Complete graph minors and the graph minor structure theorem

Gwenaël Joret\textsuperscript{a}, David R. Wood\textsuperscript{b}

\textsuperscript{a} Département d’Informatique, Université Libre de Bruxelles, Brussels, Belgium
\textsuperscript{b} Department of Mathematics and Statistics, The University of Melbourne, Melbourne, Australia

\begin{abstract}

The graph minor structure theorem by Robertson and Seymour shows that every graph that excludes a fixed minor can be constructed by a combination of four ingredients: graphs embedded in a surface of bounded genus, a bounded number of vortices of bounded width, a bounded number of apex vertices, and the clique-sum operation. This paper studies the converse question: What is the maximum order of a complete graph minor in a graph constructed using these four ingredients? Our main result answers this question up to a constant factor.

\end{abstract}

1. Introduction

Robertson and Seymour \cite{RobertsonSeymour} proved a rough structural characterization of graphs that exclude a fixed minor. It says that such a graph can be constructed by a combination of four ingredients: graphs embedded in a surface of bounded genus, a bounded number of vortices of bounded width, a bounded number of apex vertices, and the clique-sum operation. Moreover, each of these ingredients is essential.

In this paper, we consider the converse question: What is the maximum order of a complete graph minor in a graph constructed using these four ingredients? Our main result answers this question up to a constant factor.

To state this theorem, we now introduce some notation; see Section 2 for precise definitions. For a graph $G$, let $\eta(G)$ denote the maximum integer $n$ such that the complete graph $K_n$ is a minor.
of $G$, sometimes called the Hadwiger number of $G$. For integers $g, p, k \geq 0$, let $\mathcal{G}(g, p, k)$ be the set of graphs obtained by adding at most $p$ vertices, each with width at most $k$, to a graph embedded in a surface of Euler genus at most $g$. For an integer $a \geq 0$, let $\mathcal{G}(g, p, k, a)$ be the set of graphs $G$ such that $G \setminus A \in \mathcal{G}(g, p, k)$ for some set $A \subseteq V(G)$ with $|A| \leq a$. The vertices in $A$ are called apex vertices. Let $\mathcal{G}(g, p, k, a)^+$ be the set of graphs obtained from clique-sums of graphs in $\mathcal{G}(g, p, k, a)$.

The graph minor structure theorem of Robertson and Seymour [8] says that for every integer $t \geq 1$, there exist integers $g, p, k, a \geq 0$, such that every graph $G$ with $\eta(G) \leq t$ is in $\mathcal{G}(g, p, k, a)^+$. We prove the following converse result.

**Theorem 1.1.** For some constant $c > 0$, for all integers $g, p, k, a \geq 0$, for every graph $G$ in $\mathcal{G}(g, p, k, a)^+$,

$$\eta(G) \leq a + c(k + 1)\sqrt{g + p} + c.$$  

Moreover, for some constant $c' > 0$, for all integers $g, a \geq 0$ and $p \geq 1$ and $k \geq 2$, there is a graph $G$ in $\mathcal{G}(g, p, k, a)$ such that

$$\eta(G) \geq a + c'k\sqrt{g + p}.$$  

Let $\text{RS}(G)$ be the minimum integer $k$ such that $G$ is a subgraph of a graph in $\mathcal{G}(k, k, k, k)^+$. The graph minor structure theorem [8] says that $\text{RS}(G) \leq f(\eta(G))$ for some function $f$ independent of $G$. Conversely, Theorem 1.1 implies that $\eta(G) \leq f'(\text{RS}(G))$ for some (much smaller) function $f'$. In this sense, $\eta$ and $\text{RS}$ are “tied”. Note that such a function $f'$ is widely understood to exist (see for instance Diestel [2, p. 340] and Lovász [5]). However, the authors are not aware of any proof. In addition to proving the existence of $f'$, this paper determines the best possible function $f'$ (up to a constant factor).

Following the presentation of definitions and other preliminary results in Section 2, the proof of the upper and lower bounds in Theorem 1.1 are respectively presented in Sections 3 and 4.

**2. Definitions and preliminaries**

All graphs in this paper are finite and simple, unless otherwise stated. Let $V(G)$ and $E(G)$ denote the vertex and edge sets of a graph $G$. For background graph theory see [2].

A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges. (Note that, since we only consider simple graphs, loops and parallel edges created during an edge contraction are deleted.) An $H$-model in $G$ is a collection $\{S_x : x \in V(H)\}$ of pairwise vertex-disjoint connected subgraphs of $G$ (called branch sets) such that, for every edge $xy \in E(H)$, some edge in $G$ joins a vertex in $S_x$ to a vertex in $S_y$. Clearly, $H$ is a minor of $G$ if and only if $G$ contains an $H$-model. For a recent survey on graph minors see [4].

Let $G[k]$ denote the lexicographic product of $G$ with $K_k$, namely the graph obtained by replacing each vertex $v$ of $G$ with a clique $C_v$ of size $k$, where for each edge $vw \in E(G)$, each vertex in $C_v$ is adjacent to each vertex in $C_w$. Let $tw(G)$ be the treewidth of a graph $G$; see [2] for background on treewidth.

**Lemma 2.1.** For every graph $G$ and integer $k \geq 1$, every minor of $G[k]$ has minimum degree at most $k \cdot tw(G) + k - 1$.

**Proof.** A tree decomposition of $G$ can be turned into a tree decomposition of $G[k]$ in the obvious way: in each bag, replace each vertex by its $k$ copies in $G[k]$. The size of each bag is multiplied by $k$; hence the new tree decomposition has width at most $k(w + 1) - 1 = kw + k - 1$, where $w$ denotes the width of the original decomposition. Let $H$ be a minor of $G[k]$. Since treewidth is minor-monotone,

$$tw(H) \leq tw(G[k]) \leq k \cdot tw(G) + k - 1.$$  

The claim follows since the minimum degree of a graph is at most its treewidth. □

Note that Lemma 2.1 can be written in terms of contraction degeneracy; see [1,3].

Let $G$ be a graph and let $\Omega = \{v_1, v_2, \ldots, v_\ell\}$ be a circular ordering of a subset of the vertices of $G$. We write $V(\Omega)$ for the set $\{v_1, v_2, \ldots, v_\ell\}$. A circular decomposition of $G$ with perimeter $\Omega$ is
a multiset \( \{C(w) \subseteq V(G) : w \in V(\Omega) \} \) of subsets of vertices of \( G \), called bags, that satisfy the following properties:

- every vertex \( w \in V(\Omega) \) is contained in its corresponding bag \( C(w) \);
- for every vertex \( u \in V(G) \setminus V(\Omega) \), there exists \( w \in V(\Omega) \) such that \( u \) is in \( C(w) \);
- for every edge \( e \in E(G) \), there exists \( w \in V(\Omega) \) such that both endpoints of \( e \) are in \( C(w) \); and
- for each vertex \( u \in V(G) \), if \( u \in C(v_i), C(v_j) \) with \( i < j \) then \( u \in C(v_i), \ldots, C(v_j) \) or \( u \in C(v_{j+1}), \ldots, C(v_{i-1}) \).

(The last condition says that the bags in which \( u \) appears correspond to consecutive vertices of \( \Omega \).) The width of the decomposition is the maximum cardinality of a bag minus 1. The ordered pair \( (G, \Omega) \) is called a vortex; its width is the minimum width of a circular decomposition of \( G \) with perimeter \( \Omega \).

A surface is a non-null compact connected 2-manifold without boundary. Recall that the Euler genus of a surface \( \Sigma \) is \( 2 - \chi(\Sigma) \), where \( \chi(\Sigma) \) denotes the Euler characteristic of \( \Sigma \). Thus the orientable surface with \( h \) handles has Euler genus \( 2h \), and the non-orientable surface with \( c \) cross-caps has Euler genus \( c \). The boundary of an open disc \( D \subset \Sigma \) is denoted by \( \partial D \).

