Federal Reserve Bank of New York
Staff Reports

Long-Term Debt Pricing and Monetary Policy Transmission under Imperfect Knowledge

Stefano Eusepi
Marc Giannoni
Bruce Preston

Staff Report no. 547
February 2012

Long-Term Debt Pricing and Monetary Policy Transmission under Imperfect Knowledge

Stefano Eusepi, Marc Giannoni, and Bruce Preston

Federal Reserve Bank of New York Staff Reports, no. 547
February 2012
JEL classification: E32, D83, D84

Abstract

Under rational expectations, monetary policy is generally highly effective in stabilizing the economy. Aggregate demand management operates through the expectations hypothesis of the term structure: Anticipated movements in future short-term interest rates control current demand. This paper explores the effects of monetary policy under imperfect knowledge and incomplete markets. In this environment, the expectations hypothesis of the yield curve need not hold, a situation called unanchored financial market expectations. Whether or not financial market expectations are anchored, the private sector's imperfect knowledge mitigates the efficacy of optimal monetary policy. Under anchored expectations, slow adjustment of interest rate beliefs limits scope to adjust current interest rate policy in response to evolving macroeconomic conditions. Imperfect knowledge represents an additional distortion confronting policy, leading to greater inflation and output volatility relative to rational expectations. Under unanchored expectations, current interest rate policy is divorced from interest rate expectations. This permits aggressive adjustment in current interest rate policy to stabilize inflation and output. However, unanchored expectations are shown to raise significantly the probability of encountering the zero lower bound constraint on nominal interest rates. The longer the average maturity structure of the public debt, the more severe is the constraint.

Key words: long debt, optimal monetary policy, expectations stabilization, transmission of monetary policy, expectations hypothesis of the yield curve

---

Eusepi, Giannoni: Federal Reserve Bank of New York (e-mail: stefano.eusepi@ny.frb.org, marc.giannoni@ny.frb.org). Preston: Columbia University (e-mail: bp2121@columbia.edu). The authors thank participants at the 2011 Euro Area Business Cycle Network conference "Fiscal and Monetary Policy in the Aftermath of the Financial Crisis," particularly discussant Martin Ellison. The views expressed in this paper are those of the authors and do not necessarily reflect the position of the Federal Reserve Bank of New York or the Federal Reserve System.

1 Introduction

Under rational expectations monetary policy is generally highly effective in stabilizing the economy. Aggregate demand management operates through the expectations hypothesis of the term structure — anticipated movements in future short-term interest rates control current demand. This paper explores the conduct of monetary policy when this expectations channel is impaired because of imperfect knowledge.

Imperfect knowledge is introduced in a standard New Keynesian model of the kind frequently used for monetary policy evaluation — see, for example, Clarida, Gali, and Gertler (1999) and Woodford (2003). Households and firms are optimizing, have a completely specified belief system, but do not know the equilibrium mapping between observed state variables and market clearing prices. By extrapolating from historical patterns in observed data they approximate this mapping to forecast exogenous variables relevant to their decision problems, such as prices and policy variables. Beliefs are revised in response to new data using a constant-gain algorithm.[1] Because agents must learn from historical data, beliefs need not be consistent with the objective probabilities implied by the economic model. The analysis is centrally concerned with conditions under which agents' expectations are consistent with stable macroeconomic dynamics. The situation in which the model has a bounded solution is referred to as "expectational stability" or "stable expectations".

Relative to earlier analyses on imperfect knowledge by Eusepi and Preston (2010, 2011) this paper considers the consequences of imperfect knowledge in asset pricing. Under incomplete markets and imperfect knowledge there does not necessarily exist a unique forecasting model consistent with no-arbitrage in financial markets. Following Adam and Marcet (2011), if agents do not possess common knowledge of the aggregate no-arbitrage condition into the indefinite future it is not possible to write the price of an asset as a function of fundamentals — prices necessarily depend upon the one-period-ahead expectation of the price tomorrow. This approach to asset price determination is referred to as unanchored financial market expectations. In contrast, when the no-arbitrage condition is common knowledge at all points in the decision horizon, transversality implies that asset prices are the present discounted value of fundamentals. This is referred to as anchored financial market expectations.

There is only one asset in non-zero net supply — long-term government debt. The critical distinction between the two approaches to asset price determination is that unanchored finan-

[1] Milani (2007), Slobodyan and Wouters (2009) and Eusepi and Preston (2011a) provide empirical support for such belief structures.

cial market expectations do not imply satisfaction of the expectations hypothesis of the yield curve. The price of long-term debt can become divorced from fundamentals, the anticipated sequence of future short-term interest rates. The question is whether this matters for aggregate demand management policy. Can financial market expectations hinder the efficacy of monetary policy? And to what extent does the maturity structure of the public debt qualify the responses to these questions.

The analysis commences with an evaluation of the merits of various recommendations for interest-rate policy that have been prominent in the rational expectations literature on monetary policy design. Both simple Taylor rules and a target criterion implied by optimal discretion engender instability in aggregate dynamics for at least some gain coefficients regardless of whether expectations are anchored or not. The Taylor rule is particularly prone to instability at longer maturities of the public debt, while the optimal rational expectations target criterion performs worse at shorter maturities. These findings extend the 'robust stability' results of Evans and Honkapohja (2008) to a broader class of learning models in which decisions are optimal conditional on maintained beliefs and in which the pricing of long-term public debt plays a prominent role.

