Adaptive Learning, Forecast-Based Instrument Rules and Monetary Policy*

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Abstract

This paper argues that recently popular forecast-based instrument rules for monetary policy may fail to stabilize economic fluctuations. In a New Keynesian model of output gap and inflation determination in which private agents face multi-period decision problems, but have non-rational expectations and learn over time, if the monetary authority adopts a forecast-based instrument rule and responds to observed private forecasts then this class of policies frequently induce divergent learning dynamics. A central bank that correctly understands private behavior can mitigate such instability by responding to the determinants of private forecasts. This suggests gathering information on the determinants of expectations to be useful.

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1 Introduction

The recent monetary policy rules literature argues that private-sector forecasts are an important part of central bank decision procedures for the determination of the nominal interest rate. Hall and Mankiw (1994) propose nominal GDP forecast targeting. Batini and Haldane (1999) argue that a simple interest-rate rule that posits the nominal interest rate as depending on private-sector inflation forecasts provides a robust formulation of policy. Such rules are also found in a number of large-scale macroeconomic models used in policy evaluation (see, for instance, the Reserve Bank of New Zealand or Bank of England forecasting models). Clarida, Gali, and Gertler (1998, 2000) also provide evidence that reaction functions of a number of central banks find an important role for expectations in the current stance of policy. Giannoni and Woodford (2002a) demonstrate optimal targeting rules invariably imply an instrument setting that depends on expectations. More recently, Levin, Wieland, and Williams (2003) provide evidence that appropriately designed forecast-based instrument rules are robust to model uncertainty.

In evaluating the merit of such policy proposals, this literature typically assumes that both the central bank and private agents possess the same model of the economy and have rational expectations. It follows that all economic actors hold identical expectations regarding the evolution of the economic variables of interest and therefore that there is no important distinction between internal central bank forecasts and external private forecasts. To the extent that good monetary policy depends on expectations, it is sufficient for the policy problem to be cast in terms of internal central bank forecasts.

In practice, however, internal central bank forecasts and external private forecasts rarely coincide. As a result, both of these sources of forecasts provide potentially important information for the monetary policy decision process. This is evidenced by the considerable resources that central banks spend on forecasting the near-term evolution of the economy. In addition, external forecasts of various private agents are monitored using an array of surveys on the ground that policy can be improve by having more information about the state of the economy. But if these two sources of information about the near-term evolution of the

economy diverge, what then is the appropriate dependency of the central bank's instrument setting on such forecasts. Is it appropriate for monetary policy to depend on private forecasts and if so in exactly what way? Or are there reasons for a central bank to concern itself solely with internal forecasts?

This paper therefore seeks to examine whether policy rules that posit the interest rate to depend on private expectations are desirable as a means to stabilize economic fluctuations when agents and the central bank have differing expectations about the evolution of the macroeconomy. We will be interested to learn what is the appropriate dependency of optimal monetary policy decision procedures on private forecasts. In particular, we shall explore whether desirable policy can be described by an instrument rule that naively responds to observed private expectations, or whether more sophisticated uses of the information embodied in these forecasts is required in internal central bank forecasting procedures for economic stability. In so doing, it can be adjudged whether central banks need only devote resources to the measurement of forecasts themselves or whether greater resources need to be devoted to understanding the underlying determinants of such forecasts. Furthermore, if knowledge of the determinants of private forecasts is desirable, the analysis will shed light on their appropriate use in monetary policy design.

This paper proceeds as follows. Section 2 outlines the analysis of Preston (2003), which develops a model in which agents face multi-period decision problems as in the microfoundations used in recent analysis of the implications of monetary policy rules under rational expectations – see Bernanke and Woodford (1997), Clarida, Gali, and Gertler (1999) and Woodford (1999). In contrast to these papers, private agents are not assumed to possess a complete economic model in making their spending and pricing decisions and therefore cannot infer the true probability laws that govern the evolution of the economy and must instead attempt to learn them over time. They do this by use of a simple econometric model, which is updated each period as additional data become available. Section 3 discusses private agents beliefs and the learning algorithm in some detail.

Section 4 outlines the optimal policy problem. Two forecast-based instrument rules are

proposed that are consistent with implementing the resulting optimal equilibrium. Several approaches to constructing the required forecasts to implement these rules are then considered. Section 5 analyzes a decision procedure in which the monetary authority responds to observed private forecasts. It shows that this approach to policy is likely to facilitate divergent learning dynamics, and is therefore undesirable as a means to stabilizing economic fluctuations in the presence of private agent learning. Importantly, this contrasts with the analysis of Bullard and Mitra (2002), which finds the Taylor principle to be necessary and sufficient for stability under learning dynamics. However, it is also shown that if private agent's are endowed with knowledge of the monetary policy rule such instability is mitigated – the so-called Taylor principle is necessary and sufficient for stability under learning dynamics. This finding is similar in spirit to Orphanides and Williams (2004) which shows that transparency about the central bank's long-run inflation objective can engender a more favorable inflation-output trade off when agents must learn about the economy's inflation dynamics.

Section 6 proposes a second decision procedure that assumes the monetary authority to have correct knowledge of private agents' learning behavior and decision rules. This allows the monetary authority to construct optimal forecasts of the evolution of the economy conditional on agents' behavior. In this case, the instability problems associated with the first decision procedure are largely avoided. Indeed, as found by Bullard and Mitra (2002), the forecast-based instrument rules lead to stability under learning dynamics if and only if the so-called Taylor principle is satisfied. Importantly, these results highlight the importance of gathering information about the determinants of private sector expectations. Section 7 offers some remarks on alternative approaches to implementing optimal policy under learning – a topic that is more thoroughly treated in Preston (2004). The final section concludes.

2 A Simple Model

Preston (2003) analyzes the microfoundations used in several recent studies of monetary policy rules under a specific non-rational expectations assumption. This leads to important

differences in the model's implied aggregate dynamics when agents are learning relative to the predictions of rational expectations equilibrium analysis. This section recapitulates the central assumptions and results of that paper and the reader is encouraged to consult it for details. The microfoundations and resulting optimal decision rules for households and firms are also presented in the appendix.

2.1 Primitive Assumptions and Aggregate Dynamics

This paper analyzes a simple New Keynesian sticky-price model of output gap and inflation determination, comprising aggregate demand and supply relations of the form:

$$x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)x_{T+1} - \sigma(i_T - \pi_{T+1}) + r_T\right] \tag{1}$$

and

$$\pi_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa\alpha\beta \cdot x_{T+1} + (1-\alpha)\beta \cdot \pi_{T+1} + u_T\right] \tag{2}$$

where x_t is the output gap, π_t the inflation rate, i_t the nominal interest rate and r_t and u_t are exogenous disturbance terms, with all variables being properly interpreted as log-deviations from steady state values. $\sigma > 0$ is the intertemporal elasticity of substitution, $0 < \beta < 1$ the discount factor, $\kappa > 0$ and $0 < 1-\alpha < 1$ the probability that a firm will have an opportunity to change its price in any period. Preston (2003) shows that these relations can be derived from aggregation of a log-linear approximation to the optimal decision rules of the households and firms described by Bernanke and Woodford (1997), Clarida, Gali, and Gertler (1999) and Woodford (1999) when private agents have arbitrary subjective expectations. Agents are assumed not to have a complete economic model with which to derive the probability laws predicted by a rational expectations equilibrium analysis. Instead, agents form beliefs about the future paths of state variables that are relevant to their decision problems using simple econometric models described in detail in the subsequent section. Thus, $\hat{E}_t$ denotes the assumed average non-rational expectations operator.

The first equation represents the aggregation of optimal consumption decisions by households which are implications of their Euler equation and intertemporal budget constraint.

It is therefore an aggregate demand relation, specifying that output is determined by the current real rate of interest and long-horizon expectations of the output gap, the real interest rate and exogenous disturbances into the indefinite future. The presence of long-horizon expectations arise from the intertemporal nature of the household's consumption decision: to optimally allocate consumption today requires the household to plan its future consumption over time and across states of nature, which in turn requires forecasts of variables such as income and real interest rates.[1]

Relation (2) is derived from the aggregation of the optimal prices chosen by firms to maximize the expected discounted flow of profits under a Calvo-style price-setting problem. It is therefore a generalized New-Keynesian Phillips curve, specifying current inflation as depending on the contemporaneous output gap and expectations of this variable and inflation into the indefinite future. Here the presence of long-horizon expectations arise due to the pricing frictions induced by Calvo pricing. When a firm has the opportunity to change its price in period t there is probability α^{T-t} that the firm will not get to change its price in the subsequent $T-t$ periods. The firm must therefore concern itself with macroeconomic conditions relevant to marginal costs into the indefinite future when deciding the current price of its output. Future profits are also discounted at the rate β which equals the inverse of the steady-state gross real interest rate.[2]

Neither the aggregate demand relation (1) nor the Phillips curve (2) can be simplified under arbitrary subjective expectations about the evolution of state variables if agents are optimizing, as demonstrated in Preston (2003, 2004b). However, under the assumption of

[1] Indeed the connection of this relation to the predictions of permanent income theory is immediate. The first term captures precisely the basic insight of the permanent income hypothesis that agents should consume a constant fraction of the expected future discounted wealth, given a constant real interest rate equal to $\beta^{-1}-1$. The second term arises from the assumption of a time-varying real interest rate, and represents deviations from this constant real rate due to either variation in the nominal interest rate or inflation. The final term results from allowing stochastic disturbances to the economy.

[2] This expression has been modified slightly from the analysis of Preston (2003) by allowing for a cost-push shock. This is done to ensure a non-trivial stabilization problem for the monetary authority when the design of optimal monetary policy rules is considered.

rational expectations these relations simplify to give:

$$x_t = E_t x_{t+1} - \sigma^{-1}(i_t - E_t \pi_{t+1}) + r_t \tag{3}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t. \tag{4}$$

A rational expectations equilibrium analysis implies that once the true probability laws are known, only one-period-ahead expectations matter for aggregate dynamics. It is clear that learning has important implications for aggregate economic dynamics: with subjective expectations agents optimally require long-horizon expectations of macroeconomic conditions into the indefinite future. The presence of these expectational variables is important for the study of monetary policy, as expectations represent an important source of instability. Indeed, the analysis shows that these additional expectation terms provides stronger ground to avoid use of forecast-based instrument rules than does a learning analysis of the kind proposed by Bullard and Mitra (2002) where only one-period-ahead expectations matter.[3]

2.2 Adaptive Learning

Following much of the recent literature on learning in macroeconomics, this paper assumes agents learn adaptively, using a recursive least-squares algorithm. This allows application of standard convergence, or E-Stability results, outlined in Evans and Honkapohja (2001). Appendix A.2 outlines the notion of E-Stability in the context of this model and the monetary policy considered below. E-Stability provides conditions under which, if agents make small forecasting errors relative to rational expectations, their learning behavior corrects these errors over time and ensures convergence to the rational expectations dynamics.

