edges in \( G[V_i,a, V_{i+a+1}] \) using the ordering \( \overrightarrow{V_i,a}V_{i+a+1} \). Including one queue for the direct edges, we obtain a \((6\Delta + 1)\)-queue layout of \( G[V_i,a, V_{i+a+1}] \) using the ordering \( \overrightarrow{V_i,a}V_{i+a+1} \). This proves (e).

Finally we prove (f). Consider an edge \( vw \) with both end points \( v \) and \( w \) in \( V_i,a \) for some \( i, a \geq 0 \). Then \( vw \) is a level edge. Let \( vxyzw \) be the corresponding path in \( G' \). Then \( xy \) is a level non-tree edge of \( G' \) with \( v, w \in V_{i+1,b} \). By Lemma 7(a), both \( x \) and \( y \) are in \( V_{i+1,b+y} \) for some \( b \geq 0 \). Since \( vx, xy \) and \( wz \) are tree edges in \( G' \) by Lemma 7(b), we have \( b - \Delta a \in \{0, 1, \ldots, 2\Delta - 1\} \).

Assign \( vw \) to queue \( Q_{b-\Delta a} \). Suppose on the contrary that \( v < p \) for \( v < w \) for two edges \( vw \) and \( pq \) in \( Q_{b-\Delta a} \). Let \( vxyzw \) be the path in \( G' \) corresponding to \( vw \). Let \( prstq \) be the path in \( G' \) corresponding to \( pq \). Then \( x, y, s, t \in V_{i+1,b} \). Note that \( vx, wy, ps \) and \( qt \) are tree edges in \( G' \), while \( xy \) and \( st \) are level edges in \( G' \). Since \( v < p \), we have \( x < s \) in \( \overrightarrow{V_{i+1,b}} \) by Lemma 7(d). Similarly, since \( q < q \), we have \( t < y \) in \( \overrightarrow{V_{i+1,b}} \) by Lemma 7(d). Thus \( x < s < t < y \) or \( x < t < s < y \) or \( x < t < y < s \) or \( t < y < x < s \) or \( t < x < s < y \) or \( t < x < y < s \) in \( \overrightarrow{V_{i+1,b}} \). In each case, \( xy \) and \( st \) are either nest or cross, which contradicts Lemma 7(c). Thus no two edges in \( Q_{b-\Delta a} \) are nested, and \((Q_j : j \in \{0,1,\ldots,2\Delta - 1\}\) is a \(2\Delta\)-queue layout of \( G[V_i,a] \) using ordering \( \overrightarrow{V_i,a} \). This proves (f).

We now show that Lemma 8 leads to a \(O(\Delta^2)\)-queue layout of an arbitrary planar graph.
Proof of Theorem 2. Let \( \{V_{i,a} : i \in \{0,1,\ldots,t\}, a \in \{0,1,\ldots,n_i\}\} \) be the partition of \( V(G) \) from Lemma 8. Let \( \overrightarrow{V_i} \) be the ordering \( \overrightarrow{V_0V_1\ldots V_t} \) of \( V_i \). Consider the ordering \( \overrightarrow{V_0V_1\ldots V_t} \) of \( V(G) \).

An edge with both endpoints in \( V_{i,a} \) cannot nest an edge with both endpoints in \( V_{j,b} \) for \( (i,a) \neq (j,b) \), and \( 2\Delta \) queues suffice for such edges by Lemma 8(f). An edge with endpoints in \( V_{i,a} \) and \( V_{i,a+1} \) cannot nest an edge with endpoints in \( V_{j,b} \) and \( V_{j,b+1} \) for \( (i,a) \neq (j,b) \), and one queue suffice for such edges by Lemma 8(d). By Lemma 8(a) this accounts for all level edges. Thus \( 2\Delta + 1 \) queues suffice for level edges.

For \( i,a,b \geq 0 \), by Lemma 8(e), there is a queue layout \( (Q^i_{j,a,b} : j \in \{1,2,\ldots,6\Delta + 1\}) \) of \( G[V_{i,a},V_{i+1,b}] \). For \( j \in \{1,2,\ldots,6\Delta + 1\} \) and \( \alpha \in \{-1,0,\ldots,2\Delta\} \), let

\[
Q_j^\alpha := \bigcup \{Q^i_{j,a,b} : i, a, b \geq 0, \ b - \Delta a = \alpha\}.
\]

By Lemma 8(b) and (c), this accounts for all binding edges.