To address this instability consider a central bank that implements optimal monetary policy given agents' imperfect knowledge. Applying results found in Giannoni and Woodford (2010) and Eusepi, Giannoni, and Preston (2011), a proposition establishes optimal policy to induce stable aggregate dynamics for all admissible parameters. In particular, gain coefficients on the unit interval are all consistent with expectational stability. Despite this property, model dynamics are fundamentally different in the cases of anchored and unanchored expectations. The former deliver increased output and inflation variability; while the latter imply very volatile interest rates.

This difference in stabilization properties stems directly from the failure of the expectations hypothesis of the yield curve under unanchored expectations. Because long-term debt prices do not necessarily depend on the future sequence of short-term interest rates, the restraining influence of anticipated movements in the term structure is no longer a determinant of aggregate demand. Stabilization policy is shown to rest entirely on the current short rate. Imperfect knowledge leads to persistent movements in beliefs, requiring aggressive adjustment to monetary policy in response to transitory natural rate and cost push shocks. In contrast, under anchored financial market expectations, the term structure remains an important determinant of aggregate demand. But precisely because it does imposes an additional constraint

on monetary policy. Changes in current interest rates lead to revisions of beliefs about future interest rates, albeit with a lag due to learning dynamics. The revisions in beliefs in turn feedback on the state of aggregate demand in subsequent periods. Optimal policy requires small adjustments in current interest-rate policy because beliefs represent an additional distortion that policy must confront. Aggressive adjustment of current interest rates presage excessive movements in long-rates and macroeconomic volatility. The fact that anchored expectations lead to less volatile adjustment of interest rates implies increased volatility in inflation and output relative to perfect knowledge. These properties and associated intuition are developed using plots of the efficiency policy frontiers and impulse responses functions under optimal policy.

A final exercise considers the likelihood of violating the zero lower bound on nominal interest rates. This is relevant given the observed volatility of interest rates under optimal policy. Indeed, it raises the question of whether optimal policy can in fact be implemented when expectations are unanchored. Calculating the unconditional probability that nominal interest rates are negative reveals the zero lower bound to be likely problematic. Under unanchored expectations, regardless of the stabilization weight given to interest-rate volatility in the central bank's loss function, the probability of encountering the zero lower bound is bounded below at 0.14. In the case of no weight to interest-rate stabilization, this probability is close to 0.4. In contrast, for anchored expectations this probability is always small, and for moderate weights on interest-rate stabilization the probability is zero. To the extent there is expectational drift relevant to the pricing of the public debt, and, therefore, the yield curve, the zero lower bound will be a more severe constraint than suggested by rational expectations analyses of New Keynesian models. For example Schmitt-Grohe and Uribe (2007) argue in the context of their model that "the zero bound on the nominal interest rate, which is often cited as a rationale for setting positive inflation targets, is of no quantitative relevance". And Chung, Laforte, Reifschneider, and Williams (2011) adduce evidence that empirical models based on data from the Great Moderation period and which ignore parameter uncertainty understate the likelihood of the zero lower bound being an important constraint on monetary policy.

This paper builds on Eusepi and Preston (2010, 2011) which explore the consequences of monetary and fiscal policy uncertainty for macroeconomic stability under learning dynamics. The current analysis departs from these papers by considering the specific role of financial market expectations for the transmission of monetary policy. A further departure is the

characterization of fully optimal policy under learning dynamics by applying results in Eusepi, Giannoni, and Preston (2011). The latter extends the analysis of Molnar and Santoro (2005) to models in which households and firms make optimal decisions conditional on their beliefs, rather than models in which only one-period-ahead expectations matter.[2]

The paper proceeds as follows. Section 2 delineates a special case of the model developed by Eusepi and Preston (2011). Section 3 explores how different assumptions about financial market beliefs affect the stability of various simple rules that have emerged as desirable in the rational expectations literature on monetary policy. Section 4 characterizes optimal policy under learning dynamics. Section 5 investigates core properties of optimal monetary policy under anchored and unanchored financial market expectations, examining model dynamics in response to standard shocks. Section 6 further dissects the trade-offs inherent in stabilization policy under imperfect knowledge using efficient policy frontiers. Section 7 shows the zero lower bound becomes a more binding constraint under learning. Section 8 provides discussion and conclusions.

2 A Simple Model

The following section details a special case of the model studied by Eusepi and Preston (2011b). The model is similar in spirit to Clarida, Gali, and Gertler (1999) and Woodford (2003) used in many recent studies of monetary policy. The major difference is the emphasis given to details of fiscal policy and the incorporation of near-rational beliefs delivering an anticipated utility model as described by Kreps (1998) and Sargent (1999). The analysis follows Marcet and Sargent (1989) and Preston (2005b), solving for optimal decisions conditional on current beliefs. The discussion overviews key model equations. Additional detail is found in Eusepi and Preston (2011b).

2.1 Assets and fiscal policy

The are two types of assets in this economy. One-period government debt, in zero net supply, with price P_t^s; and a more general portfolio of government debt, B_t^m, in non-zero net supply with price P_t^m. The former debt instrument satisfies $P_t^s = (1 + i_t)^{-1}$ and defines the period nominal interest rate, the instrument of central bank monetary policy. Following Woodford

[2] See Preston (2005a, 2005b) for a discussion of optimal decision making under learning dynamics. Approaches based soley on one-period-ahead expectations fail to represent optimal decisions given the underlying microfoundations assumed in the New Keynesian model. Preston (2006, 2008) demonstrate these modeling choices have non-trivial implications for monetary policy design.

