3-Monotone Expanders

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Abstract. Bourgain recently constructed $O(1)$-monotone bipartite expanders. By combining this result with a generalisation of the unraveling method of Kannan, we construct 3-monotone bipartite expanders, which is best possible. Similarly, we construct bipartite expanders that have 3-page book embeddings, 2-queue layouts, and 4-track layouts. All these results are best possible.

1 Introduction

Expanders are classes of highly connected graphs that are of fundamental importance in graph theory, with numerous applications, especially in theoretical computer science [29]. While the literature contains various definitions of expanders, this paper focuses on bipartite expanders. For $\epsilon \in (0, 1]$, a bipartite graph $G$ with bipartition $V(G) = A \cup B$ is a bipartite $\epsilon$-expander if $|A| = |B|$ and $|N(S)| \geq (1 + \epsilon)|S|$ for every subset $S \subset A$ with $|S| \leq |A|^2$. Here $N(S)$ is the set of vertices adjacent to some vertex in $S$. An infinite family of bipartite $\epsilon$-expanders, for some fixed $\epsilon > 0$, is called an infinite family of bipartite expanders.

There has been much research on constructing and proving the existence of expanders with various desirable properties. The first example is that there is an infinite family of expanders with bounded degree, in fact, degree at most 3 (see [1, 29, 35] for example).

1.1 Monotone Layouts

Bourgain [4, 5] recently gave an explicit construction of an infinite family of bipartite expanders with an interesting additional property\(^1\). Say $G$ is a bipartite graph with ordered colour classes $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$. Two edges $v_i w_j$ and $v_k w_\ell$ cross if $i < k$ and $\ell < j$. A matching $M$ in $G$ is monotone if no two edges in $M$ cross. A bipartite graph with ordered coloured classes is $d$-monotone if it is the union of $d$ monotone matchings. Note that every $d$-monotone bipartite graph has maximum degree at most $d$. Motivated by connections to so-called dimension

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\(^1\) An outline of the proof was given in the original paper by Bourgain [4]. A full proof was given by Bourgain and Yehudayoff [5]. See the paper by Dvir and Wigderson [19] for more.
expander, Dvir and Shpilka [17, 18] constructed an infinite family of $O(\log n)$-monotone bipartite expanders. Bourgain [4, 5] proved the following breakthrough:

**Theorem 1** (Bourgain [4, 5]). There is an infinite family of $d$-monotone bipartite expanders, for some constant $d$.

The primary result of this paper is to manipulate the construction of Bourgain to produce 3-monotone expanders.

**Theorem 2.** There is an infinite family of 3-monotone bipartite expanders.

### 1.2 Book Embeddings

Theorem 2 has applications to book embeddings. A $k$-page book embedding of a graph $G$ consists of a linear order $(u_1, \ldots, u_n)$ of $V(G)$ and a partition $E_1, \ldots, E_k$ of $E(G)$, such that edges in each set $E_i$ do not cross with respect to $(u_1, \ldots, u_n)$. That is, for all $i \in [1, k]$, there are no edges $u_a u_b$ and $u_c u_d$ in $E_i$ with $a < c < b < d$. One may think of the vertices as being ordered along the spine of a book, with each edge drawn on one of $k$ pages, such that no two edges on the same page cross. A graph with a $k$-page book embedding is called a $k$-page graph. The page-number of a graph $G$ is the minimum integer $k$ such that there is a $k$-page book embedding of $G$. Note that page-number is also called book thickness or stack-number or fixed outer-thickness; see reference [14] for more on book embeddings. A $k$-page book embedding is $k$-pushdown if, in addition, each set $E_i$ is a matching [24].

A $d$-monotone bipartite graph has a $d$-pushdown book embedding, and thus has page-number at most $d$, since using the above notation, edges in a monotone matching do not cross in the vertex ordering $(v_1, \ldots, v_n, w_m, \ldots, w_1)$, as illustrated in Figure 1(b). Also note that in the language of Pemmaraju [34], this book embedding is ‘separated’.

![Figure 1](image)

Figure 1: Converting (a) a monotone matching to (b) a book embedding and (c) a queue layout [13].

Thus the construction of Bourgain gives an infinite family of $d$-pushdown bipartite expanders with maximum degree $d$, for some constant $d$. This result solves an old open problem of Galil
et al. [24, 25] that arose in the modelling of multi-tape Turing machines. In particular, Galil et al. [25] showed that there are $O(1)$-pushdown expanders if and only if it is not possible for a 1-tape nondeterministic Turing machine to simulate a 2-tape machine in subquadratic time.

Theorem 2 and the above observation implies:

**Theorem 3.** There is an infinite family of 3-pushdown bipartite expanders.

### 1.3 Queue Layouts

Queue layouts are dual to book embeddings. (In this setting, book embeddings are often called stack layouts.) A $k$-queue layout of a graph $G$ consists of a linear order $(u_1, \ldots, u_n)$ of $V(G)$ and a partition $E_1, \ldots, E_k$ of $E(G)$, such that edges in each set $E_i$ do not nest with respect to $(u_1, \ldots, u_n)$. That is, for all $i \in [1,k]$, there are no edges $u_axu_b$ and $u_cuxd$ in $E_i$ with $a < c < d < b$.

A graph with a $k$-queue layout is called a $k$-queue graph. The queue-number of a graph $G$ is the minimum integer $k$ such that there is a $k$-queue layout of $G$. See [7, 11–15, 27, 28] and the references therein for results on queue layouts.