Suppose that binding edges \( vw \) and \( pq \) in some \( Q_j^\alpha \) are nested with \( v < p < q < w \) in our ordering of \( V(G) \). Then \( v \in V_{i,a} \), \( p \in V_{i,b} \), \( q \in V_{i+1,c} \) and \( w \in V_{i+1,d} \) for some \( i,a,b,c,d \geq 0 \) with \( \alpha = d - \Delta a = c - \Delta b \). Thus \( d - c = \Delta(a - b) \). Since \( v < p < q < w \) in the ordering, \( d - c \geq 0 \) and \( a - b \leq 0 \). Thus \( a = b \) and \( c = d \), which contradicts Lemma 8(e).

Thus \( (2\Delta + 2)(6\Delta + 1) \) queues suffice for binding edges. In total we use \( (2\Delta + 2)(6\Delta + 1) + 2\Delta + 1 = 12\Delta^2 + 16\Delta + 3 \) queues.

We emphasise that the vertex ordering used in the proof of Theorem 2 is identical to that used by Bekos et al. [1]. Our contribution is to show that \( O(\Delta^2) \) queues suffice rather than the \( O(\Delta^6) \) queues used by Bekos et al. [1]. On the other hand, we now show that up to a constant factor our analysis is tight. That is, the above ordering can produce \( \Omega(\Delta^2) \) pairwise nested edges (a so-called 'rainbow'), which each must be assigned to a distinct queue. Start with a rooted binary tree with \( 2\Delta^2 \) leaves. Label the leaves left-right

\[
v_{1,1}, \ldots, v_{1,\Delta}; \ldots; v_{\Delta,1}, \ldots, v_{\Delta,\Delta}; w_{\Delta,\Delta}; \ldots; w_{\Delta,1}; \ldots; w_{1,\Delta}, \ldots, w_{1,1}.
\]

Subdivide the edge incident to each leaf \( v_{i,j} \). Let \( G \) be the graph obtained by adding the edge \( v_{i,j}w_{i,j} \) for \( i,j \in \{1,2,\ldots,\Delta\} \), as illustrated in Figure 3. Let \( G' \) be the well-layered graph obtained by subdividing the edges of \( G \) as described above. Thus each edge \( v_{i,j}w_{i,j} \) is replaced by a path \( v_{i,j}x_{i,j}y_{i,j}z_{i,j}w_{i,j} \). Vertices \( y_{i,j} \) and \( z_{i,j} \), which are on level 0, are joined by a level edge. Edges \( v_{i,j}x_{i,j}, x_{i,j}y_{i,j} \) and \( z_{i,j}w_{i,j} \) are tree edges. The above algorithm does not introduce any parallel edges, since each level edge joins vertices on level 0. Vertices \( v_{i,j} \) are on level 1, and vertices \( w_{i,j} \) are on level 2. It follows that \( g(w_{i,j}) = 0 \) and \( g(v_{i,j}) = i - 1 \) for all \( i,j \). Thus the vertex ordering of \( G \) produced by the above algorithm (after removing subdivision vertices) includes the sequence

\[
w_{\Delta,\Delta}; \ldots; w_{\Delta,1}; \ldots; w_{1,\Delta}; \ldots; w_{1,1}, v_{1,1}, \ldots, v_{1,\Delta}; \ldots; v_{\Delta,1}; \ldots; v_{\Delta,\Delta}.
\]

Here, \( v_{i,j}w_{i,j} \) is nested with \( v_{i',j'}w_{i',j'} \) for \( (i,j) \neq (i',j') \). Thus \( \Delta^2 \) queues are needed, as claimed. Curiously this example has maximum degree 3.
3 Graphs of Bounded Genus

This section proves our results for graphs of bounded Euler genus (Theorem 4 which implies Theorem 3). The next lemma is the key.

Lemma 9. Let $G$ be a connected graph $G$ with Euler genus $g$. For every bfs layering $V_0, V_1, \ldots, V_t$ of $G$, there is a set $Z \subseteq V(G)$ with at most $2g$ vertices in each layer $V_i$, such that $G - Z$ is planar.