(1998, 2001) the latter debt instrument has payment structure $\rho^{T-(t+1)}$ for $T > t$ and $0 \leq \rho \leq 1$. The asset can be interpreted as a portfolio of infinitely many bonds, with weights along the maturity structure given by $\rho^{T-(t+1)}$. The advantage of specifying the debt portfolio in this way is that it introduces only a single state variable whose properties are indexed by a single parameter ρ. Varying ρ varies the average maturity of debt, which is given by $\left(1 - \frac{1+\bar{\pi}}{1+\bar{\imath}}\rho\right)^{-1}$, where $\bar{\pi}$ is the steady-state inflation rate, which we assume to be approximately zero and $\bar{\imath}$ is the steady-state nominal interest rate. A central focus of the analysis will be the consequences of variations in average maturity for expectations stabilization. For example, the case of one-period debt corresponds to $\rho = 0$. A consol bond corresponds to $\rho = 1$. For simplicity, we assume that the government has zero spending at all times and runs a steady-state surplus, consistent with the positive outstanding debt. The government flow budget constraint evolves according to

$$P_t^m B_t^m = B_{t-1}^m \left(1 + \rho P_t^m\right) - T_t. \tag{1}$$

Assume that the government is understood to implement a Ricardian fiscal policy so that at any point in time the expected discounted value of fiscal surpluses backs the outstanding value of debt. Government debt is not perceived as net wealth in this economy.[3]

2.2 Households

The economy is populated by a continuum of households, indexed by i, which seeks to maximize future expected discounted utility, at rate $0 < \beta < 1$, defined in terms of a Dixit-Stiglitz consumption aggregator $C_t(i)$ and hours worked $H_t(i)$

$$\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[\ln\left(C_T(i)\right) - \frac{\phi}{1+\gamma}\left(H_T(i)\right)^{1+\gamma}\right] \tag{2}$$

subject to flow budget constraint is

$$P_t^s B_t^s(i) + P_t^m B_t^m(i) \leq \left(1 + \rho P_t^m\right) B_{t-1}^m(i) + B_{t-1}^s(i) + W_t H_t(i) + P_t \Gamma_t - T_t - P_t C_t(i) \tag{3}$$

where $B_t^s(i)$ and $B_t^m(i)$ are household i's holdings of each of the debt instruments; W_t the nominal wage determined in a perfectly competitive labor market; and Γ_t dividends from holding shares in an equal part of each firm. Initial bond holdings $B_{-1}^m(i)$ and $B_{-1}^s(i)$ are given. $\hat{E}_t^i$ denotes household i's subjective beliefs.

[3] Eusepi and Preston (2010, 2011) show that wealth effects from government debt dynamics can have important consequences for policy stabilization. The intention here is to clearly isolate the effects of financial market expectations on the transmission of monetary policy without the additional complication of demand effects arising from departures from Ricardian equivalence.

2.3 Information

Each agent in the model correctly understands their own objectives and any relevant constraints, but have no knowledge of other agents' preferences and beliefs. Despite the apparent symmetry, this knowledge assumption delivers a heterogeneous agent model. As information sets differ, the set up is formally identical to models which explicitly introduce heterogeneous preferences and beliefs. See, for example, Lorenzoni (2008). The fact that agents have no knowledge of other agents' preferences and beliefs implies that they do not know the equilibrium mapping between state variables and market clearing prices. As a result, they cannot forecast the various prices and state variables that are relevant to their decision problem, but beyond their control, without making further assumptions. We assume that agents approximate this mapping by extrapolating from historical patterns in observed data. As additional data become available the approximate model is revised. The structure of beliefs is discussed in more detail in section 2.8.

2.4 The consumption decision rule

Subsequent analysis employs a log-linear approximation in the neighborhood of a non-stochastic steady state. The optimal decision rule for household consumption is obtained by combining the optimality conditions for consumption, labor supply, the flow budget constraint and transversality. It is assumed that agents fully understand that fiscal policy is Ricardian so that government debt is not a relevant state variable in their decisions. Consumption is determined by the expected path of the short-term real interest rate and the expected evolution of labor income and profits

$$\begin{aligned} \hat{C}_t(i) &= -\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[\beta \left(\hat{\imath}_T - \pi_{T+1}\right)\right] \\ &\quad + \bar{s}_C^{-1} (1-\beta) \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[\left(\frac{\bar{\theta}-1}{\bar{\theta}}\right)\left(1+\gamma^{-1}\right)\hat{w}_T + \bar{\theta}^{-1}\hat{\Gamma}_T\right] \end{aligned} \tag{4}$$

where π_t is the inflation and $\hat{\imath}_t$ the nominal interest rate, which also denotes the one-period returns on the their asset portfolio discussed below. Finally $\bar{\theta}$ denotes the steady-state elasticity of demand in the Dixit-Stiglitz aggregator and

$$\bar{s}_C = \gamma^{-1}\left(\bar{\theta}-1\right)\bar{\theta}^{-1} + 1.$$

In the next section we focus on the asset pricing implications of the model and their consequences for the forecast of the real interest rate path, which is the main focus of the paper.