See [6] for basic terminology and results about graphs embedded in surfaces. When considering a graph \( G \) embedded in a surface \( \Sigma \), we use \( G \) both for the corresponding abstract graph and for the subset of \( \Sigma \) corresponding to the drawing of \( G \). An embedding of \( G \) in \( \Sigma \) is 2-cell if every face is homeomorphic to an open disc.

Recall Euler’s formula: if an \( n \)-vertex \( m \)-edge graph is 2-cell embedded with \( f \) faces in a surface of Euler genus \( g \), then \( n - m + f = 2 - g \). Since \( 2m \geq 3f \),

\[
m \leq 3n + 3g - 6,
\]

which in turn implies the following well-known upper bound on the Hadwiger number.

**Lemma 2.2.** If a graph \( G \) has an embedding in a surface \( \Sigma \) with Euler genus \( g \), then

\[
\eta(G) \leq \sqrt{6g + 4}.
\]

**Proof.** Let \( t := \eta(G) \). Then \( K_t \) has an embedding in \( \Sigma \). It is well known that this implies that \( K_t \) has a 2-cell embedding in a surface of Euler genus at most \( g \) (see [6]). Hence \( \left( \frac{t}{3} \right) \leq 3t + 3g - 6 \) by (1). In particular, \( t \leq \sqrt{6g + 4} \). \( \Box \)

Let \( G \) be an embedded multigraph, and let \( F \) be a facial walk of \( G \). Let \( v \) be a vertex of \( F \) with degree more than 3. Let \( e_1, \ldots, e_d \) be the edges incident to \( v \) in clockwise order around \( v \), such that \( e_1 \) and \( e_d \) are in \( F \). Let \( G' \) be the embedded multigraph obtained from \( G \) as follows. First, introduce a path \( x_1, \ldots, x_d \) of new vertices. Then for each \( i \in [1, d] \), replace \( v \) as the endpoint of \( e_i \) by \( x_i \). The clockwise ordering around \( x_1 \) is as described in Fig. 1. Finally delete \( v \). We say that \( G' \) is obtained from \( G \) by **splitting** \( v \) at \( F \). Each vertex \( x_i \) is said to belong to \( v \). By construction, \( x_i \) has degree at most 3. Observe that there is a one-to-one correspondence between facial walks of \( G \) and \( G' \). This process can be repeated at each vertex of \( F \). The embedded graph that is obtained is called
the splitting of $G$ at $F$. And more generally, if $F_1, \ldots, F_p$ are pairwise vertex-disjoint facial walks of $G$, then the embedded graph that is obtained by splitting each $F_i$ is called the splitting of $G$ at $F_1, \ldots, F_p$. (Clearly, the splitting of $G$ at $F_1, \ldots, F_p$ is unique.)

For $g, p, k \geq 0$, a graph $G$ is $(g, p, k)$-almost embeddable if there exists a graph $G_0$ embedded in a surface $\Sigma$ of Euler genus at most $g$, and there exist $q \leq p$ vertices $(G_1, \Omega_1), \ldots, (G_q, \Omega_q)$, each of width at most $k$, such that

- $G = G_0 \cup G_1 \cup \cdots \cup G_q$;
- the graphs $G_1, \ldots, G_q$ are pairwise vertex-disjoint;
- $V(G_i) \cap V(G_0) = V(\Omega_i)$ for all $i \in \{1, q\}$; and
- there exist $q$ disjoint closed discs in $\Sigma$ whose interiors $D_1, \ldots, D_q$ are disjoint from $G_0$, whose boundaries meet $G_0$ only in vertices, and such that $\text{bd}(D_i) \cap V(G_0) = V(\Omega_i)$ and the cyclic ordering $\Omega_i$ is compatible with the natural cyclic ordering of $V(\Omega_i)$ induced by $\text{bd}(D_i)$, for all $i \in \{1, q\}$.

Let $\mathcal{G}(g, p, k)$ be the set of $(g, p, k)$-almost embeddable graphs. Note that $\mathcal{G}(g, 0, 0)$ is exactly the class of graphs with Euler genus at most $g$. Also note that the literature defines a graph to be $h$-almost embeddable if it is $(h, h, h)$-almost embeddable. To enable more accurate results we distinguish the three parameters.

Let $G_1$ and $G_2$ be disjoint graphs. Let $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$ be cliques of the same cardinality in $G_1$ and $G_2$ respectively. A clique-sum of $G_1$ and $G_2$ is any graph obtained from $G_1 \cup G_2$ by identifying $v_i$ with $w_i$ for each $i \in \{1, k\}$, and possibly deleting some of the edges $v_i v_j$.

The above definitions make precise the definition of $\mathcal{G}(g, p, k, a)^+$ given in the introduction. We conclude this section with an easy lemma on clique-sums.

**Lemma 2.3.** If a graph $G$ is a clique-sum of graphs $G_1$ and $G_2$, then

$$\eta(G) \leq \max\{\eta(G_1), \eta(G_2)\}.$$

**Proof.** Let $t := \eta(G)$ and let $S_1, \ldots, S_t$ be the branch sets of a $K_t$-model in $G$. If some branch set $S_i$ were contained in $G_1 \setminus V(G_2)$, and some branch set $S_j$ were contained in $G_2 \setminus V(G_1)$, then there would be no edge between $S_i$ and $S_j$ in $G$, which is a contradiction. Thus every branch set intersects $V(G_1)$, or every branch set intersects $V(G_2)$. Suppose that every branch set intersects $V(G_1)$. For each branch set $S_i$ that intersects $G_1 \cap G_2$, remove from $S_i$ all vertices in $V(G_2) \setminus V(G_1)$. Since $V(G_1) \cap V(G_2)$ is a clique in $G_1$, the modified branch sets yield a $K_t$-model in $G_1$. Hence $t \leq \eta(G_1)$. By symmetry, $t \leq \eta(G_2)$ in the case that every branch set intersects $G_2$. Therefore $\eta(G) \leq \max\{\eta(G_1), \eta(G_2)\}$. □

3. **Proof of upper bound**

The aim of this section is to prove the following theorem.

**Theorem 3.1.** For all integers $g, p, k \geq 0$, every graph $G$ in $\mathcal{G}(g, p, k)$ satisfies

$$\eta(G) \leq 48(k + 1)\sqrt{g + p} + \sqrt{6g} + 5.$$

Combining this theorem with Lemma 2.3 gives the following quantitative version of the first part of Theorem 1.1.

**Corollary 3.2.** For every graph $G \in \mathcal{G}(g, p, k, a)^+$,

$$\eta(G) \leq a + 48(k + 1)\sqrt{g + p} + \sqrt{6g} + 5.$$

**Proof.** Let $G \in \mathcal{G}(g, p, k, a)^+$. Lemma 2.3 implies that $\eta(G) \leq \eta(G')$ for some graph $G' \in \mathcal{G}(g, p, k, a)$. Clearly, $\eta(G') \leq \eta(G' \setminus A) + a$, where $A$ denotes the (possibly empty) apex set of $G'$. Since $G' \setminus A \in \mathcal{G}(g, p, k)$, the claim follows from Theorem 3.1. □
The proof of Theorem 3.1 uses the following definitions. Two subgraphs $A$ and $B$ of a graph $G$ touch if $A$ and $B$ have at least one vertex in common or if there is an edge in $G$ between a vertex in $A$ and another vertex in $B$. We generalize the notion of minors and models as follows. For an integer $k \geq 1$, a graph $H$ is said to be an $(H, k)$-minor of a graph $G$ if there exists a collection $\{S_x \mid x \in V(H)\}$ of connected subgraphs of $G$ (called branch sets), such that $S_x$ and $S_y$ touch in $G$ for every edge $xy \in E(H)$, and every vertex of $G$ is included in at most $k$ branch sets in the collection. The collection $\{S_x \mid x \in V(H)\}$ is called an $(H, k)$-model in $G$. Note that for $k = 1$ this definition corresponds to the usual notions of $H$-minor and $H$-model. As shown in the next lemma, this generalization provides another way of considering $H$-minors in $G[k]$, the lexicographic product of $G$ with $K_k$. (The easy proof is left to the reader.)