Agents are assumed to have identical beliefs and to construct forecasts using an econometric model that uses as regressors variables that appear in the minimum-state-variable solution to the associated rational expectations problem. While this knowledge is likely to be a strong assumption, to the extent that instability arises given this knowledge, it seems unlikely that the policy would be desirable under less favorable circumstances when private

[3] See Bullard and Mitra (2002), Evans and Honkapohja (2002), Honkapohja and Mitra (2003, 2004) for analyses of this type.

agents have a misspecified or overparameterized model.[4]

Suppose that monetary policy is specified as a relation of the form

$$i_t = \psi_x \Lambda_{t-1} + \psi_u u_t + \psi_r r_t$$

where Λ_{t-1} is some exogenous variable that follows a first-order autoregressive process and that the monetary transmission mechanism is described by (3) and (4).[5] It follows immediately from standard analysis that there exists a rational expectations equilibrium that is linear in the variables $\{\Lambda_{t-1}, u_t, r_t\}$. Agents therefore estimate the linear model

$$z_t = a_t + b_t \cdot z_{t-1} + c_t \cdot u_t + d_t \cdot r_t + \epsilon_t \tag{5}$$

where $z_t = (\pi_t, x_t, i_t, \Lambda_t)'$, ϵ_t is the usual error-vector term, $\{a_t, b_t, c_t, d_t\}$ are parameters to be estimated of the form

$$a_t = \begin{bmatrix} a_{\pi,t} \\ a_{x,t} \\ a_{i,t} \\ a_{\Lambda,t} \end{bmatrix}; \; c_t = \begin{bmatrix} c_{\pi,t} \\ c_{x,t} \\ c_{i,t} \\ c_{\Lambda,t} \end{bmatrix}; \; d_t = \begin{bmatrix} d_{\pi,t} \\ d_{x,t} \\ d_{i,t} \\ d_{\Lambda,t} \end{bmatrix}$$

and

$$b_t \equiv \begin{bmatrix} 0 & 0 & 0 & b_{\pi,t} \\ 0 & 0 & 0 & b_{x,t} \\ 0 & 0 & 0 & b_{i,t} \\ 0 & 0 & 0 & b_{\Lambda,t} \end{bmatrix}.$$

The estimation procedure makes use of the entire history of available data in period t, $\{1, z_s, u_s, r_s\}_0^{t-1}$. As additional data become available, agents update their estimates of the

[4] In principle agents could make use of additional variables in constructing any relevant forecasts, or mistakenly omit variables that appear in the minimum-state-variable solution. In the former case, agents are still capable of learning the underlying REE as the forecasting equation nests the forecasting functions that obtain in an REE. In the latter, it is clear that agents can never learn the REE of interest. Analysis of these cases is clearly of interest, though beyond the scope of the paper.

[5] For the optimal policies considered in this paper, the reduced-form dynamics of the nominal interest rate will generally be of this form.

coefficients (a_t, b_t, c_t, d_t) . This is neatly represented as the recursive least squares formulation

$$\phi_t = \phi_{t-1} + t^{-1} R_t^{-1} w_{t-1}(z_{t-1} - \phi'_{t-1} w_{t-1}) \tag{6}$$

$$R_t = R_{t-1} + t^{-1}(w_{t-1} w'_{t-1} - R_{t-1}) \tag{7}$$

where the first equation describes how the forecast coefficients, $\phi_t = (a'_t, b_{\pi,t},\ b_{x,t},\ b_{i,t}, b_{\Lambda,t}, c'_t, d'_t)'$, are updated with each new data point and the second the evolution of the matrix of second moments of the appropriately stacked regressors $w_t \equiv \{1, \Lambda_{s-1}, u_s, r_s\}_0^t$. For the remainder of this paper u_t and r_t are assumed to be AR(1) processes

$$u_t = \gamma u_{t-1} + \varepsilon_{u,t}$$

$$r_t = \rho r_{t-1} + \varepsilon_{r,t}$$

with known parameters $0 < \gamma < 1$ and $0 < \rho < 1$ and $\{\varepsilon_{u,t},\ \varepsilon_{r,t}\}$ uncorrelated, bounded, i.i.d. disturbance processes. The assumption that the autoregressive parameters are known is made for algebraic convenience and is not important to the conclusions of this paper. Given homogeneity of beliefs, average forecasts can then be constructed by solving (5) backward and taking expectations to give

$$\begin{aligned} \hat{E}_t z_T = {} & (I_4 - b_t)^{-1} (I_4 - b_t^{T-t}) a_t + b_t^{T-t} z_t + \gamma u_t (\gamma I_4 - b_t)^{-1} \left(\gamma^{T-t} I_4 - b_t^{T-t}\right) c_t \\ & + \rho r_t (\rho I_4 - b_t)^{-1} \left(\rho^{T-t} I_4 - b_t^{T-t}\right) d_t \end{aligned} \tag{8}$$

for $T \geq t$, where I_4 is a (4×4) identity matrix.

To summarize, the model of the macroeconomy comprises: an aggregate demand equation, (1), a Phillips curve, (2), and the forecasting system given by (5), (6) and (7), where the latter three will vary according to the adopted econometric model of agents.

3 Optimal Monetary Policy

This section outlines the optimal commitment problem under *rational expectations* and discusses the notion of "optimality from the timeless perspective" proposed by Woodford

(1999), which serves to restrict the class of admissible policies to those that are time consistent. Subsequent sections consider a number of decision procedures that are consistent with implementing optimal policy under rational expectations, but make differing use of the information embodied in observed private forecasts. We then ask whether learning dynamics present ground to prefer one particular approach over another – that is, are any of the proposed decision procedures to be preferred from the point of view of eliminating instability from divergent learning dynamics?

The monetary authority is assumed to minimize the loss function

$$W = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} L_t \tag{9}$$

where $0 < \beta < 1$ is the household's discount factor, and the period loss is given as

$$L_t = \pi_t^2 + \lambda x_t^2$$

for some weight $\lambda > 0$. Thus the central bank wishes to stabilize variation in inflation and the output gap, and λ determines the relative importance of these stabilization objectives. Since we are concerned with monetary policies that are desirable under rational expectations, minimization of the loss (9) occurs subject to (4).

Given this policy problem, this paper further restricts attention to the class of time-invariant policies that are optimal from the so-called timeless perspective proposed by Woodford (1999). Giannoni and Woodford (2002a) and Woodford (2003, chap. 7) demonstrate that a time invariant optimal commitment can be arranged by having the central bank act subject to the requirement that the initial evolution of the economy coincides with the evolution associated with the policy. These authors demonstrate that minimizing the loss (9) subject to (4) and the additional constraint that $\pi_{t_0} = \bar{\pi}_{t_0}$ where

$$\bar{\pi}_{t_0} = (1-\mu)\frac{\lambda}{\kappa} x_{t-1} + \frac{\mu}{1-\beta\mu\gamma} u_t \tag{10}$$

delivers a set of first-order conditions that are time invariant, in the sense that they hold in all periods of the proposed commitment, and therefore characterize the optimal evolution of

the economy under the timeless perspective. Absent the constraint on the initial evolution of inflation, these optimality conditions would fail to be time invariant.

Standard methods show that the optimal state-contingent paths of $\{\pi_t, x_t, i_t\}$ are given by the following relations:

$$\pi_t = (1-\mu)\frac{\lambda}{\kappa}x_{t-1} + \frac{\mu}{1-\beta\mu\gamma}u_t \tag{11}$$

$$x_t = \mu x_{t-1} - \frac{\kappa}{\lambda}\cdot\frac{\mu}{1-\beta\mu\gamma}u_t \tag{12}$$

and

$$i_t = \frac{\sigma\lambda-\kappa}{\sigma\kappa}(1-\mu)\mu\cdot x_{t-1} + \frac{\sigma\lambda-\kappa}{\sigma\lambda}\cdot\frac{\mu(\mu+\gamma-1)}{1-\beta\mu\gamma}\cdot u_t + \frac{1}{\sigma}r_t^n \tag{13}$$

where $0<\mu<1$ is the model's only eigenvalue within the unit circle. These equations completely characterize the solution of the optimal monetary policy problem from the timeless perspective under the rational expectations assumption.

There are several points to note. First, the bounded solution for the path of inflation exactly coincides with the constraint that was imposed on the initial evolution of this variable. Thus, the constraint required for optimality from the timeless perspective is characterized by a self-consistency property – it requires the central bank to ensure that the initial evolution of the economy coincides with the evolution of the economy associated with the policy. Second, the optimal solution exhibits history dependence as evidenced by the presence of the state variable, x_{t-1}. This reflects the fact that the central bank, in committing to behave in a particular way in the future, optimally ties these promised actions to current decisions – subsequent actions then fulfill past promises. Finally, note that the cost-push shock u_t clearly makes the stabilization problem non-trivial. Given an inflationary disturbance the central bank optimally brings about a contraction in real activity. In the absence of this shock, the optimal policy (from the timeless perspective) would be to completely stabilize both output and inflation.[6]

For later analysis it will be useful to represent this solution in terms of a particular

[6] See Woodford (2003, chap. 7) for a detailed dicussion.

exogenous state variable. Solving the output gap relation backwards recursively gives

$$x_t = -\frac{\kappa}{\lambda} \cdot \frac{\mu}{1-\beta\mu\gamma} \sum_{j=0}^{\infty} \mu^j u_{t-j}.$$

Defining the exogenous state variable

$$\Lambda_t \equiv \sum_{j=0}^{\infty} \mu^j u_{t-j},$$

which satisfies the process $\Lambda_t = \mu\Lambda_{t-1} + u_t$, allows the optimal solution to be written as

$$\pi_t^* = \frac{\mu}{1-\beta\mu\gamma} \left[u_t - (1-\mu)\Lambda_{t-1}\right] \tag{14}$$

$$x_t^* = -\frac{\kappa}{\lambda} \cdot \frac{\mu}{1-\beta\mu\gamma} \left[\mu\Lambda_{t-1} + u_t\right] \tag{15}$$

$$i_t^* = \frac{\sigma\lambda - \kappa}{\sigma\lambda} \frac{\mu}{1-\beta\mu\gamma} \left[\gamma u_t - (1-\mu)(\mu\Lambda_{t-1} + u_t)\right] + \frac{1}{\sigma} r_t^n \tag{16}$$

where $*$ denotes the optimal solution when expressed as a linear function of $\{\Lambda_{t-1}, u_t, r_t\}$.