2.5 Asset pricing and beliefs formation

Under non-rational beliefs and multiple assets there are important modeling choices to be made about the precise form of financial market beliefs. In particular, the expectations hypothesis need not hold if agents have imperfect knowledge about other market participants' preferences and beliefs, as in Adam and Marcet (2011). Each household i's optimality conditions for holding the two assets provides the no-arbitrage restriction

$$\hat{E}_t^i R_{t,t+1} = \hat{E}_t^i R_{t,t+1}^m$$

where $R_{t,t+1}$ and $R_{t,t+1}^m$ denote the period returns from date t to $t+1$ on one-period government debt and the longer-term portfolio of government securities. This can expressed as

$$\hat{\imath}_t = -\hat{E}_t^i \left(\hat{P}_t^m - \rho \left(1+\bar{\imath}\right)^{-1} \hat{P}_{t+1}^m \right). \tag{5}$$

Solving (5) for $\hat{P}_t^m$ and iterating one-period forward yields

$$\hat{P}_t^m = -\hat{\imath}_t + \rho \left(1+\bar{\imath}\right)^{-1} \hat{E}_t^i \left[-\hat{\imath}_{t+1} + \rho \left(1+\bar{\imath}\right)^{-1} \hat{E}_{t+1}^{Mt+1} \hat{P}_{t+2}^m \right] \tag{6}$$

where $\hat{E}_t^{Mt+1}$ denotes the expectation of the marginal investor that determines the price of the bond a time $t+1$. Now consider two alternative models of asset price determination under incomplete information. The two models yield the same equilibrium under rational expectations. They have different implications under imperfect information and learning.

Anchored financial expectations. Under anchored financial expectations, suppose each agent i always believes that they will be the marginal investor in the future so that $\hat{E}_t^i \left(\hat{E}_{t+1}^{Mt+1} \hat{P}_{t+2}^m \right) = \hat{E}_t^i \hat{P}_{t+2}^m$. Solving (6) forward using the implication of the transversality condition associated with household optimization that

$$\lim_{T\to\infty} \hat{E}_t^i \left(\rho \left(1+\bar{\imath}\right)^{-1} \right)^{T-t} \hat{P}_{T+1}^m = 0$$

gives the price of the bond portfolio as

$$\hat{P}_t^m = -\hat{E}_t^i \sum_{T=t}^{\infty} \left(\rho \left(1+\bar{\imath}\right)^{-1} \right)^{T-t} \hat{\imath}_T. \tag{7}$$

The multiple-maturity debt portfolio is priced as the expected present discounted value of all future one-period interest rates, where the discount factor is given by $\rho \left(1+\bar{\imath}\right)^{-1}$. In this model, agents' beliefs determine a forecast of the sequence of future one-period interest rates $\{\hat{\imath}_T\}$ from which the multiple-maturity bond portfolio is priced using (7). Because the bond

pricing equation is an implication of the no-arbitrage condition, relation (5) is necessarily satisfied at all dates. In this model expectations of the future price of long-term government debt do not affect the equilibrium dynamics of the model, just like under rational expectations. All that matters is the evolution of expected future short-term interest rates. The expectations hypothesis of the term structure holds.

Unanchored financial expectations. As an alternative approach, equally consistent with the requirement of no-arbitrage, assume that agent i does not expect to be the marginal investor at all times. Because agents lack knowledge about others' beliefs, the law of iterated expectations fails to hold in (6). Hence the expectations hypothesis (7) might not be satisfied at all times. In this case, we need to replace the asset pricing equation (7) with (5), so that beliefs about the future price of long-term bonds become an important factor in determining the current bond price.[4] Agents forecast the price of long-term bonds and use it to determine a forecast of the sequence of future one-period returns $\left\{R^m_{t,t+1}\right\}$. Under such unanchored financial expectations, the price of long-term bonds might not reflect the discounted sum of expected short-term rates because agents lack common knowledge about other market participants' beliefs.[5] The price of long-term debt, $\hat{P}^m_t$, is given by the no-arbitrage condition (5), given expectations about tomorrow's bond price and current monetary policy. Note that in the special case $\rho = 0$, so that there is only one-period debt, the anchored and unanchored financial market expectations models are isomorphic.

2.6 Aggregate Demand and Supply

Aggregating across agents and imposing market-clearing conditions, the model has an aggregate demand relation that takes the form

$$\begin{aligned} x_t &= -\hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left(\hat{\imath}_T - \pi_{T+1}\right) - \hat{A}_t \\ &\quad + \bar{s}_C^{-1} (1-\beta) \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[\left(\frac{\bar{\theta}-1}{\bar{\theta}} \right) \left(1+\gamma^{-1}\right) \hat{w}_{T+1} + \bar{\theta}^{-1} \hat{\Gamma}_{T+1} \right] \end{aligned} \tag{8}$$

[4] Note that each agent i does not expect to be the marginal investor all the times which implies that one of the Euler equations characterizing asset holdings is not expected to hold with equality at all times. In this model, in order to maintain consistency with the way the consumption decision rule is computed, we assume that each investor faces constraints on short-selling of short-term bonds. The euler equation for long-term bonds is always expected to hold while for short-term bonds the constraint might be binding. As in Adam and Marcet (2011), in equilibrium, each agent is the same — they are always the marginal investor though do not know this to be true.

[5] See also Adam and Marcet (2011).

where x_t is the output gap, defined as the difference between output and efficient output, which is obtained under flexible prices in absence of markup distortions. $\hat{A}_t$ is an aggregate technology shock with properties to be described. $\hat{E}_t = \int \hat{E}_t^i$ represents average beliefs held by households. Whether financial market expectations are anchored or not will imply different forecasting models for $\hat{\imath}_T$ for $T > t$.