A $d$-monotone bipartite graph has queue-number at most $d$, since using the above notation, edges in a monotone matching do not cross in the vertex ordering $(v_1, \ldots, v_n, w_1, \ldots, w_m)$, as illustrated in Figure 1(c). Thus the construction of Bourgain provides an infinite family of bipartite expanders with bounded queue-number, as observed by Dujmović et al. [10]. And Theorem 2 gives an infinite family of 3-queue bipartite expanders. We improve this result as follows.

**Theorem 4.** There is an infinite family of 2-queue bipartite expanders with maximum degree 3.

### 1.4 Track Layouts

Finally, consider track layouts of graphs. In a graph $G$, a track is an independent set, equipped with a total ordering denoted by $\preceq$. A $k$-track layout of a graph $G$ consists of a partition $(V_1, \ldots, V_k)$ of $V(G)$ into tracks, such that between each pair of tracks, no two edges cross. That is, there are no edges $vw$ and $xy$ in $G$ with $v \prec x$ in some track $V_i$, and $y \prec w$ in some track $V_j$. The track-number is the minimum integer $k$ for which there is a $k$-track layout of $G$. See [8, 11–13, 15] and the references therein for results on track layouts. We prove the following:

**Theorem 5.** There is an infinite family of 4-track bipartite expanders with maximum degree 3.

### 1.5 Discussion

Some notes on the above theorems are in order:
Tightness: Each of Theorems 2–5 is best possible since 2-page graphs (and thus 2-monotone graphs) are planar [3], 1-queue graphs are planar [28], and 3-track graphs are planar [15], but planar graphs have $O(\sqrt{n})$ separators, and are thus far from being expanders. It is interesting that graphs that are 'close' to being planar can be expanders\(^2\).

Expansion and Separators: Nešetřil et al. [33] proved that graph classes with bounded page-number or bounded queue-number have bounded expansion (also see [32, Chapter 14]). Thus, Theorems 3 and 4 provide natural families of graphs that have bounded expansion yet contain an infinite family of expanders. The upper bound (proved in [33]) on the expansion function for graphs of bounded page-number or bounded queue-number is exponential. Nešetřil and Ossona de Mendez [32] state as an open problem whether this exponential bound is necessary. Since graphs with sub-exponential expansion have $o(n)$ separators [31, Theorem 8.3] (also see [20]), and expanders do not have $o(n)$ separators (see Appendix A), Theorems 3 and 4 imply that indeed exponential expansion is necessary for 3-page and 2-queue graphs. Since queue-number is tied to track-number [13], these same conclusions hold for track-number.

Subdivisions: Theorems slight weaker than Theorems 3–5 can be proved using subdivisions. It can be proved that if $G$ is a bipartite $\epsilon$-expander with bounded degree, then the graph obtained from $G$ by subdividing each edge twice is a bipartite $\epsilon'$-expander. Dujmović and Wood [15] proved that every $k$-page graph has a 3-page subdivision with $2 \lceil \log_2 k \rceil - 2$ division vertices per edge. Applying this result to the construction of Bourgain, we obtain an infinite family of 3-page bipartite expanders with bounded degree. Note that the degree bound here is the original degree bound from the construction of Bourgain, which is much more than 3 (the degree bound in Theorem 3). In particular, 3-monotone expanders cannot be constructed using subdivisions.

One can also construct 2-queue expanders and 4-track expanders using subdivisions. Dujmović and Wood [15] proved that every $k$-queue graph has a 2-queue subdivision with $2 \lceil \log_2 k \rceil + 1$ division vertices per edge, and has a 4-track subdivision with $2 \lceil \log_2 k \rceil + 1$ division vertices per edge. To apply these results, one must modify the relevant constructions so that each edge is subdivided an even number of times (details omitted). Again the obtained degree bound is weaker than in Theorems 4 and 5.

2 Two-Sided Bipartite Expanders

Throughout this paper, it is convenient to employ the following definition. A bipartite graph $G$ with bipartition $A, B$ is a two-sided bipartite $\epsilon$-expander if $|A| = |B|$, and for all $S \subset A$ with $|S| \leq \frac{|A|}{2}$ we have $|N(S)| \geq (1 + \epsilon)|S|$, and for all $T \subset B$ with $|T| \leq \frac{|B|}{2}$ we have\(^2\) One does not need the result of Bourgain to conclude that there are expanders that are the union of three planar graphs. In fact, there are expanders that are the union of two forests: If $G$ is any bipartite expander, and $G'$ is the graph obtained from $G$ by subdividing each edge twice, then $G'$ is the union of a star forest and a matching, and it can be shown that $G'$ is a bipartite expander.
This observation is extended as follows.

Lemma 6. If $G$ is a $k$-monotone bipartite $\epsilon$-expander with ordered bipartition $A = (v_1, \ldots, v_n)$ and $B = (w_1, \ldots, w_m)$, then the graph $G'$ with vertex set $V(G') := V(G)$ and edge set $E(G') := \{(v_i, w_j), (v_j, w_i) : (v_i, w_j) \in E(G)\}$ is a two-sided $2k$-monotone bipartite $\epsilon$-expander.