One might presume that the task of designing an optimal monetary policy is complete – the equilibrium fluctuations in endogenous variables in response to fundamental shocks has been delineated and the required path for the nominal interest rates consistent with this pattern of responses determined. However, analogously to the determinacy of rational expectations equilibria results (see Giannoni and Woodford (2002a, 2002b), Svensson and Woodford (2002) and Woodford (1999)), the results of Preston (2003) make clear that the design of optimal monetary policy is non-trivial under learning dynamics once we attempt to implement the optimal plan described by equations (14), (15) and (16). Under learning dynamics, monetary policy that depends only on the history of exogenous disturbances leads to economic instability – such policies are prone to divergent learning dynamics even though they are consistent with the optimal equilibrium. It follows that further work must be done to implement optimal monetary policy under learning dynamics.

4 Optimal Instrument Rules

This section turns to considering the desirability of forecast-based instrument rules for stabilization policy. Several recently popular Taylor-type instrument rules that are modified to be consistent with implementing optimal monetary policy outlined in section 3 are presented. The analysis then characterizes the stabilization properties of these rules under learning dynamics. Two decision procedures for the implementation of these rules are considered. First, the central bank is assumed to naively respond to observed private forecasts. Second, the central bank is assumed to correctly understand agents' learning mechanism and decision rules. In this case, optimal internal (indeed rational) forecasts can be constructed.

4.1 Some Taylor-type Rules

Preston (2003) shows in the present model that nominal interest-rate rules specified in terms of the history of exogenous disturbances are potentially destabilizing as they fail to exclude the possibility of divergent learning dynamics. It follows that nominal interest-rate rules of the form (16) are undesirable as a means to implement the optimal commitment equilibrium. Recalling the insights of the determinacy of rational expectations equilibrium literature, monetary policies that involve appropriate feedback from the model's endogenous variables do in fact lead to a determinate equilibrium.[7] One might conjecture, therefore, that what is required to implement the optimal equilibrium then is a commitment by the central bank to behave in a certain way out of equilibrium so as to exclude possible divergent learning dynamics.

Consider the following instrument rule as a means to implement the optimal equilibrium:

$$\begin{aligned} i_t &= i_t^* + \psi_x(E_t^{cb} x_{t+1} - E_t x_{t+1}^*) + \psi_\pi(E_t^{cb}\pi_{t+1} - E_t \pi_{t+1}^*) \\ &= \bar{\imath}_t^f + \psi_x E_t^{cb} x_{t+1} + \psi_\pi E_t^{cb}\pi_{t+1} \end{aligned} \tag{17}$$

[7] See McCallum (1983) for the seminal contribution and Woodford (2003) for a discussion of determinacy in the context if the model of this paper under the assumption of rational expectations.

where E_t^{cb} denotes the forecasts that the central bank intends to respond to and

$$\bar{i}_t^l = i_t^* - \psi_x E_t x_{t+1}^* - \psi_\pi E_t \pi_{t+1}^*$$

collects exogenous terms and $E_t x_{t+1}^*$ and $E_t \pi_{t+1}^*$ are the expectations of next-period's output gap and inflation that would obtain in the optimal equilibrium under rational expectations. $\{\pi_t^*, x_t^*, i_t^*\}$ denote the optimal paths for each endogenous variable given by equations (14), (15) and (16), when written as a linear function of the exogenous state variables $\{\Lambda_{t-1},\, u_t,\, r_t\}$. This rule is of the general form proposed by Batini and Haldane (1999) and Levin, Wieland, and Williams (2003) as a desirable approach to implement policy on the ground of robustness and given empirical support by Clarida, Gali and Gertler (1998, 2000).

In the case that the central bank responds to observed private forecasts then $E_t^{cb} = \hat{E}_t$ where the forecasts are given by (8). When the central bank responds to internally generated forecasts, E_t^{cb} will be determined as described below. The policy parameters (ψ_π, ψ_x) determine the response to deviations of the actual path of the inflation rate and output gap expectations from that path consistent with the optimal equilibrium. In the desired equilibrium it is clear that $E_t^{cb}\pi_{t+1} = E_t \pi_{t+1}^*$ and $E_t^{cb} x_{t+1} = E_t x_{t+1}^*$ and $i_t = i_t^*$.

Rules of the form (17) have been criticized on the ground that monetary authorities typically do not have current-dated observations on the output gap and the inflation rate when setting the current interest rate. Many researchers have responded to this criticism by modifying the information set available to the monetary authority when determining the nominal interest rate.[8] For instance, the nominal interest rate could be determined by lagged expectations of current-dated output and inflation to give an instrument rule of the form

$$\begin{aligned} i_t &= i_t^* + \psi_x (E_{t-1}^{cb} x_t - E_{t-1} x_t^*) + \psi_\pi (E_{t-1}^{cb} \pi_t - E_{t-1} \pi_t^*) \\ &= \bar{i}_t^l + \psi_x E_{t-1}^{cb} x_t + \psi_\pi E_{t-1}^{cb} \pi_t \end{aligned} \tag{18}$$

where E_{t-1}^{cb} denotes the forecasts that the central bank intends to respond to and

$$\bar{i}_t^l = i_t^* - \psi_x E_{t-1} x_t^* - \psi_\pi E_{t-1} \pi_t^*$$

[8] See, for instance, McCallum (1999).

collects exogenous terms. In the case that the central bank responds to observed private forecasts then $E_{t-1}^{cb} = \hat{E}_{t-1}$. When the central bank responds to internally generated forecasts, E_{t-1}^{cb} will be determined as described below. This instrument rule directs the monetary authority to respond to deviations of the adopted forecasts from the rational forecast that should be observed in the optimal equilibrium.[9] The forecast-based instrument rules (17) and (18) will be the focus of the remainder of this paper.

4.2 Determinacy of REE

The critical difference between these latter rules and the Taylor rule is the role of expectations: while the Taylor rule posits adjustment of the nominal interest rate to contemporaneous observations of output and prices, these variants stipulate adjustment in the nominal interest rate to expectations of the future expected path of output and prices. However, it is precisely rules of this type that have generated concern in regards to determinacy of rational expectations equilibrium. Bernanke and Woodford (1997) argue rules that "link actions to policy forecasts, thereby making the current equilibrium especially sensitive to expectations about the future, are particularly vulnerable" to the problem of indeterminacy. Before analyzing the implications of these rules under alternative central bank forecasting arrangements it is useful to recall the results for determinacy of rational expectations equilibrium.

Proposition 1 *Suppose the economy is given by the structural equations (3) and (4). Then, under the assumption of rational expectations, each of the following instrument rules will result in a unique bounded rational expectations equilibrium if and only if the specified model restrictions are satisfied.*

1. *For $i_t = \bar{\imath}_t^f + \psi_x E_t x_{t+1} + \psi_\pi E_t \pi_{t+1}$ a unique bounded rational expectations equilibrium will obtain iff*

$$0 < \kappa\left(\psi_\pi - 1\right) + \left(1 - \beta\right)\psi_x < 2\sigma\left(1 + \beta\right)$$

and

$$\psi_x < \sigma(1 + \beta^{-1})$$

[9] This formulation of policy makes especially clear that to implement the optimal commitment equilibrium, convergence of learning dynamics to the rational expectations dynamics requires private agents to adopt forecasting models that nest the true model. If the forecasting model is under-parameterized relative the minimum-state-variable rational expectations solution then agents are unable to learn the true dynamics – that is, if $\hat{E}_{t-1}x_t \neq E_{t-1}x_t^*$ and $\hat{E}_{t-1}\pi_t \neq E_{t-1}\pi_t^*$ then $i_t \neq i_t^*$.

2. *For* $i_t = \bar{i}_t^l + \psi_x E_{t-1} x_t + \psi_\pi E_{t-1} \pi_t$ *a unique bounded rational expectations equilibrium will obtain iff*

$$\kappa (\psi_\pi - 1) + (1 - \beta) \psi_x > 0.$$

Proof. See Bullard and Mitra (2002) or Woodford (2003). ■

When the monetary authority commits to a Taylor rule that responds to lagged expectations of current data, the so-called Taylor principle is necessary and sufficient for determinacy. In the case that interest rates are set in response to expectations of tomorrow's output gap and inflation rate then a stricter set of model restrictions are necessary and sufficient. Indeed, policy responses to variations in expectations of inflation or output that are too aggressive lead to an indeterminacy of equilibrium.

But how do these results translate to an economic environment in which the central bank has superior information relative to private agents who have an incomplete economic model? If the central bank responds to observed forecasts of inflation and output do similar results obtain? And if so, can policy be improved upon by having the central bank construct its own forecasts internally, so that policy responds to the determinants of private forecasts? Finally, even if we restrict attention to private agents that have forecast functions that include variables that appear in the minimum-state-variable solution, is this enough to rule out pathologies associated with indeterminacy of rational expectations equilibrium, or does it introduce difficulties of its own? The following analysis addresses these questions.

5 Responding to Private Forecasts

Suppose the monetary authority implements the forward-looking and lagged-expectations Taylor rules by responding to observed private forecasts. For simplicity it is assumed that these forecasts are accurately observed, either from a statistical agency or from surveys conducted by the monetary authority itself. The analysis could be extended to allow for noisy observations of these variables, as proposed by Evans and Honkapohja (2002), but as will be shown, there will be ground enough to obviate use of forecast-based instrument rules even when forecasts are perfectly observed.

Under rational expectations these rules induce equilibria that are linear in the state variables $\{\Lambda_{t-1},\ u_t,\ r_t\}$ (since $\{\pi_t^*, x_t^*, i_t^*\}$ were shown to take the same linear form). Therefore assume that agents use the econometric model (5) where $z_t = (\pi_t, x_t, i_t, \Lambda_t)$ and $\{a_t, b_t, c_t, d_t\}$ are as defined in Section 2. Forecasts can then be constructed according to relations (6), (7) and (8). The determinacy results for the forward-looking Taylor rule suggest that implicit instrument rules that respond to private-sector forecasts may potentially encounter stability problems under learning dynamics. This concern is well founded.