**Lemma 3.3.** Let $k \geq 1$. A graph $H$ is an $(H, k)$-minor of a graph $G$ if and only if $H$ is a minor of $G[k]$.

For a surface $\Sigma$, let $\Sigma_c$ be $\Sigma$ with $c$ cuffs added; that is, $\Sigma_c$ is obtained from $\Sigma$ by removing the interior of $c$ pairwise disjoint closed discs. (It is well known that the locations of the discs are irrelevant.) When considering graphs embedded in $\Sigma$, we require the embedding to be 2-cell. We emphasize that this is a non-standard and relatively strong requirement; in particular, it implies that the graph is connected, and the boundary of each cuff intersects the graph in a cycle. Such cycles are called cuff-cycles.

For $g \geq 0$ and $c \geq 1$, a graph $G$ is $(g, c)$-embedded if $G$ has maximum degree $\Delta(G) \leq 3$ and $G$ is embedded in a surface of Euler genus at most $g$ with at most $c$ cuffs added, such that every vertex of $G$ lies on the boundary of the surface. (Thus the cuff-cycles induce a partition of the whole vertex set.) The graph $G$ is $(g, c)$-embeddable if there exists such an embedding. Note that if $C$ is a contractible cycle in a $(g, c)$-embedded graph, then the closed disc bounded by $C$ is uniquely determined even if the underlying surface is the sphere (because there is at least one cuff).

**Lemma 3.4.** For every graph $G \in G(g, p, k)$ there exists a $(g, p)$-embeddable graph $H$ with $\eta(G) \leq \eta(H[k + 1]) + \sqrt{6g} + 4$.

**Proof.** Let $t := \eta(G)$. Let $S_1, \ldots, S_t$ be the branch sets of a $K_t$-model in $G$. Since $\eta(G)$ equals the Hadwiger number of some connected component of $G$, we may assume that $G$ is connected. Thus we may ‘grow’ the branch sets until $V(S_1) \cup \cdots \cup V(S_t) = V(G)$.

Write $G = G_0 \cup G_1 \cup \cdots \cup G_q$ as in the definition of $(g, p, k)$-almost embeddable graphs. Thus $G_0$ is embedded in a surface $\Sigma$ of Euler genus at most $g$, and $(G_1, \Omega_1), \ldots, (G_q, \Omega_q)$ are pairwise vertex-disjoint vortices of width at most $k$, for some $q \leq p$. Let $D_1, \ldots, D_q$ be the proper interiors of the closed discs of $\Sigma$ appearing in the definition.

Define $r$ and reorder the branch sets, so that each $S_i$ contains a vertex of some vortex if and only if $i \leq r$. If $t > r$, then $S_{r+1}, \ldots, S_t$ is a $K_{t-r}$-model in the embedded graph $G_0$, and hence $t - r \leq \sqrt{6g} + 4$ by Lemma 2.2. Therefore, it suffices to show that $r \leq \eta(H[k + 1])$ for some $(g, p)$-embeddable graph $H$.

Modify $G_0$, $G_0$, and the branch sets $S_1, \ldots, S_t$ as follows. First, remove from $G_0$ and $G_0$ every vertex of $S_i$ for all $i \in [r + 1, t]$. Next, while some branch set $S_i$ ($i \in [1, r]$) contains an edge $uv$ in $G_0$ where $u$ is in some vortex, but $v$ is in no vortex, contract the edge $uv$ into $u$ (this operation is done in $S_1, G, G_0$). The above operations on $G_0$ are carried out in its embedding in the natural way. Now apply a final operation on $G$ and $G_0$; for each $j \in [1, q]$ and each pair of consecutive vertices $a$ and $b$ in $\Omega_j$, remove the edge $ab$ if it exists, and embed the edge $ab$ as a curve on the boundary of $D_j$.

When the above procedure is finished, every vertex of the modified $G_0$ belongs to some vortex. It should be clear that the modified branch sets $S_1, \ldots, S_t$ still provide a model of $K_t$ in $G$. Also observe that $G_0$ is connected; this is because $V(\Omega_j)$ induces a connected subgraph for each $j \in [1, q]$, and each vortex intersects at least one branch set $S_i$ with $i \in [1, r]$. By the final operation, the boundary of the disc $D_j$ of $\Sigma$ intersects $G_0$ in a cycle $C_j$ of $G_0$ with $V(C_j) = V(\Omega_j)$ and such that $C_j$ (with the right orientation) defines the same cyclic ordering as $\Omega_j$ for every $j \in [1, q]$.

We claim that $G_0$ can be 2-cell embedded in a surface $\Sigma'$ with Euler genus at most that of $\Sigma$, such that each $C_j$ ($j \in [1, q]$) is a facial cycle of the embedding. This follows by considering the combinatorial embedding (that is, circular ordering of edges incident to each vertex, and edge signatures)
determined by the embedding in $\Sigma$ (see [6]), and observing that under the above operations, the Euler genus of the combinatorial embedding does not increase, and facial walks remain facial walks (so that each $C_j$ is a facial cycle). Now, removing the $q$ open discs corresponding to these facial cycles gives a 2-cell embedding of $G_0$ in $\Sigma'$.

We now prove that $\eta(G_0[k+1]) > r$. For every $i \in [1, q]$, let $\{C(w) \subseteq V(G_i) : w \in V(\Omega_i)\}$ denote a circular decomposition of width at most $k$ of the $i$-th vortex. For each $i \in [1, r]$, mark the vertices $w$ of $G_0$ for which $S_i$ contains at least one vertex in the bag $C(w)$ (recall that every vertex of $G_0$ is in the perimeter of some vortex), and define $S'_i$ as the subgraph of $G_0$ induced by the marked vertices. It is easily checked that $S'_i$ is a connected subgraph of $G_0$. Also, $S'_i$ and $S'_j$ touch in $G_0$ for all $i \neq j$.

Finally, a vertex of $G_0$ will be marked at most $k + 1$ times, since each bag has size at most $k + 1$. It follows that $\{S'_1, \ldots, S'_r\}$ is a $(K_r, k + 1)$-model in $G_0$, which implies by Lemma 3.3 that $K_r$ is minor of $G_0[k + 1]$, as claimed.

Finally, let $H$ be obtained from $G_0$ by splitting each vertex $v$ of degree more than 3 along the cuff boundary that contains $v$. (Clearly the notion of splitting along a face extends to splitting along a cuff.) By construction, $\Delta(H) \leq 3$ and $H$ is $(g, q)$-embedded. The $(K_r, k + 1)$-model of $G_0$ constructed above can be turned into a $(K_r, k + 1)$-model of $H$ by replacing each branch set $S'_i$ by the union, taken over the vertices $v \in V(S'_i)$, of the set of vertices in $H$ that belong to $v$. Hence $r \leq \eta(G_0[k + 1]) \leq \eta(H[k + 1])$. 

We need to introduce a few definitions. Consider a $(g, c)$-embedded graph $G$. An edge $e$ of $G$ is said to be a cuff or a non-cuff edge, depending on whether $e$ is included in a cuff-cycle. Every non-cuff edge has its two endpoints in either the same cuff-cycle or in two distinct cuff-cycles. Since $\Delta(G) \leq 3$, the set of non-cuff edges is a matching.