Aggregate supply is determined by the generalized Phillips curve

$$\hat{\pi}_t = \psi(\gamma+1)(x_t + \hat{u}_t) + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\psi\alpha\beta\left(\hat{w}_{T+1} - \hat{A}_{T+1} + \hat{u}_{T+1}\right) + (1-\alpha)\beta\pi_{T+1}\right]. \tag{9}$$

The parameter α satisfies the restrictions $0 < \alpha < 1$ and $\psi = (1-\alpha\beta)(1-\alpha)\alpha^{-1}$. Equation (9) can be derived from the aggregation of the optimal prices chosen by firms to maximize the expected discounted flow of profits under a Calvo-style price-setting problem — see Yun (1996). It is a generalized Phillips curve, specifying current inflation as depending on contemporaneous values of wages and the technology shock, and expectations for these variables and inflation into the indefinite future. The presence of long-term expectations arise due to pricing frictions embodied in Calvo pricing. When a firm has the opportunity to change its price in period t there is a probability α^{T-t} that it will not get to change its price in the subsequent $T-t$ periods. The firm must concern itself with macroeconomic conditions relevant to marginal costs into the indefinite future when deciding the current price of its output. Future profits are also discounted at the rate β, which equals the inverse of the steady-state gross real interest rate. The variable $\hat{u}_t$ represents a cost-push shock, corresponding to exogenous time-variation in the desired mark-up of firms, which in turn is related to the evolution of the households' time-varying elasticity of demand θ_t in the underlying microfoundations.

The aggregation of optimal household and firm spending and pricing plans, along with goods market clearing also deliver the following aggregate relations. Given optimal prices, firms stand ready to supply desired output which determines aggregate hours as

$$\hat{H}_t = \hat{Y}_t - \hat{A}_t \tag{10}$$

and comes from aggregation of firm production technologies, which take labor as the only input. Wages and dividends are then determined from

$$\gamma\hat{H}_t = -\hat{C}_t + \hat{w}_t \tag{11}$$

$$\hat{\Gamma}_t = \hat{Y}_t - \left(\bar{\theta} - 1\right)\left(\hat{w}_t - \hat{A}_t\right), \tag{12}$$

where the former is derived from the labor-leisure optimality condition of households, and the latter from the definition of firm profits. Finally, goods market clearing implies the log-linear restriction

$$\hat{Y}_t = \hat{C}_t. \tag{13}$$

2.7 Monetary Policy

Various arrangements for monetary policy are considered: i) simple Taylor rules; ii) an optimal target criterion derived under rational expectations; and iii) fully optimal policy under learning.

Analysis commences with rules having desirable properties under a rational expectations analysis of the model. This is an evaluation of robustness: do policies continue to perform well when agents make small forecasting errors relative to rational expectations? The first is a standard Taylor rule

$$\hat{\imath}_t = \phi_\pi \pi_t + \phi_x x_t \tag{14}$$

where $\phi_\pi, \phi_\pi \geq 0$ are policy parameters. The second is a target criterion that characterizes optimal policy under discretion assuming rational expectations

$$\pi_t = -\frac{\lambda_x}{\kappa} x_t$$

where $\lambda_x \geq 0$ is the weight given to output gap stabilization in a standard quadratic loss function.[6] Such rules are of practical import as they implicitly define an instrument rule that responds not only to output gap and inflation, but also to the price of long-term debt and, more generally, to agents' expectations about the future evolution of market prices. Comparison of this rule with simple Taylor rules permits an evaluation of the advantages of responding directly to asset prices.

Having established the stabilization properties of "simple" rational expectations rules, the fully optimal policy is characterized. The central bank is assumed to understand the structural equations describing the economy, as well as the specific form of agent's belief formation. Taking these as given, the central bank minimizes a standard quadratic loss function in inflation, output and the nominal interest rate. The details of this approach are described in the sequel.

[6] Optimal policy under commitment is not considered for reasons of simplicity. Attention is restricted to rational expectations equilibiria that are purely forward looking. This ensures that the belief structure discussed below nests all relevant rational expectations equilibria. The intertial character of optimal policy would require more general belief structures than what is considered here — though similar points could easily be established in that case.

2.8 Belief Formation

Agents construct forecasts of inflation, wages, profits, interest rates and bond prices according to

$$\hat{E}_t^i X_{t+T} = a_{t-1}^X \tag{15}$$

where $X = \left\{\pi, \hat{w}, \hat{\Gamma}, \hat{\imath}, \hat{P}^m\right\}$ for any $T > 0$. In period t forecasts are predetermined. The belief parameters constitute state variables. Beliefs are updated according to the constant gain algorithm

$$a_t^X = (1-g)\, a_{t-1}^X + g X_t \tag{16}$$

where $g > 0$ is the constant gain parameter. The belief structure is consistent with the minimum-state-variable rational expectations solution, when shocks are i.i.d. Agents learn only about the mean value of each time series. Under anchored financial expectations agents forecast $\hat{E}_t^i \hat{\imath}_{t+T}$, for all $T > 0$, while the price of the long-term bond is determined by the expectations hypothesis (7). Conversely, under unanchored financial expectations agents forecast $\hat{E}_t^i \hat{P}_{t+T}^m$, for all $T > 0$, and the short-term expected nominal return is determined by the one-period returns from long-term debt $\hat{E}_t^i R_{T,t+1}^m$. This completes the description of aggregate dynamics.

To summarize, each model comprises the six aggregate relations (8)–(13), either (5) or (7) to price long-term assets, a characterization of monetary policy such as (14), and four updating equations which determine the evolution of the variables

$$\left\{\hat{P}_t^m, \pi_t, \hat{\imath}_t, \hat{w}_t, \hat{\Gamma}_t, \hat{C}_t, \hat{Y}_t, \hat{H}_t, a_t^\pi, a_t^{\hat{w}}, a_t^{\hat{\Gamma}}, a_t^Y\right\}$$

where $Y = \{\hat{\imath}, \hat{P}^m\}$ depending on the asset price assumptions, given the exogenous processes $\left\{\hat{u}_t, \hat{A}_t\right\}$ and initial beliefs $\left\{a_{t-1}^\pi, a_{t-1}^{\hat{w}}, a_{t-1}^{\hat{\Gamma}}, a_{t-1}^Y\right\}$.