Proof. Observe that $(v_i, w_j)$ crosses $(v_a, w_b)$ if and only if $(v_j, w_i)$ crosses $(v_b, w_a)$. Thus if $M$ is a monotone matching, then $\{(v_j, w_i) : (v_i, w_j) \in M\}$ is also a monotone matching. Hence, $E(G')$ can be partitioned into $2k$ monotone matchings. Since $G$ is a spanning subgraph of $G'$, we have that $G'$ is an $\epsilon$-expander. Given $T \subseteq B$ with $|T| \leq \frac{|B|}{2}$, define $S := \{v_i \in A : v_i \in B\}$. Then $|N_{G'}(T)| \geq |N_G(S)| \geq (1 + \epsilon)|S| = (1 + \epsilon)|T|$. Thus $G'$ is a two-sided bipartite $\epsilon$-expander.

The construction of Bourgain and Lemma 6 together imply:

Theorem 7. There is an infinite family of two-sided $d$-monotone bipartite expanders, for some constant $d$.

3 Unraveling

The following construction of Kannan is the starting point for our work. Let $G$ be a graph, whose edges are $k$-coloured (not necessarily properly). Let $E_1, \ldots, E_k$ be the corresponding partition of $E(G)$. Let $G'$ be the graph with vertex set

$$V(G') := V(G) \times [1, k] = \{v_i : v \in V(G), i \in [1, k]\},$$

where $v_iw_i \in E(G')$ for each edge $vw \in E_i$ and $i \in [1, k]$, and $v_iw_{i+1} \in E(G')$ for each vertex $v \in V(G)$ and $i \in [1, k-1]$. Kannan [30] called $G'$ the unraveling of $G$, which he defined in the case that the edge colouring comes from a $k$-page book embedding, and proved that $G'$ has a 3-page book embedding. To see this, for $i \in [1, k]$, let $V_i := \{v_i : v \in V(G)\}$ ordered by the given ordering of $V(G)$. Define

$$J_1 := \{v_iw_i : vw \in E_i, i \in [1, k]\}$$

$$J_2 := \{v_iw_{i+1} : v \in V(G), i \in [1, k], i \text{ odd}\}$$

$$J_3 := \{v_iw_{i+1} : v \in V(G), i \in [1, k], i \text{ even}\}.$$

Then $J_1, J_2, J_3$ is a partition of $E(G')$, and for $i \in [3]$, no two edges in $J_i$ cross with respect to the vertex ordering $V_1, V_2, \ldots, V_k$, as illustrated in Figure 2. Thus, this is a 3-page book embedding of $G'$.

This observation is extended as follows.
Lemma 8. If a bipartite graph $G$ is $k$-monotone, then the unraveling $G'$ is 3-monotone.

Proof. Say $A, B$ is the given bipartition of $G$. Let $A_i := V_i \cap A$ and $B_i := V_i \cap B$, where $V_i$ is defined above. Say $A_i$ and $B_i$ inherit the given orderings of $A$ and $B$ respectively. Then $G'$ is bipartite with ordered bipartition given by $A_1, B_2, A_3, B_4, A_5, B_6, \ldots$ and $B_1, A_2, B_3, A_4, B_5, A_6, \ldots$. Observe that for $i \in [3]$, no two edges in $J_i$ cross with respect to these orderings, as illustrated in Figure 3. Thus $G'$ is 3-monotone. $\square$

The unraveling $G'$ has interesting expansion properties. In particular, Kannan [30] proved that if $G'$ has a small separator, then so does $G$. Thus, if $G$ is an expander, then $G$ and $G'$ have no small separator. Various results in the literature say that if every separator of an $n$-vertex graph $G$ has size at least $\epsilon n$, then $G$ contains an expander as a subgraph (for various notions of non-bipartite expansion). However, the unraveling $G'$ might not be a bipartite expander. For example, $G'$ might have a vertex of degree 1. This happens for a vertex $v_1$ where $v$ is incident to no edge coloured 1, or a vertex $v_k$ where $v$ is incident to no edge coloured $k$. The natural solution for this problem is to add the edge $v_1 v_k$ for each vertex $v$ of $G$. Now each vertex $v$ corresponds to the cycle $C_v = (v_1, v_2, \ldots, v_k)$. However, the obtained graph is still not an expander: if $S$ consists of every second vertex in some $C_v$, then it is possible for $N(S)$ to consist only of the other vertices in $C_v$, in which case $|N(S)| = |S|$, and the graph is not an expander. Moreover,
it is far from clear how to construct a 3-monotone layout of this graph. (For even \(k\), the layout in the proof of Lemma 8 is 5-monotone.)

4 Generalised Unraveling

The obstacles discussed at the end of the previous section are overcome in the following lemma. This result is reminiscent of the replacement product; see [1, 29, 35].

**Lemma 9.** Let \(G\) be a two-sided bipartite \(\epsilon\)-expander with bipartition \(A, B\) and maximum degree \(\Delta\). Let \(n := |A| = |B|\). Assume \(n \geq 3\). Let \(k \geq 2\) be an integer. For each vertex \(v\) of \(G\), let \(k_v\) be an integer with \(k \leq k_v \leq (1 + \frac{\epsilon}{2})k\). Let \(G'\) be a bipartite graph with bipartition \(X, Y\) such that:

- \(G'\) contains disjoint cycles \(\{C_v : v \in V(G)\}\),
- \(|C_v| = 2k_v\) for each vertex \(v \in V(G)\),
- \(V(G') = \cup\{V(C_v) : v \in V(G)\}\), and
- for each edge \(vw\) of \(G\) there are edges \(xy\) and \(pq\) of \(G'\) such that \(x \in C_v \cap X\) and \(y \in C_w \cap Y\) and \(p \in C_v \cap Y\) and \(q \in C_w \cap X\).