A cycle $C$ of $G$ is an $F$-cycle where $F$ is the set of non-cuff edges in $C$. A non-cuff edge $e$ is contractible if there exists a contractible $\{e\}$-cycle, and is noncontractible otherwise. Two non-cuff edges $e$ and $f$ are homotopic if $G$ contains a contractible $\{e, f\}$-cycle. Observe that if $e$ and $f$ are homotopic, then they have their endpoints in the same cuff-cycle(s), as illustrated in Fig. 2. We now prove that homotopy defines an equivalence relation on the set of noncontractible non-cuff edges of $G$.

**Lemma 3.5.** Let $G$ be a $(g, c)$-embedded graph, and let $e_1, e_2, e_3$ be distinct noncontractible non-cuff edges of $G$, such that $e_1$ is homotopic to $e_2$ and to $e_3$. Then $e_2$ and $e_3$ are also homotopic. Moreover, given a contractible $\{e_1, e_2\}$-cycle $C_{12}$ bounding a closed disc $D_{12}$, for some distinct $i, j \in \{1, 2, 3\}$, there is a contractible $\{e_i, e_j\}$-cycle bounding a closed disc containing $e_1, e_2, e_3$ and all noncontractible non-cuff edges of $G$ contained in $D_{12}$.

**Proof.** Let $C_{13}$ be a contractible $\{e_1, e_3\}$-cycle. Let $P_{12}, Q_{12}$ be the two paths in the graph $C_{12} \setminus \{e_1, e_2\}$. Let $P_{13}, Q_{13}$ be the two paths in the graph $C_{13} \setminus \{e_1, e_3\}$. Exchanging $P_{13}$ and $Q_{13}$ if necessary, we may denote the endpoints of $e_i$ ($i = 1, 2, 3$) by $u_i, v_i$ so that the endpoints of $P_{12}$ and $P_{13}$ are $u_1, u_2$ and $u_1, u_3$, respectively, and similarly, the endpoints of $Q_{12}$ and $Q_{13}$ are $v_1, v_2$ and $v_1, v_3$, respectively.

Let $D_{13}$ be the closed disc bounded by $C_{13}$. Each edge of $P_{1i}$ and $Q_{1i}$ ($i = 2, 3$) is on the boundaries of both $D_{1i}$ and a cuff; it follows that every non-cuff edge of $G$ incident to an internal vertex of $P_{1i}$
or $Q_{1i}$ is entirely contained in the disc $D_{1i}$. The paths $P_{12}$ and $P_{13}$ are subgraphs of a common cuff-cycle $C_P$, and $Q_{12}$ and $Q_{13}$ are subgraphs of a common cuff-cycle $C_Q$. Note that these two cuff-cycles could be the same.

Recall that non-cuff edges of $G$ are independent (that is, have no endpoint in common). This will be used in the arguments below. We claim that

---

**Claim:**

Every noncontractible non-cuff edge $f$ contained in $D_{1i}$ has one endpoint in $P_{1i}$ and the other in $Q_{1i}$, for each $i \in \{2, 3\}$.

The claim is immediate if $f \in \{e_1, e_3\}$. Now assume that $f \notin \{e_1, e_3\}$. The edge $f$ is incident to at least one of $P_{1i}$ and $Q_{1i}$ since there is no vertex in the proper interior of $D_{1i}$. Without loss of generality, $f$ is incident to $P_{1i}$. The edge $f$ can only be incident to internal vertices of $P_{1i}$, since $f$ is independent of $e_1$ and $e_3$. Say $f = xy$. If $x, y \in V(P_{1i})$ then the $\{f\}$-cycle obtained by combining the $x$-$y$ subpath of $P_{1i}$ with the edge $f$ is contained in $D_{1i}$ and thus is contractible. Hence $f$ is a contractible non-cuff edge, a contradiction. This proves (2).

First we prove the lemma in the case where $e_3$ is incident to $P_{12}$. Since $e_3$ is incident to an internal vertex of $P_{12}$, it follows that $e_3$ is contained in $D_{12}$. This shows the second part of the lemma. To show that $e_2$ and $e_3$ are homotopic, consider the endpoint $v_3$ of $e_3$. Since $e_3$ is in $D_{12}$ and $u_3 \in V(P_{12})$, we have $v_3 \in V(Q_{12})$ by (2). Now, combining the $u_2$-$u_3$ subpath of $P_{12}$ and the $v_2$-$v_3$ subpath of $Q_{12}$ with $e_2$ and $e_3$, we obtain an $\{e_2, e_3\}$-cycle contained in $D_{12}$, which is thus contractible. This shows that $e_2$ and $e_3$ are homotopic.

By symmetry, the above argument also handles the case where $e_3$ is incident to $Q_{12}$. Thus we may assume that $e_3$ is incident to neither $P_{12}$ nor $Q_{12}$.

Suppose $P_{12} \subseteq P_{13}$. Then, by (2), all noncontractible non-cuff edges contained in $D_{12}$ are incident to $P_{12}$, and thus also to $P_{13}$. Hence they are all contained in the disc $D_{13}$. Moreover, a contractible $\{e_2, e_3\}$-cycle can be found in the obvious way. Therefore the lemma holds in this case. Using symmetry, the same argument can be used if $P_{12} \subseteq Q_{13}$, $Q_{12} \subseteq P_{13}$, or $Q_{12} \subseteq Q_{13}$. Thus we may assume

$$P_{12} \not\subseteq P_{13}; \quad P_{12} \not\subseteq Q_{13}; \quad Q_{12} \not\subseteq P_{13}; \quad Q_{12} \not\subseteq Q_{13}. \quad (3)$$

Next consider $P_{12}$ and $P_{13}$. If we orient these paths starting at $u_1$, then they either go in the same direction around $C_P$, or in opposite directions. Suppose the former. Then one path is a subpath of the other. Since by our assumption $u_3$ is not in $P_{12}$, we have $P_{12} \subseteq P_{13}$, which contradicts (3). Hence the paths $P_{12}$ and $P_{13}$ go in opposite directions around $C_P$. If $V(P_{12}) \cap V(P_{13}) \neq \{u_1\}$, then $u_3$ is an internal vertex of $P_{12}$, which contradicts our assumption on $e_3$. Hence

$$V(P_{12}) \cap V(P_{13}) = \{u_1\}. \quad (4)$$

By symmetry, the above argument shows that $Q_{12}$ and $Q_{13}$ go in opposite directions around $C_Q$ (starting from $v_1$), which similarly implies

$$V(Q_{12}) \cap V(Q_{13}) = \{v_1\}. \quad (5)$$

Now consider $P_{12}$ and $Q_{13}$. These two paths do not share any endpoint. If $C_P \neq C_Q$, then obviously the two paths are vertex-disjoint. If $C_P = C_Q$ and $V(P_{12}) \cap V(Q_{13}) \neq \emptyset$, then at least one of $v_1$ and $v_3$ is an internal vertex of $P_{12}$, because otherwise $P_{12} \subseteq Q_{13}$, which contradicts (3). However $v_1 \notin V(P_{12})$ since $v_1 \in V(Q_{12})$, and $v_3 \notin V(P_{12})$ by our assumption that $e_3$ is not incident to $P_{12}$. Hence, in all cases,

$$V(P_{12}) \cap V(Q_{13}) = \emptyset. \quad (6)$$

By symmetry,

$$V(Q_{12}) \cap V(P_{13}) = \emptyset. \quad (7)$$

It follows from (4)–(7) that $C_{12}$ and $C_{13}$ only have $e_1$ in common. This implies in turn that $D_{12}$ and $D_{13}$ have disjoint proper interiors. Thus the cycle $C_{23} := (C_{12} \cup C_{13}) - e_1$ bounds the disc obtained...
by gluing $D_{12}$ and $D_{13}$ along $e_1$. Hence $C_{23}$ is an $\{e_2, e_3\}$-cycle of $G$ bounding a disc containing $e_3$ and all edges contained in $D_{12}$. This concludes the proof. □

The next lemma is a direct consequence of Lemma 3.5. An equivalence class $Q$ for the homotopy relation on the noncontractible non-cuff edges of $G$ is trivial if $|Q| = 1$, and non-trivial otherwise.