2.9 Calibration

Assuming a quarterly model the benchmark parameterization follows, with departures noted as they arise in subsequent text. Household decisions: the discount factor is $\beta = 0.99$; the inverse Frisch elasticity of labor supply $\gamma = 0.5$ and the elasticity of demand across differentiated goods $\theta = 8$. Firm decisions: nominal rigidities are determined by $\alpha = 0.75$.[7] Fiscal policy: the only fiscal parameter relevant to decisions is ρ in the unanchored financial expectations model. The benchmark value is $\rho = 0.96$, consistent with an average maturity of

[7] The parameter ψ is determined by the choice of α.

US government debt held by the public of approximately five years. Finally we assume that technology and cost-push shocks are i.i.d. An assumption that turns out to be useful when studying optimal policy under learning.

3 Experiments with Simple Policy Rules

This paper is centrally concerned with the transmission of monetary policy. The New Keynesian literature on monetary policy design emphasizes the role of expectations of future interest-rate movements rather than movements in current interest rates for aggregate demand management. Given a commitment to a systematic approach to policy, changes in current interest-rate policy herald adjustments in future policy. These changes are linked through the expectations hypothesis of the term structure. The following sections analyze the properties of the model under various monetary policy arrangements. The question to be addressed here is whether imperfect knowledge and the pricing of the public debt have consequences for the efficacy of monetary policy? Do unanchored financial expectations require new thinking about monetary policy design? And, specifically: how does this advice depend upon the composition of the public debt? Commencing with sub-optimal policies we show how these different assumptions regarding asset pricing have important consequences for stabilization policy. Optimal policy is then considered under learning. It is shown that even in this policy framework monetary policy is not as effective as under rational expectations.

3.1 Simple Taylor Rules

Consider the simple Taylor rule given by (14). We are interested in understanding whether such rules can lead to expectational stability — can they prevent unstable dynamics under learning?[8] Following Evans and Honkapohja (2008), stability results are provided for different constant gains. The special case of a zero gain corresponds to E-Stability — see Evans and Honkapohja (2001). Figure 1 plots stability regions in the case of a simple Taylor rule given by (14) in policy-maturity space for unanchored financial expectations. Results for the anchored financial market expectations models can be inferred as a special case of the unanchored expectations model when $\rho = 0$.[9] The gain is assumed equal to 0.02. The horizontal axis

[8] Given beliefs, the model has a standard state-space representation. Stability requires all model eigenvalues to lie inside the unit circle. If this requirement is met the model is referred to as having "stable" or "bounded" dynamics.

[9] When the average maturity of debt is one period, so that $\rho = 0$, the model is isomorphic to the model under anchored financial expectations. Here the multiple-maturity debt portfolio collapses to one-period bonds, which satisfy $\hat{P}_t^s = \hat{P}_t^m = -\hat{\imath}_t$. Even though agents only have a forecasting model in the bond price, this is

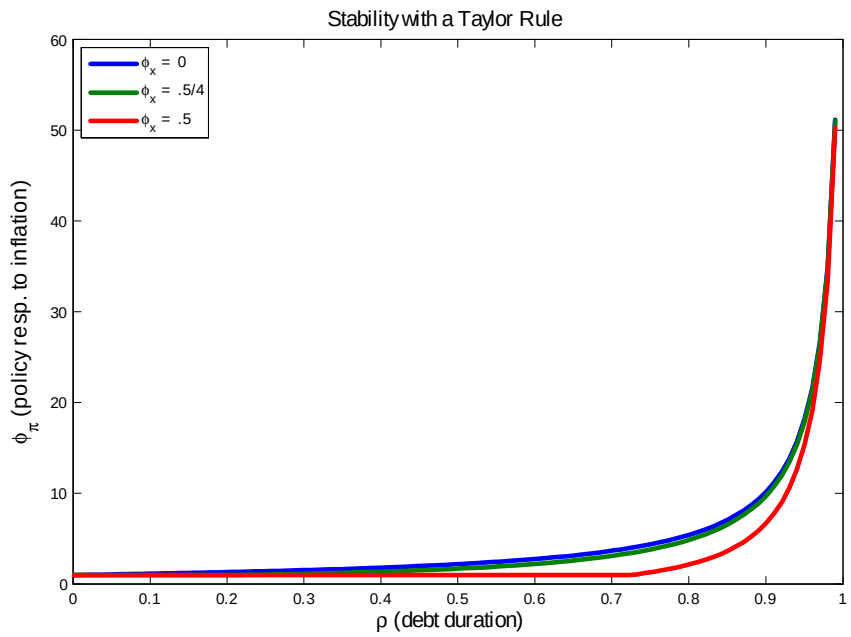

Figure 1: Robust stability regions for different maturity structures. The three contours correspond to different Taylor distinguished by their respond to the current output gap.

plots different average maturities of debt, indexed by ρ, while the vertical axis gives the policy coefficient ϕ_π. Points above each contour denote regions of stability — the model has eigenvalues inside the unit circle.

Three contours are plotted corresponding to different output responses in the Taylor rule. The greater is the average maturity of debt, the more aggressive must be the central bank's response to inflation for stability. In the limit of consol bonds — infinite-maturity debt — the required inflation response becomes substantial, with policy coefficient values just over 50. The degree of response to the output gap changes these observations little.