Proposition 2 *Suppose the economy is given by the structural equations (1) and (2) and agents forecast the evolution of state variables using (8). Then under the Taylor rule with forward-expectations (17) and the Taylor rule with lagged-expectations (18) the Taylor principle*

$$\kappa(\psi_\pi - 1) + (1-\beta)\psi_x > 0$$

is necessary but ***NOT*** *sufficient for E-Stability under least-squares learning dynamics.*

This result is important: to the extent that good policy is argued to be one that satisfies the Taylor principle, common prescriptions for monetary policy are not robust to learning dynamics. Details of the proof are relegated to the appendix B, where it is shown that E-Stability requires twelve restrictions on model parameters to be satisfied – three pertaining to learning the model's constant coefficients and three pertaining to each set of coefficients on the three state variables.[10] The following discussion considers only those restrictions arising from learning the former, with the understanding that the remaining constraints can only further restrict the space of policy parameters consistent with stability under learning dynamics.[11] The discussion also applies to both rules as the proof demonstrates the constant dynamics to be the same under rules (17) and (18).

Appendix B shows the Jacobian matrix associated with the E-Stability mapping for the constant dynamics has characteristic equation

$$P(\lambda) = \lambda^3 + A_2\lambda^2 + A_1\lambda^1 + A_0 \tag{19}$$

[10] For these rules it is difficult to obtain a complete analytical characterization of the conditions for stability under learning dynamics and the proof proceeds by contradiction.

[11] An important property of all models considered in this paper is that the conditions for stability arising from learning the set of constants or any set of coefficients on a given state variable are independent. The stability properties can then be established by considering the dynamics of each set of coefficients in turn.

where A_i are composites of model primitives and stability under learning dynamics requires $A_2, A_0 > 0$ and $A_4 = -A_0 + A_1 A_2 > 0$. The restriction $A_0 > 0$ delivers the Taylor principle. The restriction $A_2 > 0$ implies the following inequality:

$$\psi_\pi + \frac{\psi_x}{\kappa} > \frac{1}{1-\beta} - \frac{2-\beta-\alpha\beta}{\kappa\sigma(1-\alpha\beta)}.$$

Now consider taking the limit of the right hand side as $\beta \to 1$. It is immediate that the second term converges to $(\sigma\kappa)^{-1}$, while the first goes to infinity. Hence the Taylor principle cannot be sufficient for E-Stability.

In order to make clear the practical relevance of this insight it is useful to pursue a graphical analysis. To this end, assume $\psi_x = 0$ so that the Taylor principle is given by the relation $\psi_\pi > 1$, and make the following benchmark assumptions on remaining model parameters: $\sigma = 6.25$, $\beta = 0.99$, $\alpha = 0.66$ and $\kappa = 0.024$. The assumption on the policy parameter ψ_x is made purely for convenience and matters little for the substantive points that follow. The remaining parameter assumptions are taken from the estimates provided by Rotemberg and Woodford (1999).

Figure 1 plots the values of A_2 and A_4 as a function of $(\psi_\pi - 1)$ so that the positive orthant graphs (A_i, ψ_π) pairs that are consistent with E-Stability (and satisfy the Taylor principle). It is immediate that positive values for A_2 and A_4 require policy parameter coefficients in the vicinity of 100 – substantially above any empirically reasonable value.[12] This suggests nominal interest-rate rules that depend on private-sector expectations to be particularly undesirable in terms of robustness. These basic insights show little sensitivity to alternative parameter assumptions.[13]

These results contrast with the conclusions obtained by Bullard and Mitra (2002) who consider an economy given by (3) and (4) under the learning assumption to give a model of

[12] The downward sloping curve is a parabola which eventually intersects the ordinate axes for large values of the policy parameter.

[13] For instance, with $\sigma = 1$ and all other parameters taking the same values, E-Stability requires $\psi_\pi > 50$ – still an implausibly large value.

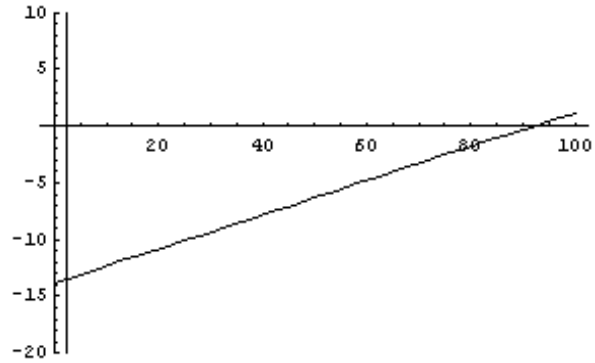

Figure 1: A_2 and A_4 as a function of the policy parameter $\psi_\pi - 1$.

the monetary transmission mechanism of the form:

$$x_t = \hat{E}_t x_{t+1} - \sigma^{-1}\left(i_t - \hat{E}_t \pi_{t+1}\right) + r_t \tag{20}$$

$$\pi_t = \beta \hat{E}_t \pi_{t+1} + \kappa x_t + u_t. \tag{21}$$

They find that the Taylor principle is necessary and sufficient for stability under the rules (17) and (18).[14] As underscored by Preston (2003), when the microfoundations underpinning this model are solved under the non-rational expectations assumption, the predicted aggregate dynamics depend on long-horizon forecasts of inflation, output and the nominal interest rate into the indefinite future. It is the presence of these additional expectational variables that present a non-trivial source of instability. By requiring agents to forecast the future path of the nominal interest rate (that is, by requiring agents to learn the rational expectations restrictions implied by the policy rule, which is not a property of the Bullard and Mitra model), in addition to the output gap and inflation rate, a more general learning dynamic is admitted which in turn requires a stricter set of requirements on admissible policy parameters consistent with stability under learning. Indeed, numerical analysis reveals that the three roots associated with the constant dynamics depend on private-sector beliefs.

[14] Bullard and Mitra (2002) do not actually consider a Taylor rule that is capabale of implementing the optimal equilibrium. However, the rules proposed here differ only in terms of the exogenous disturbances that the bank is responding to. As a result, the learning dynamics associated with agents' estimated constants are equivalent to a rule of the form cosidered by these authors since exogenous variables do not affect learning dynamics. See Evans and Honkapohja (2003).

Thus, forecast-based instrument rules in an economy given by (1) and (2) are unlikely to assist learnability of rational expectations equilibrium. Indeed, such rules may strengthen the effects of private forecasts on the economy's evolution by making central bank behavior, as well as private behavior, respond to them. Such central bank behavior makes it easier for expectations to be a source of instability.

To make clear that forecasting the future path of nominal interest rates is an important source of instability, consider the following: suppose agents know the form of the policy rule (17). This means that agents know the restriction on nominal interest rates and one-period-ahead forecasts of inflation and the output gap that is required by the monetary policy rule to hold in a rational expectations equilibrium. Since agents understand this restriction to hold in all future periods we can substitute (17) for i_T in (1) which combined with (2) gives the system

$$x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta-\sigma\psi_x) x_{T+1} + \sigma(1-\psi_\pi)\pi_{T+1} - \sigma \bar{i}_T^f + r_T \right] \tag{22}$$

$$\pi_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa\alpha\beta \cdot x_{T+1} + (1-\alpha)\beta \cdot \pi_{T+1} + u_T \right]. \tag{23}$$

The crucial difference here, in contrast to (1) and (2), is that private agents need not independently forecast the future path of the nominal interest rate — they need only forecast i_T as a function of the forecasts of the future paths of inflation and the output gap. We therefore have a system of two equations in the unknowns (π_t, x_t). Under the assumption that agents know the policy rule (17) or (18) learning analysis yields the following proposition.

Proposition 3 *Suppose monetary policy is conducted according to either the rule (17) or (18) and the form of the particular rule is understood by agents to hold in all periods. The economy is then given by the structural equations (1) and (2) and agents forecast the evolution of state variables using (8). The Taylor principle*

$$\kappa(\psi_\pi - 1) + (1-\beta)\psi_x > 0$$

is necessary and sufficient for E-Stability under least-squares learning dynamics.

The proof is contained in the appendix. Importantly, these stability results coincide with the findings of Bullard and Mitra (2002) which considers forecast-based instrument

rules of the form of (17) or (18) in an economy described by relations (20) and (21). The contrasting findings of proposition 2 and proposition 3 stem from agents having to forecast the future path of nominal interest rates in the model presented in this paper. It follows immediately that requiring agents to learn the rational expectations restrictions embodied in the policy rule is an important source of instability; moreover, it underscores the source of difference in the stability conditions obtained here and those of Bullard and Mitra (2002) and directs attention to the potential importance of transparency in monetary policy design. We conclude that the policy rules (17) and (18), by depending on observed private-sector forecasts, can serve to make the equilibrium more susceptible to economic instability in the presence of learning dynamics.

It is worth noting that transparency has been at the forefront of discussion on monetary policy design particularly in the recent inflation targeting literature. Indeed, Faust and Svensson (2001) present a model in which the central bank has an idiosyncractic employment target which is imperfectly observed by the public. Fluctuations in this target lead to central bank temptation to deviate from pre-announced inflation goals. However, increased transparency allows the private sector to observe the employment target with greater precision and therefore raises the costs to the central bank of deviating from its pre-announced inflation objectives. Transparency is therefore desirable as it provides a commitment mechanism. Svensson (1999) further argues on the ground of this result that for inflation targeting central banks it is generally desirable for detailed information on policy objectives, including forecasts, to be published. Such transparency enhances the public's understanding of the monetary policy process and raises the costs to a central bank from deviating from its stated objectives. Proposition 3 shares this property: clearly articulating monetary policy strategy helps anchor private expectations and consequently assists in managing economic fluctuations.

More recently, and closely related to the present analysis, Orphanides and Williams (2004), and the introductory remarks on this paper by Bernanke and Woodford (2004) in the same volume, highlight the advantages of publishing an inflation target. Orphanides and

Williams (2004) show in a simple model of the output-inflation trade off that if private agents must learn about the inflation dynamics – in much the same way as in the framework of this paper – a more favorable trade off between inflation and output can be achieved if private agents are assumed to know the central bank's long-run inflation target rather than having to learn this quantity. Hence transparency of inflation objectives helps anchor inflation expectations and facilitates stabilization of aggregate dynamics. Again, proposition 3 is a result that is almost identical in spirit to this analysis. By publishing the adopted policy rule, private agents gain information that greatly assists determining expectations that are consistent with the equilibrium the central bank is seeking to implement. Importantly, such transparency about the monetary policy process helps guard against the possibility that expectations are a source of instability.