Lemma 3.6. Let $G$ be a $(g,c)$-embedded graph and let $Q$ be a non-trivial equivalence class of the noncontractible non-cuff edges of $G$. Then there are distinct edges $e, f \in Q$ and a contractible $(e, f)$-cycle $C$ of $G$, such that the closed disc bounded by $C$ contains every edge in $Q$.

Our main tool in proving Theorem 3.1 is the following lemma, whose inductive proof is enabled by the following definition. Let $G$ be a $(g,c)$-embedded graph and let $k \geq 1$. A graph $H$ is a $k$-minor of $G$ if there exists an $(H,4k)$-model $\{S_x: x \in V(H)\}$ in $G$ such that, for every vertex $u \in V(G)$ incident to a noncontractible non-cuff edge $e$ in a non-trivial equivalence class, the number of subgraphs in the model including $u$ is at most $k$. Such a collection $\{S_x: x \in V(H)\}$ is said to be a $k$-model of $H$ in $G$. This provides a relaxation of the notion of $(H,k)$-minor since some vertices of $G$ could appear in up to $4k$ branch sets (instead of $k$). We emphasize that this definition depends heavily on the embedding of $G$.

Lemma 3.7. Let $G$ be a $(g,c)$-embedded graph and let $k \geq 1$. Then every $k$-minor $H$ of $G$ has minimum degree at most $48k\sqrt{c} + g$.

Proof. Let $q(G)$ be the number of non-trivial equivalence classes of noncontractible non-cuff edges in $G$. We proceed by induction, firstly on $g + c$, then on $q(G)$, and then on $|V(G)|$. Now $G$ is embedded in a surface of Euler genus $g' \leq g$ with $c' \leq c$ cuffs added. If $g' < g$ or $c' < c$ then we are done by induction. Now assume that $g' = g$ and $c' = c$.

We repeatedly use the following observation: If $C$ is a contractible cycle of $G$, then the subgraph of $G$ consisting of the vertices and edges contained in the closed disc $D$ bounded by $C$ is outerplanar, and thus has treewidth at most 2. This is because the proper interior of $D$ contains no vertex of $G$ (since all the vertices in $G$ are on the cuff boundaries).

Let $\{S_x: x \in V(H)\}$ be a $k$-model of $H$ in $G$. Let $d$ be the minimum degree of $H$. We may assume that $d \geq 20k$, otherwise $d \leq 48k\sqrt{c} + g$ (since $c \geq 1$) and we are done. Also, it is enough to prove the lemma when $H$ is connected, so assume this is the case.

Case 1: Some non-cuff edge $e$ of $G$ is contractible. Let $C$ be a contractible $[e]$-cycle. Let $u, v$ be the endpoints of $e$. Remove from $G$ every vertex in $V(C) \setminus \{u, v\}$ and modify the embedding of $G$ by redrawing the edge $e$ where the path $C - e$ was. Thus $e$ becomes a cuff-edge in the resulting graph $G'$, and $u$ and $v$ both have degree 2. Also observe that $G'$ is connected and remains simple (that is, this operation does not create loops or parallel edges). Since the embedding of $G'$ is 2-cell, $G'$ is $(g, c)$-embedded also.

If $e_1$ and $e_2$ are noncontractible non-cuff edges of $G'$ that are homotopic in $G'$, then $e_1$ and $e_2$ were also noncontractible and homotopic in $G$. Hence, $q(G') \leq q(G)$. Also, $|V(G')| < |V(G)|$ since we removed at least one vertex from $G$. Thus, by induction, every $k$-minor of $G'$ has minimum degree at most $48k\sqrt{c} + g$. Therefore, it is enough to show that $H$ is also a $k$-minor of $G'$.

Let $G_1$ be the subgraph of $G$ lying in the closed disc bounded by $C$; observe that $G_1$ is outerplanar. Moreover, $(G_1, G')$ is a separation of $G$ with $V(G_1) \cap V(G') = \{u, v\}$. (That is, $G_1 \cup G' = G$ and $V(G_1) \setminus V(G') \neq \emptyset$ and $V(G') \setminus V(G_1) \neq \emptyset$.)

First suppose that $S_x \subseteq G_1 \setminus \{u, v\}$ for some vertex $x \in V(H)$. Let $H'$ be the subgraph of $H$ induced by the set of such vertices $x$. In $H$, the only neighbors of a vertex $x \in V(H')$ that are not in $H'$ are vertices $y$ such that $S_y$ includes at least one of $u, v$. There are at most $2 \cdot 4k = 8k$ such branch sets $S_y$. Hence, $H'$ has minimum degree at least $d - 8k \geq 12k$. However, $H'$ is a minor of $G_1[4k]$ and hence has minimum degree at most $4k \cdot tw(G_1) + 4k - 1 \leq 12k - 1$ by Lemma 2.1, a contradiction.

It follows that every branch set $S_x (x \in V(H'))$ contains at least one vertex in $V(G')$. Let $S_i' = S_i \cap G'$. Using the fact that $u v \in E(G')$, it is easily seen that the collection $\{S_i': x \in V(H')\}$ is a $k$-model of $H$ in $G'$. 


Case 2: Some equivalence class $Q$ is non-trivial. By Lemma 3.6, there are two edges $e, f \in Q$ and a contractible $[e, f]$-cycle $C$ such that every edge in $Q$ is contained in the disc bounded by $C$. Let $P_1, P_2$ be the two components of $C \setminus [e, f]$. These two paths either belong to the same cuff-cycle or to two distinct cuff-cycles of $G$.

Our aim is to eventually contract each of $P_1, P_2$ into a single vertex. However, before doing so we slightly modify $G$ as follows. For each cuff-cycle $C'$ intersecting $C$, select an arbitrary edge in $E(C') \setminus E(C)$ and subdivide it twice. Let $G'$ be the resulting $(g, c)$-embedded graph. Clearly $q(G') = q(G)$, and there is an obvious $k$-model $\{S'_x : x \in V(H)\}$ of $H$ in $G'$: simply apply the same subdivision operation on the branch sets $S_x$.

Let $G'_1$ be the subgraph of $G'$ lying in the closed disc $D$ bounded by $C$. Observe that $G'_1$ is outer-planar with outercycle $C$. Suppose that some edge $xy$ in $E(G'_1) \setminus E(C)$ has both its endpoints in the same path $P_i$, for some $i \in \{1, 2\}$. Then the cycle obtained by combining $xy$ and the $x$-$y$ path in $P_i$ is a contractible cycle of $G'$, and its only non-cuff edge is $xy$. The edge $xy$ is thus a contractible edge of $G'$, and hence also of $G$, a contradiction.

It follows that every non-cuff edge included in $G'_1$ has one endpoint in $P_1$ and the other in $P_2$. Hence, every such edge is homotopic to $e$ and therefore belongs to $Q$.

Consider the $k$-model $\{S'_x : x \in V(H)\}$ of $H$ in $G'$ mentioned above. Let $e = uv$ and $f = u'v'$, with $u, u' \in V(P_1)$ and $v, v' \in V(P_2)$. Let $X := \{u, u', v, v'\}$. For each $w \in X$, the number of branch sets $S'_w$ that include $w$ is at most $k$, since $e$ and $f$ are homotopic noncontractible non-cuff edges.