For the case of anchored financial expectations ($\rho = 0$), satisfaction of the Taylor principle ensures stability regardless of the composition of the public debt, as is the case for the model under rational expectations. In the anchored financial expectations model, changing interest rates directly impact beliefs about future interest rates, representing a restraining influence on aggregate demand.

What is the source of instability under unanchored financial expectations? In this model, changes in interest rates only affect beliefs to the extent that they affect current and ex-

equivalent to forecasting the period interest rate when there is only one-period debt. As the average maturity structure of debt increases this equivalence breaks down.

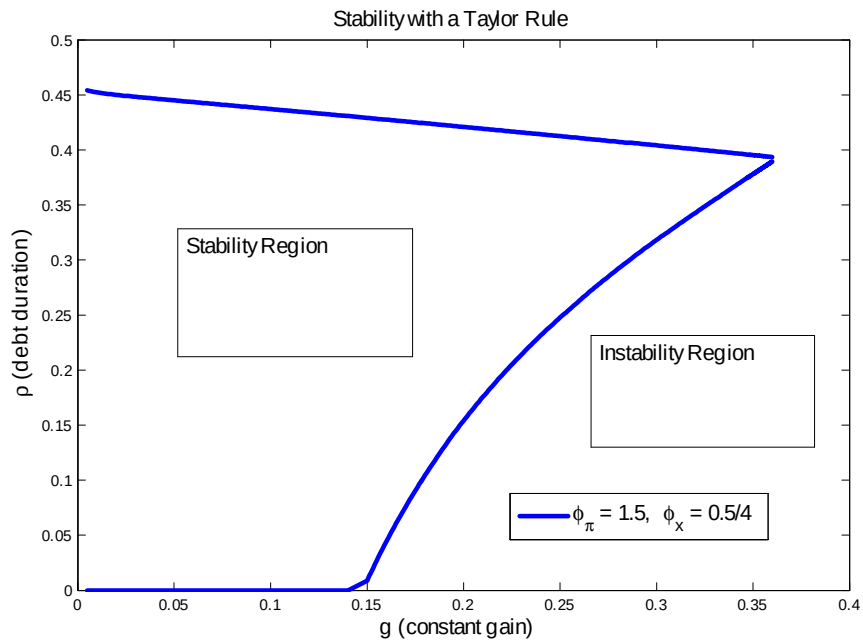

Figure 2: Stability regions in gain-maturity space for a Taylor rule.

pected bond prices in equilibrium. This substantially weakens the restraining influence of future interest-rate policy on aggregate demand. To see this more clearly, we can re-write the aggregate demand relation (8) using the arbitrage condition (5) as[10]

$$\begin{aligned} x_t &= -\hat{\imath}_t + \hat{E}_t \rho\beta \hat{P}^m_{t+1} + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[\beta (1-\rho) \hat{P}^m_{T+1} + \pi_{T+1} \right] - \hat{A}_t \\ &\quad + \bar{s}_C^{-1} (1-\beta) \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[\left(\frac{\bar{\theta}-1}{\bar{\theta}} \right) \left(1+\gamma^{-1}\right) \hat{w}_{T+1} + \bar{\theta}^{-1} \hat{\Gamma}_{T+1} \right]. \end{aligned} \tag{17}$$

The long-term bond price can become unanchored from the expected evolution of short-rates *consistent* with the monetary policy rule. Given that the price of the bond is not expected to reflect the expected discounted sum of future policy rates, the restraining influence of anticipated future interest rates is diminished. The central bank thus needs to move the current policy rate more aggressively in response to changes in the output gap and inflation to stabilize aggregate demand – hence the higher values of ϕ_π required to guarantee stability in Figure 1.

Figure 2 plots stability regions for the Taylor rule in gain-maturity space. Comparison to Figure 1 reveals a different impression on the stability properties of simple Taylor rules.

[10] Note that is steady state $\beta = (1+\bar{\imath})^{-1}$.

For average maturities satisfying $\rho \gtrsim 0.45$ the Taylor rule is never stable. This is consistent with the findings of Figure 1 — for stability, policy must be more aggressive as the maturity structure increases from this point. At short maturities, the model can be stable, but the degree to which it is, is non-monotonic.

The source of non-monotonicity comes from the interplay of two basic mechanisms, one stabilizing, one destabilizing. The first mechanism can be understood as follows. When $\rho = 0$ the average maturity of debt is unity and the expectations hypothesis of the term structure holds. Changes in current interest rates lead to changes in long-term interest rates — equivalently, long-term bond prices — through the revision of interest-rate expectations. These revisions are larger, the larger is the constant gain coefficient. Hence, for fixed Taylor rule coefficient ϕ_π, higher gains translate into larger movements in long-term interest rates with concomitantly larger impacts on aggregate demand. All else equal higher gains are destabilizing. However, as the average maturity structure of debt rises, the arbitrage relationships that define the term structure weaken — movements in current interest rates are less strongly related to movements in long-term interest rates. In consequence, movements in long-bond prices become divorced from current interest-rate changes. Alternatively stated, shifts in interest-rate expectations are less important for aggregate demand. This permits higher gains as the maturity structure rises, but only so far.

The second mechanism is simple: higher gains imply larger shifts in expectations about all prices when revised in the light of new data. For sufficiently large gains, monetary policy, characterized by fixed policy coefficients (ϕ_π, ϕ_x), is not aggressive enough to off-set their consequences on inflation and output. Self-fulfilling expectations become possible in much the same way that indeterminacy of rational expectations arises in this model when the Taylor principle is not satisfied. For average debt maturities with $\rho > 0.45$, this latter effect tends to dominate, so much so, the model is not stable for any gain.