6 Optimal Internal Forecasts

Suppose instead that the monetary authority constructs its own rational forecasts of the future path of the economy. The Taylor rules (17) and (18) are therefore interpreted as stipulating adjustment of the nominal interest rate in response to these internal forecasts. Does this give cause to alter our conclusion about the desirability of such monetary policies?

To understand how such an internal forecast-based decision procedure for monetary policy might be implemented, suppose that the monetary authority knows the true model of the economy, and knows the way in which private agents construct forecasts. It follows that the monetary authority can determine private-sector forecasts and solve for the temporary equilibrium for inflation and output conditional on its current interest-rate setting. These relations in conjunction with the instrument rule then provide a rational expectations model of the central bank's interest rate setting procedure, which can be solved using standard methods. This solution determines the central bank's current instrument choice and the values of output and inflation. These relations can then be used to construct the E-Stability mapping from the private-sector's forecast parameters to the coefficients that would be rational given this evolution of the endogenous variables.

More formally, take the case of the forward-looking Taylor rule. Substituting the private forecasts (8) into the structural relations (1) and (2) implies the output gap and inflation to be given by

$$\pi_t = \bar{a}_\pi + \bar{b}_\pi \Lambda_{t-1} + \bar{c}_\pi u_t + \bar{d}_\pi r_t + \bar{e}_\pi i_t \tag{24}$$

$$x_t = \bar{a}_x + \bar{b}_x \Lambda_{t-1} + \bar{c}_x u_t + \bar{d}_x r_t + \bar{e}_x i_t \tag{25}$$

where the coefficients $\left(\bar{a}_i, \bar{b}_i, \bar{c}_i, \bar{d}_i, \bar{e}_i\right)$ for $i \in \{\pi, x\}$ are functions of the coefficients (a_t, b_t, c_t, d_t) of the current private-sector forecasting rule. These equations give the period t inflation rate and output gap realizations, conditional on the current choice of the interest rate. Under the assumption that the monetary authority correctly understands the model of the private sector, it can then construct forecasts of next period's inflation and output gap by leading (24) and (25) and taking expectations (rational) to give

$$\begin{aligned} E_t^{cb}\pi_{t+1} &= \bar{a}_\pi + \bar{b}_\pi \Lambda_t + \bar{c}_\pi \gamma u_t + \bar{d}_\pi \rho r_t + \bar{e}_\pi E_t^{cb} i_{t+1} \\ E_t^{cb} x_{t+1} &= \bar{a}_x + \bar{b}_x \Lambda_t + \bar{c}_x \gamma u_t + \bar{d}_x \rho r_t + \bar{e}_x E_t^{cb} i_{t+1}. \end{aligned}$$

Substitution of these relations into the nominal interest-rate rule (17) yields an equation of the form

$$\begin{aligned} i_t = {} & (\psi_\pi \bar{a}_\pi + \psi_x \bar{a}_x) + (\psi_\pi \bar{b}_\pi + \psi_x \bar{b}_x)\Lambda_t + (\psi_\pi \bar{c}_\pi + \psi_x \bar{c}_x)\gamma u_t \\ & + (\psi_\pi \bar{d}_\pi + \psi_x \bar{d}_x)\rho r_t + (\psi_\pi \bar{e}_\pi + \psi_x \bar{e}_x) E_t^{cb} i_{t+1} \end{aligned}$$

which is a linear rational expectations model of the central bank's own behavior. If the condition $|\psi_\pi \bar{e}_\pi + \psi_x \bar{e}_x| < 1$ is satisfied, the model has a unique bounded solution, which gives the interest-rate setting that the monetary authority chooses given its correct understanding of the model's structural relations and the current parameters of private agents' forecasting rules. However, since we are describing a decision procedure of a single actor in this economy, it is equally plausible to assume that the central bank solves this model concerning itself only with the minimum-state-variable solution regardless of whether $|\psi_\pi \bar{e}_\pi + \psi_x \bar{e}_x| < 1$ is true or not. Note also that the model's coefficients are time varying as they are functions of private

forecast parameters. It follows that the central bank must solve this model each period for the appropriate instrument setting given the current forecast parameters. A similar logic can be applied in the case that the monetary authority sets interest rates according to (18).

Proposition 4 *Suppose the economy is given by the structural equations (1) and (2), that private agents forecast using (8) and that the monetary authority correctly understands the model and the manner in which private-sector forecasts are constructed. If the monetary authority constructs optimal internal forecasts then the Taylor principle*

$$\kappa(\psi_\pi - 1) + (1 - \beta)\,\psi_x > 0$$

is necessary and sufficient for stability under learning dynamics.

The proof is contained in the appendix. An interesting feature of this result are the properties of the eigenvalues. The appendix shows for both instrument rules, the four roots corresponding to learning the interest-rate dynamics in a rational expectations equilibrium are equal to negative unity and therefore cannot be a source of instability. In contrast, in proposition 2, when the central bank responds to observed forecasts, all roots were found to depend on private agents' forecast parameters, implying that the interest rate rule itself is a source of instability under learning dynamics. Thus we find some evidence that rules that make the nominal interest rate depend on observed private-sector expectations can make the economy prone to divergent learning dynamics. However, if a central bank seeks to obtain information about the underlying determinants of private-sector forecasts then this problem can be mitigated.[15]

The intuition for these contrasting results is straightforward. Recall the agents' learning problem: they are attempting to learn the true rational expectations dynamics implied by the structural relations (1) and (2) and the monetary policy rule at hand by use of econometric models. When responding to observed forecasts, an instrument rule that depends on private

[15] It is also of interest to note the parallels of these results to the corresponding determinacy results of proposition 1. To exclude the possibility of self-fulfilling expectations under a forward-looking Taylor rule proposition 4 shows that policy parameters must satisfy the Taylor principle. Moreover, for a unique bounded solution to the central bank's decision procedure these policy parameters can not be too large. It follows that if the central bank responds too aggressively the economy will be subject to indeterminacy. In contrast, for the lagged expectations Taylor rule the Taylor principle is necessary and sufficient to exclude indeterminacy under rational expectations and self-fulfilling expectations under learning dynamics. These observations are closely related to the determinacy results for these rules.

agents' forecasts directly introduces instability via the aggregate demand equation. Since agents are attempting to learn the rational expectations restrictions implied by the aggregate demand and Phillip's curve relations, the additional source of instability that results from the instrument setting can only serve to make the task of identifying the true dynamics embodied in the aggregate demand relation more difficult. Most importantly, the instrument setting is such that it validates agents' expectations. By making central bank behavior as well as private behavior respond to private forecasts, forecast-based instrument rules strengthen the effects of private forecasts on the economy's evolution – therefore making it easier for expectations to be a source of instability.

In contrast, when internal forecasts are generated, the central bank makes use of its knowledge of both the form of the monetary policy rule *and* the true structural relations in choosing its current instrument setting. Forecasts of output and inflation are formed optimally, and, importantly, conditional on both its expected next-period choice of the instrument and the current private forecast parameters. By taking explicit account of the effects of learning dynamics and its own future instrument choice on the evolution of the economy, the monetary authority can offset any instability induced by a monetary policy rule that depends on private-sector forecasts. Thus the monetary authority, by constructing forecasts optimally given private-sector behavior, chooses its instrument to accommodate instability arising from private agents' learning.

7 Alternative Forecast-based Instrument Rules

The results presented thus far cast doubt on the usefulness of forecast-based instrument rules for implementing optimal policy. Naive uses of observed private forecasts are susceptible to divergent learning dynamics. While such instability problems can be mitigated – either by complete transparency with regards to the form of the monetary policy rule or detailed and accurate information on private agents' long-horizon forecasts – it is natural to inquire as to whether there are other approaches to implementing optimal monetary policy that are less informationally demanding.

To this end, consider the following forecast-based instrument rule:

$$i_t = \psi_{lp} p_{t-1} + \psi_x \hat{E}_t x_{t+1} + \psi_p \hat{E}_t p_{t+1} + \psi_u u_t + \psi_r r_t \tag{26}$$

where

$$\psi_x = \psi_r = 1/\sigma; \qquad \psi_u = \psi_{lp}$$

and

$$\psi_{lp} = \frac{\kappa - \lambda\sigma}{(\kappa^2 + \lambda + \beta\lambda)\,\sigma}; \quad \psi_p = \frac{\beta\kappa + (\kappa^2 + \lambda)\sigma}{(\kappa^2 + \lambda + \beta\lambda)\,\sigma}.$$

Preston (2004) demonstrates this rule is consistent with implementing a timelessly optimal equilibrium that results when expectations are formed rationally and when the central bank optimizes subject to a condition analogous to (10) written in terms if the price level. Note that this rule exhibits a different kind of history dependence relative to rules (17) and (18). Rather than responding to a lagged exogenous disturbance, it responds to the lagged price level. Similarly, the response coefficients on observed forecasts are of a very different kind than in the rules (17) and (18), only introducing one free policy parameter that determines the central bank's objective function. Nonetheless, this instrument rule is consistent with implementing the timelessly optimal monetary policy.

Preston (2004) shows that when the central bank observes private forecasts and interprets the above rule as naively responding to such forecasts the adoption of (26) ensures stability under learning dynamics for a large range of plausible parameter values (see proposition 6 of that paper). For example, when α, β, θ, γ and ρ take the values 0.66, 0.99, 7.88, 0.35, 0.35, $\kappa \in (0, 1]$ and $\sigma \in (0, 2]$ E-stability always results.[16] The rule (26) therefore provides an example of a class of rule that is less informationally demanding than say the results of proposition 4 but nonetheless achieves the central bank's objectives.