Let $J := G'_1 \setminus X$. Note that $tw(J) \leq 2$ since $G'_1$ is outerplanar. Let $Z := \{x \in V(H) : S'_x \subseteq J\}$. First, suppose that $Z \neq \emptyset$. Every vertex of $J$ is in at most $4k$ branch sets $S'_x (x \in Z)$. It follows that the induced subgraph $H[Z]$ is a minor of $J[4k]$.

If a non-cuff edge is contractible in $G''$ then it is also contractible in $G'$, implying all non-cuff edges in $G''$ are noncontractible. Two non-cuff edges of $G''$ are homotopic in $G''$ if and only if they are in $G'$. It follows $q(G'') = q(G') - 1 = q(G) - 1$, since $e$ is not homotopic to another non-cuff edge in $G''$. By induction, every $k$-minor of $G''$ has minimum degree at most $48k\sqrt{c + g}$. Thus, it suffices to show that $H$ is also a $k$-minor of $G''$.

For $x \in V(H)$, let $S''_x$ be obtained from $S'_x$ by performing the same contraction operation as when defining $G''$ from $G'$: every edge in $Q \setminus \{e\}$ is removed and every edge in $E(P_1) \cup E(P_2)$ is contracted. Using that every subgraph $S''_x$ either is disjoint from $V(G'_1)$ or contains some vertex in $X$, it can be checked that $S''_x$ is connected.

Consider an edge $xy \in E(H)$. We now show that the two subgraphs $S''_x$ and $S''_y$ touch in $G''$. Suppose $S''_x$ and $S''_y$ share a common vertex $w$. If $w \notin V(G'_1)$, then $w$ is trivially included in both $S''_x$ and $S''_y$. If $w \in V(G'_1)$, then each of $S''_x$ and $S''_y$ contains a vertex from $X$, and hence either $u$ or $v$ is included in both $S''_x$ and $S''_y$, or $u$ is included in one and $v$ in the other. In the latter case $uv$ is an edge of $G'$ joining $S''_x$ and $S''_y$. Now assume $S''_x$ and $S''_y$ are vertex-disjoint. Thus there is an edge $ww' \in E(G')$ joining these two subgraphs in $G'$. Again, if neither $w$ nor $w'$ belongs to $V(G'_1)$, then obviously $ww'$ joins $S''_x$ and $S''_y$ in $G''$. If $w, w' \in V(G'_1)$, then each of $S''_x$ and $S''_y$ contains a vertex from $X$, and we are done exactly as previously. If exactly one of $w, w'$ belongs to $V(G'_1)$, say $w$, then $w \in X$ and $w'$ is the unique neighbor of $w$ in $G'$ outside $V(G'_1)$. The contraction operation naturally maps $w$ to a vertex $m(w) \in \{u, v\}$. The edge $w'm(w)$ is included in $G''$ and thus joins $S''_x$ and $S''_y$.

In order to conclude that $\{S''_x : x \in V(H)\}$ is a $k$-model of $H$ in $G''$, it remains to show that, for every vertex $w \in V(G'')$, the number of branch sets including $w$ is at most $4k$, and is at most $k$ if
$w$ is incident to a non-cuff edge homotopic to another non-cuff edge. This condition is satisfied if $w \notin \{u, v\}$, because two non-cuff edges of $G'$ are homotopic in $G''$ if and only if they are in $G'$. Thus assume $w \in \{u, v\}$. By the definition of $G''$, the edge $e = uv$ is not homotopic to another non-cuff edge of $G''$. Moreover, for each $z \in X$, there are at most $k$ branch sets $S_x'$ ($x \in V(H)$) containing $z$. Since $|X| = 4$, it follows that there are at most $4k$ branch sets $S_x'$ ($x \in V(H)$) containing $w$. Therefore, the condition holds also for $w$, and $H$ is a $k$-minor of $G''$.

**Case 3:** There is at most one non-cuff edge. Because $G$ is connected, this implies that $G$ consists either of a unique cuff-cycle, or of two cuff-cycles joined by a non-cuff edge. In both cases, $G$ has treewidth exactly 2. Since $H$ is a minor of $G[4k]$, Lemma 2.1 implies that $H$ has minimum degree at most $4k \cdot \text{tw}(G) + 4k - 1 = 12k - 1 \leq 48k \sqrt{\epsilon + g}$, as desired.

**Case 4:** Some cuff-cycle $C$ contains three consecutive degree-$2$ vertices. Let $u$, $v$, $w$ be three such vertices (in order). Note that $C$ has at least four vertices, as otherwise $G = C$ and the previous case would apply. It follows $uw \notin E(G)$. Let $G'$ be obtained from $G$ by contracting the edge $uv$ into the vertex $u$. In the embedding of $G'$, the edge $uw$ is drawn where the path $uvw$ was; thus $uw$ is a cuff-edge, and $G'$ is $(g, c)$-embedded. We have $q(G') = q(G)$ and $|V(G')| < |V(G)|$, hence by induction, $G'$ satisfies the lemma, and it is enough to show that $H$ is a $k$-minor of $G'$.

Consider the $k$-model $(S_x; x \in V(H))$ of $H$ in $G$. If $V(S_x) = \{v\}$ for some $x \in V(H)$, then $x$ has degree at most $3 \cdot 4k - 1 = 12k - 1$, because $xv \in E(H)$ implies that $S_x$ contains at least one of $u$, $v$, $w$. However this contradicts the assumption that $H$ has minimum degree $d \geq 20k$. Thus every branch set $S_x$ that includes $v$ also contains at least one of $u$, $w$ (since $S_x$ is connected).

For $x \in V(H)$, let $S_x'$ be obtained from $S_x$ as expected: contract the edge $uv$ if $uv \in E(S_x)$. Clearly $S_x'$ is connected. Consider an edge $xy \in E(H)$. If $S_x$ and $S_y$ had a common vertex then so do $S_x'$ and $S_y'$. If $S_x$ and $S_y$ were joined by an edge $e$, then either $e$ is still in $G'$ and joins $S_x'$ and $S_y'$, or $e = uv$ and $u \in V(S_x')$, $v \in V(S_y')$. Hence in each case $S_x'$ and $S_y'$ touch in $G'$. Finally, it is clear that $(S_x'; x \in V(H))$ meets remaining requirements to be a $k$-model of $H$ in $G'$, since $V(S_x') \subseteq V(S_x)$ for every $x \in V(H)$ and the homotopy properties of the non-cuff edges have not changed. Therefore, $H$ is a $k$-minor of $G'$.

**Case 5:** None of the previous cases apply. Let $t$ be the number of non-cuff edges in $G$ (thus $t \geq 2$). Since there are no three consecutive degree-$2$ vertices, every cuff-edge is at distance at most $1$ from a non-cuff edge. It follows that

$$|E(G)| \leq 9t.$$  

(This inequality can be improved but is good enough for our purposes.)

For a facial walk $F$ of the embedded graph $G$, let $nc(F)$ denote the number of occurrences of non-cuff edges in $F$. (A non-cuff edge that appears twice in $F$ is counted twice.) We claim that $nc(F) \geq 3$. Suppose on the contrary that $nc(F) \leq 2$.

First suppose that $F$ has no repeated vertex. Thus $F$ is a cycle. If $nc(F) = 0$, then $F$ is a cuff-cycle, every vertex of which is not incident to a non-cuff edge, contradicting the fact that $G$ is connected with at least two non-cuff edges. If $nc(F) = 1$ then $F$ is a contractible cycle that contains exactly one non-cuff edge $e$. Thus $e$ is contractible, and Case 1 applies. If $nc(F) = 2$ then $F$ is a contractible cycle containing exactly two non-cuff edges $e$ and $f$. Thus $e$ and $f$ are homotopic. Hence there is a non-trivial equivalence class, and Case 2 applies.