Then \(G'\) is a two-sided bipartite \(\epsilon'\)-expander, for some \(\epsilon'\) depending only on \(\epsilon, k\) and \(\Delta\).

**Proof.** For each vertex \(v\) of \(G\), we have \(|C_v \cap X| = |C_v \cap Y| = k_v\). Thus
\[
|X| = |Y| = \sum_{v \in V(G)} k_v \leq (1 + \frac{\epsilon}{2})kn.
\]

Let \(S \subseteq X\) with \(|S| \leq \frac{|X|}{2}\), which is at most \((1 + \frac{\epsilon}{2})kn\). By the symmetry between \(X\) and \(Y\), it suffices to prove that \(|N_{G'}(S)| \geq (1 + \epsilon')|S|\).

For each vertex \(v\) of \(G\), observe that \(|C_v \cap S| \leq |C_v \cap X| = k_v\). Say \(v\) is **heavy** if \(|C_v \cap S| = k_v\). Say \(v\) is **light** if \(1 \leq |C_v \cap S| \leq k_v - 1\). Say \(v\) is **unused** if \(|C_v \cap S| = 0\). Each vertex of \(G\) is either heavy, light or unused.

Say a heavy vertex \(v\) of \(G\) is **fat** if every neighbour of \(v\) is also heavy. Let \(F\) be the set of fat vertices in \(G\). Let \(H\) be the set of non-fat heavy vertices in \(G\). Let \(L\) be the set of light vertices in \(G\). Let \(U\) be the set of unused vertices in \(G\). Thus \(F, H, L, U\) is a partition of \(V(G)\). Let \(f := |F|\) and \(h := |H|\) and \(l := |L|\). Let \(f_A := |F \cap A|\) and \(f_B := |F \cap B|\) and \(h_A := |H \cap A|\) and \(h_B := |H \cap B|\).

Since the vertices in \(H\) are not fat, every vertex in \(H\) has a neighbour in \(L \cup U\). Let \(H'\) be the set of vertices in \(H\) adjacent to no vertex in \(U\) (and thus with a neighbour in \(L\)). Let \(H''\) be the set of vertices in \(H\) adjacent to some vertex in \(U\). Define \(h' := |H'|\) and \(h'' := |H''|\).
For each vertex $v$ of $G$, let $c(v)$ be the number of vertices in $C_v$ adjacent to some vertex in $C_v \cap S$. Since $C_v \cap S$ is an independent set in $C_v$, by Lemma 10 below, if $v$ is heavy, then every second vertex of $C_v$ is in $S$ and $c(v) = k_v = |C_v \cap S|$, and if $v$ is light then $c(v) \geq |C_v \cap S| + 1$. Thus

$$|N_{G'}(S)| \geq \sum_{v \in F \cup H \cup L} c(v) \geq \ell + \sum_{v \in F \cup H \cup L} |C_v \cap S| = \ell + |S|. \quad (1)$$

Moreover, each vertex $v$ in $H''$ is adjacent in $G$ to some vertex $w$ in $U$. By assumption, there is an edge $xy$ of $G'$ such that $x \in C_v \cap X$ and $y \in C_w \cap Y$. Since $v$ is heavy, $x$ is in $S$ and $y$ is in $N_{G'}(S)$. And since $w$ is unused, $y$ is adjacent to no vertex in $C_w \cap S$. Thus $y$ is not counted in the lower bound on $N_{G'}(S)$ in (1). Each such vertex $y$ is adjacent to at most $\Delta$ vertices in $H''$. Hence

$$|N_{G'}(S)| \geq |S| + \ell + \frac{h''}{\Delta}. \quad (2)$$

Our goal now is to prove that $\ell + \frac{h''}{\Delta} \geq e'|S|$, where $e' := (k + k\Delta(1 + \frac{1}{\epsilon}))^{-1}$. Since $1 - e'(k + k\Delta(1 + \frac{1}{\epsilon})) = 0$ and $\frac{1}{\Delta} - e'k(1 + \frac{1}{\epsilon}) \geq 0$,

$$(1 - e'(k + k\Delta(1 + \frac{1}{\epsilon})))\ell + \left(\frac{1}{\Delta} - e'k(1 + \frac{1}{\epsilon})\right)h'' \geq 0.$$

That is,

$$\ell + \frac{h''}{\Delta} \geq e'k\ell + e'k(1 + \frac{1}{\epsilon})(\Delta \ell + h'').$$

Every vertex in $H'$ has a neighbour in $L$, each of which has degree at most $\Delta$. Thus $h' \leq \Delta \ell$, implying

$$\ell + \frac{h''}{\Delta} \geq e'k\ell + e'k(1 + \frac{1}{\epsilon})(h' + h'') = e'k(\ell + h + \frac{h''}{\Delta}).$$

Suppose, on the contrary, that $f_A \geq \lfloor \frac{n}{2} \rfloor + 1$. Let $Q$ be a subset of $F \cap A$ of size $\lfloor \frac{n}{2} \rfloor$. Since $G$ is a two-sided $\epsilon$-expander, and since every neighbour of each vertex in $F \cap A$ is in $(F \cup H) \cap B$, we have $f_B + h_B \geq |N_G(Q)| \geq (1 + \epsilon)\lfloor \frac{n}{2} \rfloor$. Thus

$$f_A + f_B + h_B \geq \lfloor \frac{n}{2} \rfloor + 1 + (1 + \epsilon)\lfloor \frac{n}{2} \rfloor \geq (2 + \epsilon)\lfloor \frac{n}{2} \rfloor + 1 \geq n + \frac{\epsilon}{3}(n - 1).$$

However, $(1 + \frac{\epsilon}{3})kn \geq |S| \geq k(f_A + f_B + h_B)$, implying $f_A + f_B + h_B \leq (1 + \frac{\epsilon}{3})n$, which is a contradiction (since $n \geq 3$).