While a thorough examination of the performance of this rule relative to (17) and (18) is beyond the scope of this paper, several points are worthy of note. First, such stability results highlight that rules that depend on one-period-ahead forecasts need not necessarily

[16] Regions are considered for the parameters κ and σ as there is least agreement on their empirical value and the results exhibit less senstitivity to the other parameter assumptions.

lead to divergent learning dynamics. Second, it underscores that there are advantages to having instrument rules exhibit a particular kind of history dependence and a particular kind of dependence on available forecasts. Thus while the recent rational expectations literature promoting forecast-based instrument rules has typically eschewed rules with history dependence – presumably on the ground of simplicity – there may be good reasons for having such dependence: not only because optimal monetary policy is necessarily history dependent in forward-looking rational expectations models but also because they may have the desirable characteristic of promoting stability under non-rational expectations. Indeed, results of this kind are also found in Bullard and Mitra (2000) which shows that inertial Taylor rules – policies exhibiting dependencies on the previous period's nominal interest rate – can help promote stability under learning. Third, and somewhat more speculatively, it remains a challenge to identify whether rules exist that induce both determinacy of REE and stability under learning dynamics for all parameter values and for central banks which only have limited information on the private expectations. The price-level targeting rules discussed in Preston (2004a) and associated implied instrument rules such as (26) are promising in this regard.

8 Conclusions

This paper applies the framework of Preston (2003) to understand the appropriate use of private forecasts in the design of monetary policy. The analysis demonstrates that recently popular forecast-based instrument rules may give rise to divergent learning dynamics and therefore be undesirable as a means to stabilize economic fluctuations. In particular, the Taylor principle is not a sufficient condition for private agents to be able to learn the associated rational expectations equilibrium. This result contrasts with recent work by Bullard and Mitra (2002) which finds in an economy where only one-period-ahead expectations matter, that the Taylor principle is in fact necessary and sufficient conditions for stability under learning dynamics. Evidence on the importance of a transparent monetary policy is also adduced by demonstrating that such instability can be mitigated if private agents are informed

about the form of the central bank's instrument rule. In this case, the Taylor principle is once more necessary and sufficient for stability under learning.

However, if the central bank correctly understands the learning mechanism of private agents, it can construct optimal forecasts conditional on private agents' behavior. In this case, forecast-based instrument rules have the central bank respond to the determinants of private forecasts, rather than the actual forecasts themselves, and this mitigates observed instability problems of the former decision procedure. Indeed, the Taylor principle is again necessary and sufficient for E-Stability. This underscores the importance of gathering information on the nature and form of private forecasting methods. Moreover, it emphasizes that the concern of Bernanke and Woodford (1997), that policy rules which naively depend on observed private forecasts might be susceptible to problems of indeterminacy, has greater ambit than rational expectations models.

A Appendix

This appendix first outlines the microfoundations of the model adopted in the main text. The general approach to analyzing learning dynamics in the context of this framework is then discussed. It then turns to sketching the proofs of the central results which are all applications of this general methodology. Since the algebra underpinning these results is at times tedious, it is largely omitted. Most calculations were performed in Mathematica.

A.1 A Simple Model

A.1.1 Household and Firm Decision Problems

The economy is populated by a continuum of households which seek to maximize future expected discounted utility

$$\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[U(C_T^i; \xi_T) - \int_0^1 v(h_T^i(j); \xi_T) dj \right] \tag{27}$$

where utility depends on a consumption index, C_t^i, of the economy's available goods (to be specified), a vector of aggregate preference shocks, ξ_t, and the amount of labor supplied for the production of each good j, $h^i(j)$. The second term in the brackets captures the total disutility of labor supply. The consumption index, C_t^i, is the Dixit-Stiglitz constant-elasticity-of-substitution aggregator of the economy's available goods and has an associated price index written, respectively, as

$$C_t^i \equiv \left[\int_0^1 c_t^i(j)^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \quad \text{and} \quad P_t \equiv \left[\int_0^1 p_t(j)^{1-\theta} dj \right]^{\frac{1}{1-\theta}}$$

where $\theta > 1$ is the elasticity of substitution between any two goods and $c_t^i(j)$ and $p_t(j)$ denote household i's consumption and the price of good j.[17]

$\hat{E}_t^i$ denotes the subjective beliefs of household i about the probability distribution of the model's state variables: that is, variables that are beyond agents' control though relevant

[17] The absence of real money balances from the period utility function (27) reflects the assumption that there are no transaction frictions that can be mitigated by holding money balances. However, agents may nonetheless choose to hold money if it provides comparable returns to other available financial assets.

to their decision problems – i.e. prices and exogenous variables. Beliefs are assumed to be homogenous across households, though each household has no knowledge of the beliefs of other households. Agents therefore do not have a complete economic model of the determination of aggregate state variables and it is about the evolution of such variables that agents are attempting to learn. The learning algorithm is discussed below. The discount factor is assumed to satisfy $0 < \beta < 1$.

Asset markets are assumed to be *incomplete*: there is a single one-period riskless non-monetary asset available to transfer wealth intertemporally. Under this assumption, the household's flow budget constraint can be written as

$$M_t^i + B_t^i \leq \left(1 + i_{t-1}^m\right) M_{t-1}^i + \left(1 + i_{t-1}\right) B_{t-1}^i + P_t Y_t^i - T_t - P_t C_t^i \tag{28}$$

where M_t^i denotes the household's end-of-period holdings of money, B_t^i the household's end-of-period nominal holdings of risk-less bonds, i_t^m and i_t are the nominal interest rates paid on money balances and bonds held at the end of period t, Y_t^i the period income (real) of households and T_t denotes lump sum taxes and transfers. The household receives income in the form of wages paid, $w(j)$, for labor supplied in the production of each good, j. All households are assumed to own an equal part of each firm and therefore receive a common share of profits $\Pi_t(j)$ from the sale of each firm's good j. Period nominal income is therefore determined as $P_t Y_t^i = \int_0^1 [w_t(j) h_t^i(j) + \Pi_t(j)] dj$ for each household i. Fiscal policy is assumed to be Ricardian. To summarize, the household's problem in each period t is to choose $\{c_t^i(j), h_t^i(j), M_t^i, B_t^i\}$ for all $j \in [0,1]$ so as to maximize (27) subject to the constraint (28) taking as parametric the variables $\left\{p_T(j),\ w_T(j),\ \Pi_T,\ i_{T-1},\ i_{T-1}^m,\ \xi_T\right\}$ for $T \geq t$.

Firms face a Calvo-style price-setting problem in period t and maximize the expected present discounted value of profits

$$\hat{E}_t^i \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left[\Pi_T^i\left(p_t(i)\right)\right] \tag{29}$$

where

$$\Pi_T^i(p) = y_t(i)\, p_t(i) - w_t(i) h_t(i) \tag{30}$$

and $y_t(i) = Y_t(p_t(i)/P_t)^{-\theta}$ is the demand curve faced by producer i and output is produced using technology $y_t(i) = A_t f(h_t(i))$ where A_t is an exogenous technology shock. The factor α^{T-t} in the firm's objective function is the probability that the firm will not be able to adjust its price for the next $(T-t)$ periods. Firms are assumed to value future streams of income at the marginal value of aggregate income in terms of the marginal value of an additional unit of aggregate income today. That is, a unit of income in each state and date T is valued by the stochastic discount factor $Q_{t,T} = \beta^{T-t} P_t U_c(Y_T, \xi_T)/(P_T U_c(Y_t; \xi_t)$. This simplifying assumption is appealing in the context of the symmetric equilibrium that is examined in this model. To summarize, the firm's problem is to choose $\{p_t(i)\}$ to maximize (29) taking as given $\{Y_T,\ P_T,\ w_T(j),\ A_T,\ Q_{t,T}\}$ for $T \geq t$ and $j \in [0,1]$.

A.1.2 Optimal Decision Rules

Preston (2003) shows that a log-linear approximation to the first order conditions of the above infinite horizon decision problems lead to following optimal decision rules. Each household allocates consumption according to

$$\hat{C}_t^i - Y_t^n = (1-\beta)\,\varpi_t^i + \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)x_T - \beta\sigma(\hat{\imath}_T - \hat{\pi}_{T+1}) + \beta r_T\right].$$

where

$$\hat{C}_t^i \equiv \ln(C_t^i/\bar{Y});\ \hat{Y}_t \equiv \ln(Y_t/\bar{Y});\quad \hat{\imath}_t \equiv \ln[(1+i_t)/(1+\bar{\imath})]; \tag{31}$$

$$\hat{\pi}_t = \ln(P_t/P_{t-1});\quad \varpi_t^i = W_t^i/(P_t\bar{Y})$$

and for any variable g, $\bar{g}$ denotes the variables steady state value. The output gap $x_t = \hat{Y}_t - \hat{Y}_t^n$ is the difference between output and the economy's natural rate of output that obtains in this model with fully flexible prices. It is a function of model disturbances as is the natural rate of interest r_t. Household wealth is defined as $W_{t+1}^i = (1+i_t^m)\,M_t^i + (1+i_t)\,B_t^i$ and $\sigma = -U_{cc}/U_c\bar{Y} > 0$ is the intertemporal elasticity of substitution.

Each firm determines its optimal price according to

$$\widehat{p}_t^*(i) = \hat{E}_t^i \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\frac{1-\alpha\beta}{1+\omega\theta} \cdot (\omega + \sigma^{-1})x_T + \hat{\pi}_T\right]$$

where $\widehat{p}_t^*(i) \equiv \ln(p_t^*(i)/P_t)$ and $\omega > 0$ is the elasticity of real marginal costs with respect to own output. Here the presence of long-horizon expectations arise due to the pricing frictions induced by Calvo pricing. Integrating over the optimal decision rules of the above household and firm decision problems; applying the market clearing condition $\int_0^1 \hat{C}_t^i = \hat{Y}_t$; the definition of x_t; noting that prices satisfy the relation $\hat{\pi}_t = \widehat{p}_t^* \cdot (1-\alpha)/\alpha$; and dropping the '^' notation, gives the New Keynesian sticky-price model of output gap and inflation determination reported in the main text.