Proof. Fix an embedding of $G$ in a surface of Euler genus $g$. Say $G$ has $n$ vertices, $m$ edges, and $f$ faces. By Euler’s formula, $n - m + f = 2 - g$. Let $V_0, V_1, \ldots, V_t$ be a bfs layering of $G$ rooted at some vertex $r$. Let $T$ be the corresponding bfs spanning tree. Let $D$ be the graph with $V(D) = F(G)$, where for each edge $e$ of $G - E(T)$, if $f_1$ and $f_2$ are the faces of $G$ with $e$ on their boundary, then there is an edge $f_1f_2$ in $D$. (Think of $D$ as the spanning subgraph of $G^*$ consisting of those edges that do not cross edges in $T$.) Note that $|V(D)| = f = 2 - g - n + m$ and $|E(D)| = m - (n - 1) = |V(D)| - 1 + g$. Since $T$ is a tree, $D$ is connected; see [6, Lemma 11] for a proof. Let $T^*$ be a spanning tree of $D$. Let $Q := E(D) \setminus E(T^*)$. Thus $|Q| = g$.
Say $Q = \{v_1w_1, v_2w_2, \ldots, v_gw_g\}$. For $i \in \{1, 2, \ldots, g\}$, let $Z_i$ be the union of the $v_ir$-path and the $w_i, r$-path in $T$, plus the edge $v_iw_i$. Let $Z$ be $Z_1 \cup Z_2 \cup \cdots \cup Z_g$. Say $Z$ has $p$ vertices and $q$ edges. Since $Z$ consists of a subtree of $T$ plus the $g$ edges in $Q$, we have $q = p - 1 + g$.

We now describe how to ‘cut’ along the edges of $Z$ to obtain a new graph $G'$; see Figure 4. First, each edge $e$ of $Z$ is replaced by two edges $e'$ and $e''$ in $G'$. Each vertex of $G$ incident with no edges in $Z$ is untouched. Consider a vertex $v$ of $G$ incident with edges $e_1, e_2, \ldots, e_d$ in $Z$ in clockwise order. In $G'$ replace $v$ by new vertices $v_1, v_2, \ldots, v_d$, where $v_i$ is incident with $e_i, e_{i+1}$ and all the edges incident with $v$ clockwise from $e_i$ to $e_{i+1}$ (exclusive). Here
$e_{d+1}$ means $e_1$ and $e'_{d+1}$ means $e'_1$. This operation defines a cyclic ordering of the edges in $G'$ incident with each vertex (where $e'_{i+1}$ is followed by $e'_i$ in the cyclic order at $v_i$). This in turn defines an embedding of $G'$ in some orientable surface. (Note that if $G$ is embedded in a non-orientable surface, then the edge signatures for $G$ are ignored in the embedding of $G'$.)

\[
\begin{align*}
\text{deg}_Z(v) &= 1 & e_1 \quad & e'_1 \\
\text{deg}_Z(v) &= 2 & e_1 \quad & e'_1 \\
\text{deg}_Z(v) &= 3 & e_1 \quad & e'_1
\end{align*}
\]

Figure 4: Cutting the blue edges in $Z$ at each vertex.

Say $G'$ has $n'$ vertices and $m'$ edges, and the embedding of $G'$ has $f'$ faces and Euler genus $g'$. Each vertex $v$ in $G$ with degree $d$ in $Z$ is replaced by $d$ vertices in $G'$. Each edge in $Z$ is replaced by two edges in $G'$, while each edge of $G - E(Z)$ is maintained in $G'$. Thus

\[
n' = n - p + \sum_{v \in V(G)} \text{deg}_Z(v) = n + 2q - p = n + 2(p - 1 + g) - p = n + p - 2 + 2g
\]

and $m' = m + q = m - p - 1 + g$. Each face of $G$ is preserved in $G'$. Say $r$ new faces are created by the cutting. Thus $f' = f + r$. Since $D$ is connected, it follows that $G'$ is connected. By Euler's formula, $n' - m' + f' = 2 - g'$. Thus

\[
(n + p - 2 + 2g) - (m + p + 1 + g) + (f + r) = 2 - g',
\]

implying

\[
(n - m + f) - 1 + g + r = 2 - g'.
\]

Hence $(2 - g) - 1 + g + r = 2 - g'$, implying $g' = 1 - r$. Since $r \geq 1$ and $g' \geq 0$, we have $g' = 0$ and $r = 1$. Therefore $G'$ is planar.
We now prove Theorem 4, which says that if \( G - V(Z) \) is a subgraph of \( G' \), and \( G - V(Z) \) is planar. By construction, each path \( Z_i \) has at most two vertices in each layer \( V_j \). Thus \( Z \) has at most \( 2g \) vertices in each \( V_j \). \( \square \)

We need the following lemma of independent interest.