Now assume that $f_A \leq \frac{n}{2}$. Since $G$ is a two-sided $\epsilon$-expander, and since every neighbour of a vertex in $F \cap A$ is in $(F \cup H) \cap B$, we have $f_B + h_B \geq |N_G(F \cap A)| \geq (1 + \epsilon)f_A$. By symmetry, $f_A + h_A \geq (1 + \epsilon)f_B$. Thus $f + h \geq (1 + \epsilon)f$ and $h \geq \epsilon f$. Hence

$$\ell + \frac{h''}{\Delta} \geq e'k(\ell + h + f).$$

Since $|S| \leq k(f + h + \ell)$,

$$\ell + \frac{h''}{\Delta} \geq e'|S|,$$

and $N_{G'}(S) \geq (1 + e')|S|$, as desired. \qed
Lemma 10. Let $I$ be an independent set in a cycle graph $C$. Then $|N_C(I)| \geq |I|$ with equality only if $I = \emptyset$ or $|C| = 2|I|$.

Proof. For each vertex $x$ in $N_C(I)$, if $x$ is adjacent to exactly one vertex $v$ in $I$, then send the charge of 1 from $x$ to $v$, and if $x$ is adjacent to exactly two vertices $v$ and $w$ in $I$, then send a charge of $\frac{1}{2}$ from $x$ to each of $v$ and $w$. Each vertex in $I$ receives a charge of at least $\frac{1}{2}$ from each of its neighbours in $C$. Thus the total charge, $|N_C(I)|$, is at least $I$, as claimed. If the total charge equals $|I|$, then each vertex $v$ in $I$ receives a charge of exactly 1, which implies that both neighbours of $v$ sent a charge of $\frac{1}{2}$ to $v$. Thus both neighbours of $v$ are adjacent to two vertices in $I$. It follows that $I$ consists of every second vertex in $C$, and $|C| = 2|I|$.

5 The Wall

The following example is a key to our main proofs, and is of independent interest. The wall is the infinite graph $W$ with vertex set $\mathbb{Z}^2$ and edge set
\[
\{(x,y)(x+1,y) : x, y \in \mathbb{Z}\} \cup \{(x,y)(x,y+1) : x, y \in \mathbb{Z}^+, x + y \text{ even}\}.
\]

As illustrated in Figure 4, the wall is 3-regular and planar.

The next two results depend on the following vertex ordering of $W$. For vertices $(x,y)$ and $(x',y')$ of $W$, define $(x,y) \preceq (x',y')$ if $x + y < x' + y'$, or $x + y = x' + y'$ and $x \leq x'$.

Lemma 11. The wall is 3-monotone bipartite.

Proof. Let $A := \{(x,y) \in \mathbb{Z}^2 : x + y \text{ even}\}$ and $B := \{(x,y) \in \mathbb{Z}^2 : x + y \text{ odd}\}$. Observe that $A,B$ is a bipartition of $W$. Consider $A$ and $B$ to be ordered by $\preceq$. Colour the edges of
$W$ as follows. For each vertex $(x, y)$ where $x + y$ is even, colour $(x, y)(x + 1, y)$ red, colour $(x, y)(x - 1, y)$ blue, and colour $(x, y)(x, y + 1)$ green, as illustrated in Figure 4. Each edge of $W$ is thus coloured. If $(x, y) < (x', y')$ in $A$, then $(x + 1, y) < (x' + 1, y)$ in $B$. Thus the red edges form a monotone matching. Similarly, the green edges form a monotone matching, and the blue edges form a monotone matching. Thus $W$ is 3-monotone.

Lemma 12. The wall has a 2-queue layout, such that for all edges pq and pr with $p < q < r$ or $r < q < p$, the edges pq and pr are in distinct queues (called a 'strict' 2-queue layout in [36]).

Proof. We first prove that no two edges of $W$ are nested with respect to $\preceq$. Suppose that some edge $(x_2, y_2)(x_3, y_3)$ is nested inside another edge $(x_1, y_1)(x_4, y_4)$, where $(x_1, y_1) < (x_2, y_2) < (x_3, y_3) < (x_4, y_4)$. By the definition of $\preceq$, we have $x_1 + y_1 \leq x_2 + y_2 \leq x_3 + y_3 \leq x_4 + y_4$. Since $(x_2, y_2)(x_3, y_3)$ and $(x_1, y_1)(x_4, y_4)$ are edges, $x_4 + y_4 = x_1 + y_1 + 1$ and $x_3 + y_3 = x_2 + y_2 + 1$. Hence $x_1 + y_1 = x_2 + y_2$ and $x_3 + y_3 = x_4 + y_4$. By the definition of $\preceq$, we have $x_1 < x_2$ and $y_2 < y_1$, and $x_3 < x_4$ and $y_4 < y_3$. Since $(x_1, y_1)(x_4, y_4)$ is an edge with $(x_1, y_1) < (x_4, y_4)$, either $x_4 = x_1 + 1$ or $y_4 = y_1 + 1$. First suppose that $x_4 = x_1 + 1$. Then $x_1 < x_2 \leq x_3$, implying $x_4 = x_1 + 1 \leq x_3$, which is a contradiction. Now assume that $y_4 = y_1 + 1$. Then $y_1 + 1 = y_4 < y_3$. Since $(x_2, y_2)(x_3, y_3)$ is an edge, $y_3 \leq y_2 + 1$, implying $y_1 < y_2$, which is a contradiction. Hence no two edges are nested.