A.2 Expectational Stability

Suppose monetary policy is conducted according to the rule

$$i_t = \psi_x x_{t-1} + \psi_u u_t + \psi_r r_t$$

where x_{t-1} is the lagged output gap. Standard analysis implies there exists a rational expectations equilibrium that is linear in the variables $\{x_{t-1}, u_t, r_t\}$. If agents know the form of the minimum-state-variable solution they estimate a linear model

$$z_t = a_t + b_t \cdot z_{t-1} + c_t \cdot u_t + d_t \cdot r_t + \epsilon_t \tag{32}$$

where $z_t = (\pi_t, x_t, i_t)'$, ϵ_t is the usual error term, $\{a_t, b_t, c_t, d_t\}$ are coefficient parameter vectors to be estimated. Relation (32) is called the agents' perceived law of motion. Forecasts can then be constructed by solving this model forward and taking expectations to give

$$\begin{aligned} \hat{E}_t z_T &= (I_3 - b_t)(I_3 - b_t^{T-t})a_t + b_t^{T-t} z_t + \gamma u_t (\gamma I_3 - b_t)^{-1} \left(\gamma^{T-t} I_3 - b_t^{T-t}\right) c_t \\ &\quad + \rho r_t (\rho I_3 - b_t)^{-1} \left(\rho^{T-t} I_3 - b_t^{T-t}\right) d_t \end{aligned} \tag{33}$$

for $T \geq t$. To obtain the actual law of motion, substitute (33) into the system of equations (1) and (2). Collecting like terms gives a general expression of the form

$$z_t = \bar{a}_t + \bar{b}_t x_{t-1} + \bar{c}_t u_t + \bar{d}_t r_t$$

where $\{\bar{a}_t, \bar{b}_t, \bar{c}_t, \bar{d}_t\}$ are functions of the current private forecast parameters $\{a_t, b_t, c_t, d_t\}$. Leading this expression one period and taking expectations (rational) provides

$$E_t z_{t+1} = \bar{a}_t + \bar{b}_t x_t + \bar{c}_t \gamma u_t + \bar{d}_t \rho r_t$$

which describes the optimal rational forecast conditional on private-sector behavior. Taken together with (33) at $T = t + 1$ it defines a mapping that determines the optimal forecast coefficients given the current private-sector forecast parameters (a_t', b_t'), written as

$$T(a_t, b_t, c_t, d_t) = (\bar{a}_t, \bar{b}_t, \bar{c}_t, \bar{d}_t). \tag{34}$$

A rational expectations equilibrium (REE) is a fixed point of this mapping. For such REE, we are interested in asking under what conditions does an economy with learning dynamics converge to this equilibrium. Using stochastic approximation methods, Evans and Honkapohja show that the conditions for convergence of the learning algorithm (6) and (7) are neatly characterized by the local stability properties of the associated ordinary differential equation

$$\frac{d}{d\tau}(a, b, c, d) = T(a, b, c, d) - (a, b, c, d), \tag{35}$$

where τ denotes "notional" time. The REE is said to be expectationally stable, or E-Stable, if this differential equation is locally stable in the neighborhood of the REE. From standard results for ordinary differential equations, a fixed point is locally asymptotically stable if all eigenvalues of the Jacobian matrix $D\left[T(a, b, c, d) - (a, b, c, d)\right]$ have negative real parts (where D denotes the differentiation operator and the Jacobian is understood to be evaluated at the rational expectations equilibrium of interest.) See Evans and Honkapohja (2001) for further details on expectational stability.

B Proof of Proposition 2

Consider the case of the forward-looking Taylor rule. We assume that private agents an econometric model that is linear in $\{\Lambda_{t-1},\, u_t,\, r_t\}$. Forecasts are therefore constructed according to the relation

$$\begin{aligned} \hat{E}_t z_T &= (I_4 - b_t)^{-1} (I_4 - b_t^{T-t})^{-1} a_t + b_t^{T-t} z_t + \gamma u_t (\gamma I_4 - b_t)^{-1} \left(\gamma^{T-t} I_4 - b_t^{T-t}\right) c_t \\ &\quad + \rho r_t (\rho I_4 - b_t)^{-1} \left(\rho^{T-t} I_4 - b_t^{T-t}\right) d_t \end{aligned} \tag{36}$$

for $T > t$, where $z_T = (\pi_T, x_T, i_T, \Lambda_T)$, and $(a_t, b_t,\, c_t,\, d_t)$ are the estimated coefficient matrices defined section 2.3. Substituting the forecasts into the structural relations gives

the temporary equilibrium from which the E-Stability mapping can be constructed. The conditions for E-Stability are then given by the local stability properties of the Jacobian of the associated ordinary differential equation

$$T'(\phi) - I_{16} = \begin{bmatrix} A_1 - I_4 & A_5 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_2 - I_4 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_6 & A_3 - I_4 & \mathbf{0} \\ \mathbf{0} & A_7 & \mathbf{0} & A_4 - I_4 \end{bmatrix}$$

where $\phi = (a_t', b_{\pi,t}, b_{x,t}, b_{i,t}, b_{\Lambda,t}, c_t', d_t')'$, I_4 and I_{16} identity matrices of stated dimension and the A_i matrices have dimension (4×4) and elements that are composites of model parameters. E-Stability requires all roots of this system to have negative real parts. The roots are clearly determined by the properties of the matrices $A_1 - I_4, A_2 - I_4$, $A_3 - I_4$and $A_4 - I_4$. A_1 can be shown to be given by

$$A_1 = \begin{bmatrix} \frac{(1-\alpha)\beta}{1-\alpha\beta} + \kappa\left(\frac{\sigma}{1-\beta} - \sigma\psi_\pi\right) & \frac{\alpha\beta\kappa}{1-\alpha\beta} + \kappa(1 - \sigma\psi_x) & -\frac{\kappa\sigma\beta}{1-\beta} & a_{14} \\ \frac{\sigma}{1-\beta} - \sigma\psi_\pi & 1 - \sigma\psi_x & -\frac{\sigma\beta}{1-\beta} & a_{24} \\ \psi_\pi & \psi_x & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where a_{14}, a_{24} and a_{34} are composites of model primitives. It follows that $A_1 - I_4$ has one root equal to negative unity and the remaining roots determined by the characteristic equation

$$P(\lambda) = \lambda^3 + \bar{A}_2\lambda^2 + \bar{A}_1\lambda + \bar{A}_0.$$

and stability under learning dynamics requires $\bar{A}_0, \bar{A}_2 > 0$ and $\bar{A}_4 = -\bar{A}_0 + \bar{A}_1\bar{A}_2 > 0$, where $\bar{A}_i$ are composites of model primitives. The first two conditions can be shown to imply the restrictions

$$\psi_\pi + \frac{1-\beta}{\kappa}\psi_x > 1$$

and

$$\psi_\pi + \frac{\psi_x}{\kappa} > \frac{\kappa\sigma(1-\alpha\beta) - (1-\beta)^2 - (1-\beta)(1-\alpha\beta)}{\kappa\sigma(1-\beta)(1-\alpha\beta)}.$$

The former clearly established the Taylor principle to be necessary for E-Stability. However, it is not sufficient for all parameter values under the maintained assumptions. To see this consider the RHS constant term of the second restriction. It can be written as

$$\frac{1}{1-\beta}-\frac{2-\beta-\alpha\beta}{\kappa\sigma\left(1-\alpha\beta\right)}.$$

As $\beta \to 1$ this expression tends to infinity so that the Taylor principle is necessarily violated for a given policy setting (ψ_π, ψ_x). Since the graphical analysis suggest that for reasonable parameter values E-Stability is unlikely to obtain given these conditions alone, the analysis of the conditions from $A_2 - I_4$, $A_3 - I_4$and $A_4 - I_4$ that relate to learning the slope coefficients on each state variable are not developed. In the author's experience, the constant coefficients typically imply the strictest conditions.

Now consider the case of the lagged-expectations Taylor rule. The economy is given by the system of equations (1), (2) and the monetary policy rule (18). In contrast to the forward-looking Taylor rule, rational expectations equilibrium is linear in the variables $\{\Lambda_{t-1},\, u_{t-1},\, r_{t-1}\}$. Thus agents use an econometric model of the form

$$z_t = a_t + b_t z_{t-1} + c_t u_{t-1} + d_t r_{t-1} + \varepsilon_t$$

$z_T = (\pi_T, x_T, i_T, \Lambda_T)$ to construct forecasts as

$$\begin{aligned}\hat{E}_t z_T &= (I_4 - b_t)^{-1}(I_4 - b_t^{T-t})a_t + b_t^{T-t} z_t + u_t\left(\gamma I_4 - b_t\right)^{-1}\left(\gamma^{T-t} I_4 - b_t^{T-t}\right) c_t \\ &\quad + r_t\left(\rho I_4 - b_t\right)^{-1}\left(\rho^{T-t} I_4 - b_t^{T-t}\right) d_t. \qquad (37)\end{aligned}$$

It is immediate that the forecasts of the constant dynamics are identical to the forward-looking Taylor rule. It follows that the results derived for that case apply here, so that the Taylor principle is again necessary but not sufficient.

C Proof of Proposition 3

Consider the case of the forward-looking Taylor rule. We assume that private agents an econometric model that is linear in $\{\Lambda_{t-1},\, u_t,\, r_t\}$. Forecasts are therefore constructed according to the relation (36) for $T > t$, where $z_T = (\pi_T, x_T, \Lambda_T)$ is redefined recalling private

agents no longer need to forecast the future path of interest rates, and $(a_t, b_t,\ c_t,\ d_t)$ are the estimated coefficient matrices defined section 2. Substituting the forecasts into the structural relations gives the temporary equilibrium from which the E-Stability mapping can be constructed. The conditions for E-Stability are then given by the local stability properties of the Jacobian of the associated ordinary differential equation

$$T'(\phi) - I_{12} = \begin{bmatrix} A_1 - I_3 & A_5 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_2 - I_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_6 & A_3 - I_3 & \mathbf{0} \\ \mathbf{0} & A_7 & \mathbf{0} & A_3 - I_3 \end{bmatrix}$$

where $\phi = (a_t', b_{\pi,t}, b_{x,t}, , b_{\Lambda,t},\ c_t', d_t')'$ when evaluated at the rational expectations equilibrium of interest, I_3 and I_{12} identity matrices of stated dimension and the A_i matrices have dimension (3×3) and elements that are composites of model parameters and $\mathbf{0}$ is a null matrix of similar dimension. E-Stability requires all roots of this system to have negative real parts. The roots are clearly determined by the properties of the matrices $A_1 - I_4$, $A_2 - I_4$, $A_3 - I_4$and $A_4 - I_4$. All $A_i - I_3$ have characteristic equations of the form

$$P(\lambda) = (1 + \lambda)(\lambda^2 + \bar{A}_1\lambda + \bar{A}_0)$$

therefore having one eigenvalue equal to negative unity. The remain two roots will be negative if and only if $\bar{A}_0, \bar{A}_2 > 0$. In the case of $A_1 - I_3$ these latter two inequalities can be shown to imply the restrictions

$$\psi_\pi + \frac{1-\beta}{\kappa}\psi_x > 1$$

and

$$\psi_\pi + \frac{\psi_x}{\kappa} > 1 - \frac{(1-\beta)^2}{(1-\alpha\beta)\kappa\sigma}.$$

The former clearly established the Taylor principle to be necessary for E-Stability. Similar arguments to those employed in the proof of proposition 2 establish satisfaction of the Taylor principle as being sufficient the latter to hold.