Now assume that $F$ contains a repeated vertex $v$. Let

$$F = (x_1, x_2, \ldots, x_{t-1}, x_t = v, x_{t+1}, x_{t+2}, \ldots, x_{2t-1}, x_j = v).$$

All of $x_1, x_{t+1}, x_{t+2}, x_{2t-1}$ are adjacent to $v$. Since $x_1 \neq x_{t-1}$ and $x_{t+1} \neq x_{2t-1}$ and $\deg(v) \leq 3$, we have $x_i = x_{i+1}$ or $x_i = x_{i-1}$. Without loss of generality, $x_{j-1} = x_{j+1}$. Thus the path $x_{j-1}v x_{j+1}$ is in the boundary of the cuff-cycle $C$ that contains $v$. Moreover, the edge $v x_{j+1} = v x_{j-1}$ counts twice in $nc(F)$. Since $nc(F) \leq 2$, every edge on $F$ except $v x_{j+1} = v x_{j-1}$ is a cuff-edge. Thus every edge in the walk $v, x_1, x_2, \ldots, x_{t-1}, x_t = v$ is in $C$, and hence $v, x_1, x_2, \ldots, x_{t-1}, x_t = v$ is the cycle $C$. Similarly, $x_1, x_2, \ldots, x_{j-2}, x_{j-1} = x_{j+1}$ is a cycle $C'$ bounding some other cuff. Hence $v x_{j+1}$ is the only
non-cuff edge incident to $C$, and the same for $C'$. Therefore $G$ consists of two cuff-cycles joined by a non-cuff edge, and Case 3 applies.

Therefore, $nc(F) \geq 3$, as claimed.

Let $n := |V(G)|$, $m := |E(G)|$, and $f$ be the number of faces of $G$. It follows from Euler’s formula that

$$n - m + f + c = 2 - g.$$  \hspace{1cm} (9)

Every non-cuff edge appears exactly twice in faces of $G$ (either twice in the same face, or once in two distinct faces). Thus

$$2t = \sum_{F \text{ face of } G} nc(F) \geq 3f.$$  \hspace{1cm} (10)

Since $n = m - t$, we deduce from (9) and (10) that

$$t = f + c + g - 2 \leq \frac{2}{3}t + c + g - 2.$$  \hspace{1cm}

Thus $t \leq 3(c + g)$, and $m \leq 9t \leq 27(c + g)$ by (8). This allows us, in turn, to bound the number of edges in $G[4k]$:

$$|E(G[4k])| = \binom{4k}{2} n + (4k)^2 m \leq (4k)^2 \cdot 2m \leq 54(4k)^2 (c + g) \leq 2(24k)^2 (c + g).$$

Since $H$ is a minor of $G[4k]$, we have $|E(H)| \leq |E(G[4k])|$. Thus the minimum degree $d$ of $H$ can be upper bounded as follows:

$$2|E(H)| \geq d|V(H)| \geq d^2,$$

and hence

$$d \leq \sqrt{2|E(H)|} \leq \sqrt{2|E(G[4k])|} \leq \sqrt{2 \cdot 2(24k)^2 (c + g)} = 48k \sqrt{c + g},$$

as desired. \hfill \Box

Now we put everything together and prove Theorem 3.1.

**Proof of Theorem 3.1.** Let $G \in \mathcal{G}(g, p, k)$. By Lemma 3.4, there exists a $(g, p)$-embedded graph $G'$ with $\eta(G) \leq \eta(G'[k + 1]) + \sqrt{6g} + 4$.

Let $t := \eta(G'[k + 1])$. Thus $K_t$ is a $(k + 1)$-minor of $G'$ by Lemma 3.3. Lemma 3.7 with $H = K_t$ implies that

$$\eta(G'[k + 1]) - 1 = t - 1 \leq 48(k + 1) \sqrt{g + p}.$$  \hspace{1cm}

Hence $\eta(G) \leq 48(k + 1) \sqrt{g + p} + \sqrt{6g} + 5$, as desired. \hfill \Box

4. Constructions

This section describes constructions of graphs in $\mathcal{G}(g, p, k, a)$ that contain large complete graph minors. The following lemma, which in some sense, is converse to Lemma 3.4 will be useful.

**Lemma 4.1.** Let $G$ be a graph embedded in a surface with Euler genus at most $g$. Let $F_1, \ldots, F_p$ be pairwise vertex-disjoint facial cycles of $G$, such that $V(F_1) \cup \cdots \cup V(F_p) = V(G)$. Then for all $k \geq 1$, some graph in $\mathcal{G}(g, p, k)$ contains $G[k]$ as a minor.

**Proof.** Let $G'$ be the embedded multigraph obtained from $G$ by replacing each edge $vw$ of $G$ by $k^2$ edges between $v$ and $w$ bijectively labeled from $\{(i, j) : i, j \in [1, k]\}$. Embed these new edges ‘parallel’
to the original edge $vw$. Let $H_0$ be the splitting of $G'$ at $F_1, \ldots, F_p$. Edges in $H_0$ inherit their label in $G'$. For each $\ell \in [1, p]$, let $J_\ell$ be the face of $H_0$ that corresponds to $F_\ell$ (see Fig. 3).

Let $H_\ell$ be the graph with vertex set $V(J_\ell) \cup \{(v, i) : v \in V(F_\ell), i \in [1, k]\}$, where:

(a) each vertex $x$ in $J_\ell$ that belongs to a vertex $v$ in $F_\ell$ is adjacent to each vertex $(v, i)$ in $H_\ell$; and
(b) vertices $(v, i)$ and $(w, j)$ in $H_\ell$ are adjacent if and only if $v = w$ and $i \neq j$.

We now construct a circular decomposition $\{B(x) : x \in V(J_\ell)\}$ of $H_\ell$ with perimeter $J_\ell$. For each vertex $x$ in $J_\ell$ that belongs to a vertex $v$ in $F_\ell$, let $B(x)$ be the set $\{x\} \cup \{(v, i) : i \in [1, k]\}$ of vertices in $H_\ell$. Thus $|B(x)| = k + 1$. For each type-(a) edge between $x$ and $(v, i)$, the endpoints are both in bag $B(x)$. For each type-(b) edge between $(v, i)$ and $(v, j)$ in $H_\ell$, the endpoints are in every bag $B(x)$ where $x$ belongs to $v$. Thus the endpoints of every edge in $H_\ell$ are in some bag $B(x)$. Thus $\{B(x) : x \in V(J_\ell)\}$ is a circular decomposition of $H_\ell$ with perimeter $J_\ell$ and width at most $k$.

Let $H$ be the graph $H_0 \cup H_1 \cup \cdots \cup H_p$. Thus $V(H_0) \cap V(H_\ell) = V(J_\ell)$ for each $\ell \in [1, p]$. Since $J_1, \ldots, J_p$ are pairwise vertex-disjoint facial cycles of $H_0$, the subgraphs $H_1, \ldots, H_p$ are pairwise vertex-disjoint. Hence $H$ is $(g, p, k)$-almost embeddable.

To complete the proof, we now construct a model $\{D_{v,i} : v^{(i)} \in V(G[k])\}$ of $G[k]$ in $H$, where $v^{(i)}$ is the $i$-th vertex in the $k$-clique of $G[k]$ corresponding to $v$. Fix an arbitrary total order $\prec$ on $V(G)$. Consider a vertex $v^{(i)}$ of $G[k]$. Say $v$ is in face $F_\ell$. Add the vertex $(v, i)$ of $H_\ell$ to $D_{v,i}$. For each edge $v^{(i)}w^{(j)}$ of $G[k]$ with $v \prec w$, by construction, there is an edge $xy$ of $H_0$ labeled $(i, j)$, such that $x$ belongs to $v$ and $y$ belongs to $w$. Add the vertex $x$ to $D_{v,j}$. Thus $D_{v,i}$ induces a connected star subgraph of $H$ consisting of type-(a) edges in $H_\ell$. Since every vertex in $J_\ell$ is incident to at most one labeled edge, $D_{v,i} \cap D_{w,j} = \emptyset$ for distinct vertices $v^{(i)}$ and $w^{(j)}$ of $G[k]$.