For each vertex $(x, y)$ where $x + y$ is even, assign the edges $(x, y)(x + 1, y)$ and $(x, y)(x - 1, y)$ to the first queue, and assign the edge $(x, y)(x, y + 1)$ to the second queue. If $x + y$ is even, then $(x, y)$ has two neighbours $(x + 1, y)$ and $(x, y + 1)$ to the right of $(x, y)$ in $\preceq$, and one neighbour $(x - 1, y)$ to the left. On the other hand, if $x + y$ is odd, then $(x, y)$ has two neighbours $(x - 1, y)$ and $(x, y - 1)$ to the left of $(x, y)$ in $\preceq$, and one neighbour $(x + 1, y)$ to the right. Consider a vertex $p = (x, y)$ incident to distinct edges pq and pr. If $p < q < r$, then $x + y$ is even and $q = (x, y + 1)$ and $r = (x + 1, y)$, implying that pq and pr and in distinct queues. If $r < q < p$, then $x + y$ is odd and $r = (x - 1, y)$ and $q = (x, y - 1)$, implying that pq and pr and in distinct queues.

Lemma 13. The wall has a 4-track layout, such that for all distinct edges pq and pr, the vertices $q$ and $r$ are in distinct tracks.

Proof. Consider the following vertex ordering of $W$. For vertices $(x, y)$ and $(x', y')$ of $W$, define $(x, y) \preceq (x', y')$ if $x < x'$, or $x = x'$ and $y \leq y'$.

Colour each vertex $(x, y)$ of $W$ by $(x + 2y) \mod 4$, as illustrated in Figure 5. Observe that this defines a proper vertex colouring of $W$. Order each colour class by $\preceq$. Each colour class is now a track. Observe that for all distinct edges pq and pr, the vertices $q$ and $r$ are in distinct tracks. Put another way, this is a 4-colouring of the square of $W$.

Suppose on the contrary that edges $(x_1, y_1)(x_4, y_4)$ and $(x_2, y_2)(x_3, y_3)$ cross, where $(x_1, y_1) < (x_2, y_2)$ in some track, and $(x_3, y_3) < (x_4, y_4)$ in some other track. Thus $x_1 \leq x_2$ and $x_3 \leq x_4$. 

Without loss of generality, \((x_1, y_1) \prec (x_3, y_3)\). Thus \((x_1, y_1) \prec (x_3, y_3) \prec (x_4, y_4)\). Hence \(x_1 \leq x_3 \leq x_4\).

Suppose that \(x_1 = x_4\). Thus \(y_1 < y_3 < y_4\), implying \(y_4 \geq y_1 + 2\) and \((x_1, y_1)(x_4, y_4)\) is not an edge. Now assume that \(x_1 < x_4\). Since \((x_1, y_1)(x_4, y_4)\) is an edge, \(x_4 = x_1 + 1\) and \(y_1 = y_4\).

In what follows, all congruences are modulo 4. We have \(x_3 + 2y_3 \equiv x_4 + 2y_4\). Thus \(x_3 - x_4 \equiv 2(y_4 - y_3)\), implying \(x_3 - x_4\) is even. Since \(x_1 \leq x_3 \leq x_4 = x_1 + 1\), we have \(x_3 = x_4\). Since \((x_3, y_3) \prec (x_4, y_4)\), we have \(y_3 < y_4\). Since \(x_1 + 2y_1 \equiv x_2 + 2y_2\), we have \(x_1 - x_2 \equiv 2(y_2 - y_1)\), implying \(x_1 - x_2\) is even.

Suppose that \(x_2 \leq x_4\). Then \(x_1 \leq x_2 \leq x_4 = x_1 + 1\). Since \(x_1 - x_2\) is even, \(x_1 = x_2\). Since \((x_1, y_1) \prec (x_2, y_2)\), we have \(y_1 < y_2\). Since \((x_2, y_2)(x_3, y_3)\) is an edge and \(x_3 = x_4 = x_1 + 1 = x_2 + 1\), we have \(y_2 = y_3\). Similarly, since \((x_1, y_1)(x_4, y_4)\) is an edge and \(x_4 = x_1 + 1\), we have \(y_1 = y_4\). Since \(y_3 < y_4\), we have \(y_2 < y_1\), which is a contradiction. Now assume that \(x_2 > x_4\).