Similarly, $A_2 - I_3$ has the remaining two roots being negative if and only if the restrictions

$$\psi_\pi + \frac{(1-\beta\mu)}{\kappa}\psi_x > 1 - \frac{(1-\mu)(1-\mu\beta)}{\kappa\sigma}$$

and

$$\psi_\pi + \frac{\psi_x}{\kappa} > 1 - \frac{(1-\beta\mu)^2 + (1-\mu)(1-\alpha\beta\mu)}{(1-\alpha\beta\mu)\kappa\sigma\mu}$$

which are again satisfied if the Taylor principle is satisfied (using the same arguments as Proposition 2) and noting $0 < \mu < 1$. The proof is complete by observing that $A_3 - I_3$ and $A_4 - I_3$ take an identical form to $A_2 - I_3$ once μ is replaced by ρ and γ respectively. Since $0 < \rho, \gamma < 1$ the result follows immediately.

In the case of the lagged expectations Taylor rule, almost identical calculations give the result that the Taylor principle is necessary and sufficient for E-Stability.

D Proof of Proposition 4

Consider the case of the forward-looking Taylor rule. Section 6 showed that the monetary authority's decision problem each period is to solve the rational expectations problem

$$\begin{aligned} i_t &= (\psi_\pi \bar{a}_\pi + \psi_x \bar{a}_x) + (\psi_\pi \bar{b}_\pi + \psi_x \bar{b}_x)\Lambda_t + (\psi_\pi \bar{c}_\pi + \psi_x \bar{c}_x)\gamma u_t \\ &\quad + (\psi_\pi \bar{d}_\pi + \psi_x \bar{d}_x)\rho r_t + (\psi_\pi \bar{e}_\pi + \psi_x \bar{e}_x) E_t^{cb} i_{t+1} \\ &= \bar{a}_i + \bar{b}_i \Lambda_{t-1} + \bar{c}_i u_t + \bar{d}_i r_t + \bar{e}_i E_t^{cb} i_{t+1} \end{aligned}$$

where

$$\begin{aligned} \bar{a}_i &= (\psi_\pi \bar{a}_\pi + \psi_x \bar{a}_x) \quad ; \qquad \bar{b}_i = (\psi_\pi \bar{b}_\pi + \psi_x \bar{b}_x)\mu \\ \bar{d}_i &= (\psi_\pi \bar{d}_\pi + \psi_x \bar{d}_x)\rho \; ; \qquad \bar{e}_i = (\psi_\pi \bar{e}_\pi + \psi_x \bar{e}_x) \\ \bar{c}_i &= (\psi_\pi \bar{c}_\pi + \psi_x \bar{c}_x)\gamma + (\psi_\pi \bar{b}_\pi + \psi_x \bar{b}_x) \end{aligned}$$

If $|\bar{e}_i| < 1$ there is a unique bounded solution. Solving forward gives

$$i_t = E_t \sum_{s=0}^{\infty} \bar{e}_i^s \left[\bar{a}_i + \bar{b}_i \Lambda_{t+s-1} + \bar{c}_i u_{t+s} + \bar{d}_i r_{t+s} \right]. \tag{38}$$

Noting

$$\begin{aligned} E_t \Lambda_{t+s} &= \mu^s \sum_{j=0}^{\infty} \mu^j u_{t-j} + E_t \sum_{j=0}^{s-1} \mu^j u_{t+s-j} \\ &= \mu^s \Lambda_t + \frac{\gamma(\gamma^s - \mu^s)}{\gamma - \mu} u_t. \end{aligned}$$

and evaluating expectations in (38) gives

$$i_t = \frac{\bar{a}_i}{1-\bar{e}_i} + \frac{\bar{b}_i}{1-\bar{e}_i\mu}\Lambda_{t-1} + \frac{\bar{b}_i\bar{e}_i + \bar{c}_i\left(1-\bar{e}_i\mu\right)}{\left(1-\bar{e}_i\mu\right)\left(1-\bar{e}_i\gamma\right)}u_t + \frac{\bar{d}_i}{1-\bar{e}_i\rho}r_t$$

as the required solution. This, together with

$$\pi_t = \bar{a}_\pi + \bar{b}_\pi\Lambda_{t-1} + \bar{c}_\pi u_t + \bar{d}_\pi r_t + \bar{e}_\pi i_t \tag{39}$$

$$x_t = \bar{a}_x + \bar{b}_x\Lambda_{t-1} + \bar{c}_x u_t + \bar{d}_x r_t + \bar{e}_x i_t \tag{40}$$

solves for the temporary equilibrium values of $\{\pi_t, x_t, i_t, \Lambda_t\}$. Leading these equations one period and taking expectations gives the optimal forecasts given this evolution of the endogenous variables.

The Jacobian matrix of the associated ordinary differential equation can be shown to be given by

$$T'(\phi) - I_{16} = \begin{bmatrix} A_1 - I_4 & A_5 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_2 - I_4 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_6 & A_3 - I_4 & \mathbf{0} \\ \mathbf{0} & A_7 & \mathbf{0} & A_4 - I_4 \end{bmatrix}$$

where $\phi' = (a_t', b_{\pi,t}, b_{x,t}, b_{i,t}, b_{\Lambda,t},\ c_t',\ d_t')$, I_4 and I_{16} are identity matrices of the indicated dimension, $\mathbf{0}$ a (4×4) null matrix and the A_i matrices have elements that are composites of model primitives.

For E-Stability $A_1 - I_4$, $A_2 - I_4$, $A_3 - I_4$ and $A_4 - I_4$ must each have four roots with negative real parts. Consider $A_1 - I_4$. It is easily shown that two roots are equal to negative unity, while the remaining two roots are the solution to the quadratic equation

$$a\lambda^2 + b\lambda + c = 0$$

where

$$\begin{aligned} a &= (1-\beta)(1-\alpha\beta)(1+\sigma(\psi_x + \kappa\psi_\pi)) \\ b &= \Gamma_0 + \Gamma_1\psi_1 + \Gamma_2\psi_\pi \\ c &= \sigma\kappa(\psi_\pi - 1) + \sigma(1-\beta)\psi_x \end{aligned}$$

and

$$\begin{aligned}
\Gamma_0 &= (1-\beta)^2 - \kappa\sigma(1-\alpha\beta) \\
\Gamma_1 &= \sigma\left[(1-\beta)^2 + (1-\alpha\beta)\right] \\
\Gamma_2 &= \kappa\sigma\left[(1-\beta) + (1-\alpha\beta)\right].
\end{aligned}$$

Under the maintained model assumptions $a > 0$. For a two roots with negative real parts we require $b, c > 0$. The latter restriction requires

$$\psi_\pi + \psi_x \cdot \frac{1-\beta}{\kappa} > 1.$$

establishing the Taylor principle to be necessary for E-Stability. $b > 0$ implies

$$\psi_\pi + \psi_x \cdot \frac{\left[(1-\beta)^2 + (1-\alpha\beta)\right]}{\kappa\left[(1-\beta) + (1-\alpha\beta)\right]} > \frac{\kappa\sigma(1-\alpha\beta) - (1-\beta)^2}{\kappa\sigma\left[(1-\beta) + (1-\alpha\beta)\right]}.$$

Since the right-hand side constant is clearly less than unity and the coefficient on the policy parameter ψ_x greater than $(1-\beta)/\kappa$ we infer that the Taylor principle is also sufficient for satisfaction of this restriction.

For $A_2 - I_4$ identical manipulations show that two roots are equal to negative unity, while the remaining two roots are the solution to the quadratic equation

$$a\lambda^2 + b\lambda + c = 0$$

where

$$\begin{aligned}
a &= (1-\beta\mu)(1-\alpha\beta\mu)(1+\mu\sigma\psi_x + \mu\kappa\sigma\psi_\pi) \\
b &= \Gamma_0 + \Gamma_1\psi_x + \Gamma_2\psi_\pi \\
c &= \kappa\sigma(\psi_\pi - 1) + \sigma\mu(1-\beta\mu)\psi_x + (1-\beta\mu)(1-\mu)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_0 &= (1-\beta\mu)^2 + (1-\mu)(1-a\beta\mu) - \kappa\sigma\mu(1-\alpha\beta\mu) \\
\Gamma_1 &= \sigma\mu\left[(1-\beta\mu)^2 + (1-\alpha\beta\mu)\right] \\
\Gamma_2 &= \kappa\sigma\mu\left[(1-\beta\mu) + (1-\alpha\beta\mu)\right].
\end{aligned}$$

Again under the maintained model assumptions $a > 0$. For the final two roots to have negative real parts we require $b, c > 0$ implying the restrictions

$$\psi_\pi + \psi_x \cdot \frac{\left[(1-\beta\mu)^2 + (1-\alpha\beta\mu)\right]}{\kappa\left[(1-\beta\mu) + (1-\alpha\beta\mu)\right]} > \frac{\kappa\sigma\mu\,(1-\alpha\beta\mu) - \left[(1-\beta\mu)^2 + (1-\mu)\,(1-a\beta\mu)\right]}{\kappa\sigma\mu\left[(1-\beta\mu) + (1-\alpha\beta\mu)\right]}$$

and

$$\psi_\pi + \psi_x \cdot \frac{(1-\beta\mu)}{\kappa} > \frac{\kappa\sigma\mu - (1-\mu)\,(1-a\beta\mu)}{\kappa\sigma\mu}.$$

It is again easy to show that under the maintained model assumptions both these inequalities have right-hand side constants that are less than unity and slope coefficients on the parameter ψ_x that are greater than $(1-\beta)/\kappa$ which establishes the Taylor principle as being necessary and sufficient for E-Stability conditions on $A_2 - I_4$ to be satisfied. Similar calculations demonstrate the matrices $A_3 - I_4$ and $A_4 - I_4$ to imply the same conditions as $A_2 - I_4$. Therefore the Taylor principle is necessary and sufficient for E-Stability.

For the lagged expectations Taylor rule the proof is virtually identical to the case of the forward-looking Taylor rule. It is omitted here, and available from the author on request.

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