**Lemma 10.** If a graph \( G \) has a \( k \)-queue layout, and \( V_0, V_1, \ldots, V_t \) is a layering of \( G \), then \( G \) has a \( 3k \)-queue layout using ordering \( V_0, V_1, \ldots, V_t \).

**Proof.** Say \( E_1, E_2, \ldots, E_k \) is the edge-partition and \( \preceq \) is the ordering of \( V(G) \) in a \( k \)-queue layout of \( G \). For \( a \in \{1, 2, \ldots, k\} \), let \( X_a \) be the set of edges \( vw \in Q_a \) with \( v, w \in V_i \) for some \( i \); let \( Y_a \) be the set of edges \( vw \in Q_a \) with \( v \in V_i \) and \( w \in V_{i+1} \) for some \( i \); and let \( Z_a \) be the set of edges \( vw \in Q_a \) with \( v, w \in V_i \) for some \( i \). Then \( X_1, Y_1, Z_1, X_2, Y_2, Z_2, \ldots, X_k, Y_k, Z_k \) is a partition of \( E(G) \).

Let \( \preceq' \) be the ordering \( V_0, V_1, \ldots, V_t \) of \( V(G) \) where each \( V_i \) is ordered by \( \preceq \). No two edges in some set \( X_a \) are nested in \( \preceq' \), as otherwise the same two edges would be in \( Q_a \) and would be nested in \( \preceq \). Suppose that \( v \preceq' p \preceq' q \preceq' w \) for some edges \( vw, pq \in Y_a \) and \( p, q \in V_{i+1} \) for some \( i \) and \( v, w \) are in \( V_i \) and \( v, w \) are in \( V_{i+1} \) for some \( i \). Then \( X_1, Y_1, Z_1, X_2, Y_2, Z_2, \ldots, X_k, Y_k, Z_k \) is a \( 3k \)-queue layout of \( G \).

We now prove Theorem 4, which says that if \( G \) is a hereditary class of graphs, such that every planar graph in \( G \) has queue-number at most \( k \), then every graph in \( G \) with Euler genus \( g \) has queue-number at most \( 3k + 4g \).

**Proof of Theorem 4.** Let \( G \) be a graph in \( G \) with Euler genus \( g \). Since the queue-number of \( G \) equals the maximum queue-number of the connected components of \( G \), we may assume that \( G \) is connected. Let \( V_0, V_1, \ldots, V_t \) be a bfs layering of \( G \). By Lemma 9, there is a set \( Z \subseteq V(G) \) with at most \( 2g \) vertices in each layer \( V_a \) such that \( G - Z \) is planar. Since \( G \) is hereditary, \( G - Z \in \mathcal{G} \), and by assumption \( G - Z \) has a \( k \)-queue layout. Note that \( V_0 \setminus Z, V_1 \setminus Z, \ldots, V_t \setminus Z \) is a layering of \( G - Z \). By Lemma 10, \( G - Z \) has a \( 3k \)-queue layout using ordering \( V_0 \setminus Z, V_1 \setminus Z, \ldots, V_t \setminus Z \). Recall that \( |V_j \cap Z| \leq 2g \) for all \( j \in \{0, 1, \ldots, t\} \). Let \( \preceq \) be the ordering

\[
V_0 \cap Z, V_0 \setminus Z, V_1 \cap Z, V_1 \setminus Z, \ldots, V_t \cap Z, V_t \setminus Z
\]