Since \(x_2 > x_4 = x_3\) and \((x_2, y_2)(x_3, y_3)\) is an edge, \(y_2 = y_3\). Thus \(x_2 = x_3 + 1 = x_4 + 1 = x_1 + 2\). Since \(x_1 + 2y_1 \equiv x_2 + 2y_2\) we have \(2y_1 \equiv 2 + 2y_2\), implying \(y_1 - y_2\) is odd. Since \(y_1 = y_4\) and \(y_3 = y_2\), we have \(y_4 - y_3\) is odd. However, since \(x_3 = x_4\) and \(x_3 + 2y_3 \equiv x_4 + 2y_4\), we have \(y_3 - y_4\) is even. This contradiction proves that no two edges between the same pair of tracks cross. \(\square\)

6 The Main Proofs

Here we give a unified proof of Theorem 2, Theorem 4 and Theorem 5. Let \(G\) be a two-sided 2k-monotone bipartite \(\epsilon\)-expander with bipartition \(A, B\). An infinite family of such graphs exist by Theorem 7 for fixed \(\epsilon\) and \(k\). We may assume that \(k \geq \frac{2}{\epsilon}\). Let \(E_1, \ldots, E_{2k}\) be the corresponding partition of \(E(G)\). Now define a graph \(G'\). For each vertex \(v \in A\), introduce the following cycle
in $G'$:

$$C_v := (v_0, v_1, v_2, \ldots, v_k, v_k', v_{k-1}', \ldots, v_1', v_0').$$

For each vertex $w \in B$, introduce the following cycle in $G'$:

$$C_w := (w_{-1}, \omega_{-1}, w_0, \omega_0, w_1, \omega_1, w_2, \omega_2, \ldots, w_k, \omega_k, w_{k+1}, \omega_{k+1}, w_{k+2}, w_{k+3}, \omega_{k+3}, w_k', \omega_k', w_{k-1}', \omega_{k-1}', w_{k-2}', \omega_{k-2}', w_1', \omega_1', w_0', \omega_0', w_{-1}, \omega_{-1}).$$

All the above cycles are pairwise disjoint in $G'$. Finally, for each edge $vw$ of $G$, if $vw \in E_i$ then add the edges $v_iw_i$ and $v_i'w_i'$ to $G'$.

Observe that $G'$ is bipartite with colour classes:

$$X := \{v_i : i \in [1, k]\} \cup \{v_i' : i \in [0, k]\} \cup \{w_i' : i \in [-1, k + 2]\} \cup \{\omega_i : i \in [-1, k + 1]\}$$

$$Y := \{v_i' : i \in [1, k]\} \cup \{v_i : i \in [0, k]\} \cup \{w_i : i \in [-1, k + 2]\} \cup \{\omega_i' : i \in [-1, k + 1]\}.$$

We now show that Lemma 9 is applicable to $G'$. For $v \in A$, let $k_v := 2k + 1$. For $w \in A$, let $k_w := 2k + 7$. Each cycle $C_v$ has length $2k_v$, as required. Since $k \geq 3/\epsilon$, we have $\frac{2k+7}{2k+1} \leq 1 + \frac{\epsilon}{4}$, as required. We now show that final requirement in Lemma 9 is satisfied. Consider an edge $vw \in E_b$ where $v \in A$ and $w \in B$. Then $v_i \in C_v \cap X$ and $w_i \in C_w \cap Y$ and $v_i' \in C_w \cap X$ and $w_i' \in C_w \cap X$. Thus the edges $v_iw_i$ and $v_i'w_i'$ in $G'$ satisfy the final requirement in Lemma 9. Hence $G'$ is an $\epsilon'$-expander for some $\epsilon'$ depending on $\epsilon, k$ and $\Delta(G) \leq 2k$.

For $i \in [1, k]$, let $A_i := \{v_i : v \in A\}$ and $A_i := \{v_i : v \in A\}$. For $i \in [0, k]$, let $A_i := \{v_i : v \in A\}$ and $A_i := \{v_i : v \in A\}$. Similarly, for $i \in [-1, k + 2]$, let $B_i := \{w_i : w \in B\}$ and $B_i := \{w_i : w \in B\}$ and $\Omega_i := \{\omega_i : w \in B\}$ and $\Omega_i := \{\omega_i : w \in B\}$. By ordering each of these sets by the given ordering of $A$ or $B$, we consider each such set to be a track. As illustrated in Figure 6, the graph $H$ obtained from $G'$ by identifying each of these tracks into a single vertex is a subgraph of the wall.

![Figure 6: The graph H. The inner cycle corresponds to vertices in A. The outer cycle corresponds to vertices in B.](image)

In other words, there is a homomorphism from $G$ to $H$, where the preimage of each vertex in $H$ is a track in $G$. For each edge $pq$ of $H$, where $pq$ is of the form $a_i b_i$ or $a_i' b_i'$, there is no crossing.
in \( G' \) between the tracks corresponding to \( p \) and \( q \) since these edges correspond to a monotone matching. For every other edge \( pq \) of \( H \), the edges between the tracks corresponding to \( p \) and \( q \) form a non-crossing perfect matching. By Lemma 11, \( H \) is 3-monotone. Replacing each vertex of \( H \) by the corresponding track gives a 3-monotone layout of \( G' \), as illustrated in Figure 7(a). This proves Theorem 2. Similarly, by Lemma 12, \( H \) has a 2-queue layout, such that for all edges \( pq \) and \( pr \) with \( p < q < r \) or \( r < q < p \), the edges \( pq \) and \( pr \) are in distinct queues. Replacing each vertex of \( H \) by the corresponding track gives a 2-queue layout of \( G' \), as illustrated in Figure 7(b). This proves Theorem 4. Finally, by Lemma 13, \( H \) has a 4-track layout, such that for all edges \( pq \) and \( pr \), the vertices \( q \) and \( r \) are in distinct tracks. Replacing each vertex of \( H \) by the corresponding track gives a 4-track layout of \( G' \), as illustrated in Figure 7(c). This proves Theorem 5. Note that, in fact, between each pair of tracks, the edges form a monotone matching.