Consider an edge $v^{(i)}w^{(j)}$ of $G[k]$. If $v = w$ then $i \neq j$ and $v$ is in some face $F_\ell$, in which case a type-(b) edge in $H_\ell$ joins the vertex $(v, i)$ in $D_{v,i}$ with the vertex $(w, j)$ in $D_{w,j}$. Otherwise, without loss of generality, $v \prec w$ and by construction, there is an edge $xy$ of $H_0$ labeled $(i, j)$, such that $x$ belongs to $v$ and $y$ belongs to $w$. By construction, $x$ is in $D_{v,i}$ and $y$ is in $D_{w,j}$. In both cases there is an edge of $H$ between $D_{v,i}$ and $D_{w,j}$. Hence the $D_{v,i}$ are the branch sets of a $G[k]$-model in $H$. □

Our first construction employs just one vortex and is based on an embedding of a complete graph.

**Lemma 4.2.** For all integers $g \geq 0$ and $k \geq 1$, there is an integer $n \geq k\sqrt{6g}$ such that $K_n$ is a minor of some $(g, 1, k)$-almost embeddable graph.

**Proof.** The claim is vacuous if $g = 0$. Assume that $g \geq 1$. The map color theorem [7] implies that $K_m$ triangulates some surface if and only if $m \bmod 6 \in \{0, 1, 3, 4\}$, in which case the surface has Euler...
genus $\frac{1}{6}(m - 3)(m - 4)$. It follows that for every real number $m_0 \geq 2$ there is an integer $m$ such that 
$m_0 \leq m \leq m_0 + 2$ and $K_m$ triangulates some surface of Euler genus $\frac{1}{6}(m - 3)(m - 4)$. Apply this result 
with $m_0 = \sqrt{6g} + 1$ for the given value of $g$. We obtain an integer $m$ such that $\sqrt{6g} + 1 \leq m \leq \sqrt{6g} + 3$ 
and $K_m$ triangulates a surface $\Sigma$ of Euler genus $g' := \frac{1}{6}(m - 3)(m - 4)$. Since $m - 4 < m - 3 \leq \sqrt{6g}$, 
we have $g' \leq g$. Every triangulation has facewidth at least 3. Thus, deleting one vertex from the 
embedding of $K_m$ in $\Sigma$ gives an embedding of $K_{m-1}$ in $\Sigma$, such that some facial cycle contains every 
vertex. Let $n := (m - 1)k \geq k\sqrt{6g}$. Lemma 4.1 implies that $K_{m-1}|k| \cong K_n$ is a minor of some $(g', 1,k)$-
almost embeddable graph. \hfill $\Box$

Now we give a construction based on grids. Let $L_n$ be the $n \times n$ planar grid graph. This graph has 
vertex set $[1, n] \times [1, n]$ and edge set $\{(x, y)(x', y') : |x - x'| + |y - y'| = 1\}$. The following lemma is 
well known; see [9].

**Lemma 4.3.** $K_{nk}$ is a minor of $L_n[2k]$ for all $k \geq 1$.

**Proof.** For $x, y \in [1, n]$ and $z \in [1, 2k]$, let $(x, y, z)$ be the $z$-th vertex in the $2k$-clique corresponding 
to the vertex $(x, y)$ in $L_n[2k]$. For $x \in [1, n]$ and $z \in [1, k]$, let $B_{x,z}$ be the subgraph of $L_n[2k]$ 
induced by $\{(x, y, 2z - 1), (y, x, 2z): y \in [1, n]\}$. Clearly $B_{x,z}$ is connected. For all $x, x' \in [1, n]$ and 
z, $z' \in [1, k]$ with $(x, z) \neq (x', z')$, the subgraphs $B_{x,z}$ and $B_{x',z'}$ are disjoint, and the vertex $(x, x', 2z - 1)$ 
in $B_{x,z}$ is adjacent to the vertex $(x, x', 2z')$ in $B_{x',z'}$. Thus the $B_{x,z}$ are the branch sets of a $K_{nk}$
minor in $L_n[2k]$. \hfill $\Box$

**Lemma 4.4.** For all integers $k \geq 2$ and $p \geq 1$, there is an integer $n \geq \frac{2 \ell}{3\sqrt{3}}\sqrt[k]{p}$, such that $K_n$ is a minor of some 
$(0, p, k)$-almost embeddable graph.

**Proof.** Let $m := \lfloor \sqrt[p]{p} \rfloor$ and $k := \lfloor \frac{1}{2} \rfloor$. Let $n := 2m\ell \geq 2 \cdot \sqrt[p]{p} \cdot \frac{k}{2} = \frac{2 \ell}{3\sqrt{3}}\sqrt[k]{p}$. For $x, y \in [1, m]$, let $F_{x,y}$ be 
the face of $L_{2m}$ with vertex set $\{(2x - 1, 2y - 1), (2x, 2y - 1), (2x, 2y), (2x - 1, 2y)\}$. There are $m^2$ such 
facing vertices, and every vertex of $L_{2m}$ is in exactly one such face. By Lemma 4.3, $K_n$ is a minor of $L_{2m}[2\ell]$. 
Since $L_{2m}$ is planar, by Lemma 4.1, $K_n$ is a minor of some $(0, m^2, 2\ell)$-almost embeddable graph. The 
result follows since $p \geq m^2$ and $k \geq 2\ell$. \hfill $\Box$

The following theorem summarizes our constructions of almost embeddable graphs containing 
large complete graph minors.

**Theorem 4.5.** For all integers $g \geq 0$ and $p \geq 1$ and $k \geq 2$, there is an integer $n \geq \frac{1}{4}k\sqrt[p]{p + g}$, such that $K_n$ is a minor of some $(g, p, k)$-almost embeddable graph.

**Proof.** First suppose that $g \geq p$. By Lemma 4.2, there is an integer $n \geq k\sqrt[6g]{g}$, such that $K_n$ is a minor of some 
$(g, 1, k)$-almost embeddable graph, which is also $(g, p, k)$-embeddable (since $p \geq 1$). Since 
n \geq k\sqrt[3p + 3g]{p + g} \geq \frac{1}{4}k\sqrt[p]{p + g}$, we are done.

Now assume that $p > g$. By Lemma 4.4, there is an integer $n \geq \frac{2 \ell}{3\sqrt{3}}\sqrt[p]{p}$, such that $K_n$ is a minor of 
some $(0, p, k)$-almost embeddable graph, which is also $(g, p, k)$-embeddable (since $g \geq 0$). Since 
n \geq \frac{2 \ell}{3\sqrt{3}}\sqrt[p]{p} + g = \frac{2 \ell}{3\sqrt{3}}k\sqrt[p]{g + p} + \frac{1}{4}k\sqrt[p]{g + p}$, we are done. \hfill $\Box$

Adding a dominant vertices to a graph increases its Hadwiger number by $a$. Thus Theorem 4.5 
implies:

**Theorem 4.6.** For all integers $g, a \geq 0$ and $p \geq 1$ and $k \geq 2$, there is an integer $n \geq a + \frac{1}{4}k\sqrt[p]{p + g}$, such that 
$K_n$ is a minor of some graph in $\mathcal{G}(g, p, k, a)$.

**Corollary 3.2** and **Theorem 4.6** together prove Theorem 1.1.
Acknowledgment

Thanks to Bojan Mohar for instructive conversations.

References