of \( V(G) \), where each set \( V_j \cap Z \) is ordered arbitrarily, and each set \( V_j \setminus Z \) is ordered according to the above \( 3k \)-queue layout of \( G - Z \). Edges of \( G - Z \) inherit their queue assignment. We now assign edges incident with vertices in \( Z \) to queues. For \( i \in \{1, \ldots, 2g\} \) and odd \( j \geq 1 \), put each edge incident with the \( i \)-th vertex in \( V_j \cap Z \) in a new queue \( S_i \). For \( i \in \{1, \ldots, 2g\} \) and even \( j \geq 1 \), put each edge incident with the \( i \)-th vertex in \( V_j \cap Z \) (not already assigned to a queue) in a new queue \( T_i \). Suppose that two edges \( vw \) and \( pq \) in \( S_i \) are nested, where \( v \prec p \prec q \prec w \). Say \( v \in V_a \) and \( p \in V_b \) and \( q \in V_c \) and \( w \in V_d \). By construction, \( a \leq b \leq c \leq d \). Since \( vw \) is an edge, \( d \leq a + 1 \). At least one endpoint of \( vw \) is in \( V_j \cap Z \) for some odd \( j \), and one endpoint of \( pq \) is in \( V_l \cap Z \) for some odd \( l \). Since \( v, w, p, q \) are distinct, \( j \neq l \). Thus \( |i - j| \geq 2 \). This is a contradiction since \( a \leq b \leq c \leq d \leq a + 1 \). Thus \( S_i \) is a queue. Similarly \( T_i \) is a queue. Hence this step introduces \( 4g \) new queues. We obtain a \( (3k + 4g) \)-queue layout of \( G \). \( \square \)
4 Excluded Minors

Whether the result of Bekos et al. [1] can be generalised for arbitrary excluded minors is an interesting question. That is, do graphs excluding a fixed minor and with bounded degree have bounded queue-number? It might even be true that graphs excluding a fixed minor have bounded queue-number.

References


Dujmović and Wood [8] proved that if some \((\leq c)\)-subdivision of a graph \(G\) has a \(k\)-queue layout, then \(G\) has a \(O(k^{2c})\)-queue layout. Here we improve this bound to \(O(k^{c+1})\).

**Lemma 11.** For every \((\leq c)\)-subdivision \(G'\) of a graph \(G\), if \(G'\) has a \(k\)-queue layout using vertex ordering \(\leq\), then \(G\) has a \(\frac{2k}{2k-1}((2k)^{c+1} - 1)\)-queue layout using \(\preceq\) restricted to \(V(G)\).

**Proof.** Let \(E_1, \ldots, E_k\) be the partition of \(E(G')\) into queues. For each edge \(xy \in E_i\), let \(q(xy) := i\). For distinct vertices \(a, b \in V(G')\), let \(f(a, b) := 1\) if \(a < b\) and let \(f(a, b) := -1\) if \(b < a\). For \(\ell \in \{0, 1, \ldots, c\}\), let \(X_\ell\) be the set of edges in \(G\) that are subdivided exactly \(\ell\) times in \(G'\). We will use distinct sets of queues for the \(X_\ell\). Consider an edge \(vw\) in \(X_\ell\) with \(v < w\). Say \(v = x_0, x_1, \ldots, x_\ell, x_{\ell+1} = w\) is the corresponding path in \(G'\). Let \(f(vw) := (f(x_0, x_1), \ldots, f(x_\ell, x_{\ell+1}))\) and \(q(vw) := (q(x_0, x_1), \ldots, q(x_\ell, x_{\ell+1}))\). Consider edges \(vw, pq \in E_\ell\) with \(v, w, p, q\) distinct and \(f(vw) = f(pq)\) and \(g(vw) = g(pq)\). Assume \(v < p\). Say \(v = x_0, x_1, \ldots, x_\ell, x_{\ell+1} = w\) and \(p = y_0, y_1, \ldots, y_\ell, x_{\ell+1} = q\) are the paths respectively corresponding to \(vw\) and \(pq\) in \(G'\). Thus \(f(x_i, x_{i+1}) = f(y_i, y_{i+1})\) and \(q(x_i, x_{i+1}) = q(y_i, y_{i+1})\) for \(i \in \{0, 1, \ldots, \ell\}\). Thus \(x_i, x_{i+1}\) and \(y_i, y_{i+1}\) are not nested. Since \(v = x_0 < y_0 = p\), it follows by induction that \(x_i < y_i\) for \(i \in \{0, 1, \ldots, \ell + 1\}\). In particular, \(w = x_{\ell+1} < y_{\ell+1} = q\). Thus \(vw\) and \(pq\) are not nested. There are \(2^{\ell+1}\) values for \(f\), and \(k^{\ell+1}\) values for \(q\). Thus \((2k)^{\ell+1}\) queues suffice for \(X_\ell\). In total, \(\sum_{\ell=0}^c (2k)^{\ell+1} = \frac{2k}{2k-1}((2k)^{c+1} - 1)\) queues suffice for \(G\). \(\square\)