![Figure 7](image)

**Figure 7:** Replacing each vertex of \( H \) by a track in \( G' \).

### 7 Open Problems

Heath et al. [27, 28] conjectured that planar graphs have bounded queue-number, which holds if and only if 2-page graphs have bounded queue-number [15]. The best upper bound on the queue-number of planar graphs is \( O(\log n) \) due to Dujmo\'vi\'\'c [9]; see [12] for recent extensions.

More generally (since planar graphs have bounded page-number [6, 37]), Dujmo\'vi\'\'c and Wood [15] asked whether queue-number is bounded by a function of page-number. This is equivalent to whether 3-page graphs have bounded queue-number [15].

Dujmo\'vi\'\'c and Wood [15] also asked whether page-number is bounded by a function of queue-number, which holds if and only if 2-queue graphs have bounded page-number [15].
Grohe and Marx [26] established a close connection between expanders and linear treewidth that, with Theorem 3, gives an infinite family of \( n \)-vertex 3-page graphs with \( \Omega(n) \) treewidth (and maximum degree 3). This observation seems relevant to a question of Dujmović and Wood [16], who asked whether there is a polynomial time algorithm to determine the book thickness of a graph with bounded treewidth; see [2] for related results and questions.

A final thought: 3-page graphs arise in knot theory, where they are called Dynnikov Diagrams [21–23]. It would be interesting to see if the existence of 3-page expanders has applications in this domain.

References


A Separators in Bipartite Expanders

A separator in a graph $G$ is a set $Z \subseteq V(G)$ such that each component of $G - Z$ has at most $\frac{|V(G)|}{2}$ vertices. The following connection between expanders and separators is well known, although we are unaware of an explicit proof for bipartite expanders, so we include it for completeness.

**Lemma 14.** If $G$ is bipartite $\epsilon$-expander with $2n$ vertices, then every separator in $G$ has size at least $\frac{\epsilon}{2}(n - 1) - 1$.

**Proof.** Let $A, B$ be the bipartition of $G$ with $|A| = |B| = n$. Let $Z$ be a separator of $G$. Our goal is to prove that $|Z| \geq \frac{\epsilon}{2}(n - 1) - 1$. Let $Z_1 := Z \cap A$ and $Z_2 := Z \cap B$.

Let $X_1, \ldots, X_k$ be a partition of $V(G - Z)$ such that each $X_i$ is the union of some subset of the components of $G - Z$ with at most $n$ vertices in total, and subject to this condition, $k$ is minimal. This is well-defined, since each component of $G - Z$ has at most $n$ vertices. By minimality, $|X_i| + |X_j| > n$ for all distinct $i, j \in [1, k]$. If $k \geq 4$ then $|X_1| + |X_2| > n$ and $|X_3| + |X_4| > n$, which contradicts the fact that $|V(G)| = 2n$. Hence $k \leq 3$. Let $A_i := X_i \cap A$ for $i \in [1, k]$.

First suppose that $|A_i| \geq \frac{n}{2}$ for some $i \in [1, k]$. Let $S$ be a subset of $A_i$ with exactly $\lfloor \frac{n}{2} \rfloor$ vertices. Observe that $N(S) \subseteq (X_i \setminus S) \cup Z$. Thus

$$(1 + \epsilon)|S| \leq |N(S)| \leq |X_i| - |S| + |Z| \leq n - |S| + |Z|,$$

and

$$|Z| \geq (2 + \epsilon)|S| - n = (2 + \epsilon)\left(\frac{n}{2}\right) - n \geq (2 + \epsilon)\left(\frac{n - 1}{2}\right) - n = \epsilon\frac{n}{2}(n - 1) - 1,$$

as desired.

Now assume that $|A_i| < \frac{n}{2}$ for all $i \in [1, k]$. Observe that

$$\sum_i (2 + \epsilon)|A_i| - |X_i| = (2 + \epsilon)\sum_i |A_i| - \sum_i |X_i|$$

$$= (2 + \epsilon)(n - |Z_1|) - (2n - |Z_1| + |Z_2|)$$

$$= \epsilon n - (1 + \epsilon)|Z_1| + |Z_2|.$$  

Thus, for some $i \in [1, k]$, we have $(2 + \epsilon)|A_i| - |X_i| \geq \frac{1}{k}(\epsilon n - (1 + \epsilon)|Z_1| + |Z_2|)$. Observe that $N(A_i) \subseteq (X_i \setminus A_i) \cup Z_2$. Thus

$$(1 + \epsilon)|A_i| \leq |N(A_i)| \leq |X_i| - |A_i| + |Z_2|,$$

and

$$|Z_2| \geq (2 + \epsilon)|A_i| - |X_i| \geq \frac{1}{k}(\epsilon n - (1 + \epsilon)|Z_1| + |Z_2|),$$

implying

$$(k - 1)|Z_2| + (1 + \epsilon)|Z_1| \geq \epsilon n.$$  

Since $k \leq 3$ and $1 + \epsilon \leq 2$, we have $2|Z| \geq (k - 1)|Z_2| + (1 + \epsilon)|Z_1| \geq \epsilon n$, implying $|Z| \geq \frac{\epsilon n}{2}$ as desired. \qed