Finally note that the parameter values $\rho \in [0, 0.45]$ span average maturities from 0 to 1.8 quarters, which are considerably shorter than typical debt portfolios in advanced economies. This suggests that the Taylor rule is particularly prone to instability from unanchored financial expectations. The Taylor rule appears to provide an unpromising approach to implement monetary policy, to the extent that expectations can be inconsistent with the expectations hypothesis of the yield curve.

Are there other prescriptions from rational expectations analyses that yield better outcomes? Evans and Honkapohja (2003, 2006), Woodford (2007), Preston (2008) and Eusepi

and Preston (2010, 2011) argue that adjusting policy instruments so as to satisfy particular target criteria exhibit improved stabilization properties in economies where agents have imperfect knowledge. To this end, we examine a simple example first proposed by Evans and Honkapohja (2003) in a model with one-period-ahead expectations and decreasing gain learning.

3.2 Optimal Rational Expectations Target Criteria

To gain further understanding of the role of financial market expectations, it is instructive to study target criteria that emerge from optimal policy problems under rational expectations. Consider a policy maker seeking to minimize the loss function, which corresponds to the second-order approximation to household utility,

$$\bar{E}_t^{RE} \sum_{T=t}^{\infty} \beta^{T-t} \left(\pi_T^2 + \lambda_x x_T^2\right)$$

where $\lambda_x = \psi(\gamma+1)/\bar{\theta} \geq 0$ indexes the relative priority given to output stabilization versus inflation stabilization and $\bar{E}_t^{RE}$ denotes rational expectations. The central bank's state-contingent choices over inflation and the output gap must satisfy the constraint (9). Under rational expectations the Phillips curve collapses to

$$\pi_t = \kappa x_t + \beta \bar{E}_t^{RE} \pi_{t+1} + \kappa \hat{u}_t$$

where $\kappa = \psi(\gamma+1)$. Minimization of the loss gives the familiar consolidated first-order condition under optimal discretion

$$\pi_t = -\bar{\theta}^{-1} x_t \tag{18}$$

requiring that inflation be proportional to the output gap, with constant of proportionality determined by the weight given to output gap stabilization and the slope of the Phillips curve, $\lambda_x/\kappa = \bar{\theta}^{-1}$.[11]

Following Evans and Honkapohja (2003) and Preston (2008), an implicit instrument rule can be derived as follows. The target criterion and Phillips curve (9) together provide

$$x_t = -\hat{u}_t - \frac{1}{\bar{\theta}^{-1} + \kappa} \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\psi\alpha\beta\left(\hat{w}_{T+1} - \hat{A}_{T+1}\right) + (1-\alpha)\beta\pi_{T+1}\right].$$

This determines the level of the output gap that jointly satisfies the aggregate supply relation and target criterion conditional on arbitrary beliefs about future inflation, wages, cost-push

[11] Attention is restricted to discretion to limit the state variables relevant to beliefs in equilibrium. This facilitates comparison across policies as all associated rational expectations equilibria are nested in the assumed belief structure (16).

shocks and technology. Denote this value of the output gap as x_t^*. Substitution into the aggregate demand curve (17) and solving for the current-period interest rate gives

$$\begin{aligned}\hat{\imath}_t &= -x_t^* + \rho \hat{E}_t \hat{P}_{t+1}^m - \hat{A}_t \\ &\quad + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[\beta (1-\rho) \hat{P}_{T+1}^m + \pi_{T+1} \right] \\ &\quad + \bar{s}_C^{-1} (1-\beta) \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[\left(\frac{\bar{\theta}-1}{\bar{\theta}} \right) \left(1+\gamma^{-1}\right) \hat{w}_{T+1} + \bar{\theta}^{-1} \hat{\Gamma}_{T+1} \right]. \end{aligned} \tag{19}$$

As before, assuming $\rho = 0$ delivers the model under anchored financial expectations. Relation (19) is an expectations-based instrument rule implicitly defined by the target criterion (18). It has the property that interest rates are adjusted in response to expectations about inflation, dividends, wages and long-bond prices. This instrument rule guarantees satisfaction of the target criterion (18) regardless of how expectations are formed about future prices. This characteristic is argued by Preston (2008) and Woodford (2007) to be an important strength of the target criterion approach to implementing optimal monetary policy. Such policies might have certain advantages over simple Taylor-type rules: monetary policy responds not only to current conditions but also to shifting expectations about inflation, wages, profits and the price of long-term debt.

Figure 3 gives stability regions in (g, ρ) space for the target criterion (18). In contrast with the Taylor rule, instability occurs for gain-maturity pairs that lie *below* the plotted contour. For one-period debt the model is stable for gains less than 0.02. As the average maturity rises the stability region expands. While not shown, as $\rho \to 1$ giving consol bonds, the model is stable for all gains on the unit interval. Because the model is always stable for small positive gains, it is also expectationally stable in the sense of Evans and Honkapohja (2001) for all average maturities of public debt. That is, as the gain goes to zero, the model is E-Stable for all parameter values. Finally, for anchored financial expectations the stability region is independent of the maturity structure of debt. For maintained parameter assumptions, stability obtains for all gains satisfying $g < 0.021$. Hence, for small gains the model is stable independently of the assumptions about asset pricing.

The intuition for the instability at low values of ρ is similar to that in our discussion of the Taylor rule. Potential instability in long-term bond prices constrains the degree to which current monetary policy can respond to evolving economic conditions. This contrasts markedly with a rational expectations analysis of such policies, where the target criterion guarantees determinacy of equilibrium in output and inflation dynamics. In such a case, interest-rate

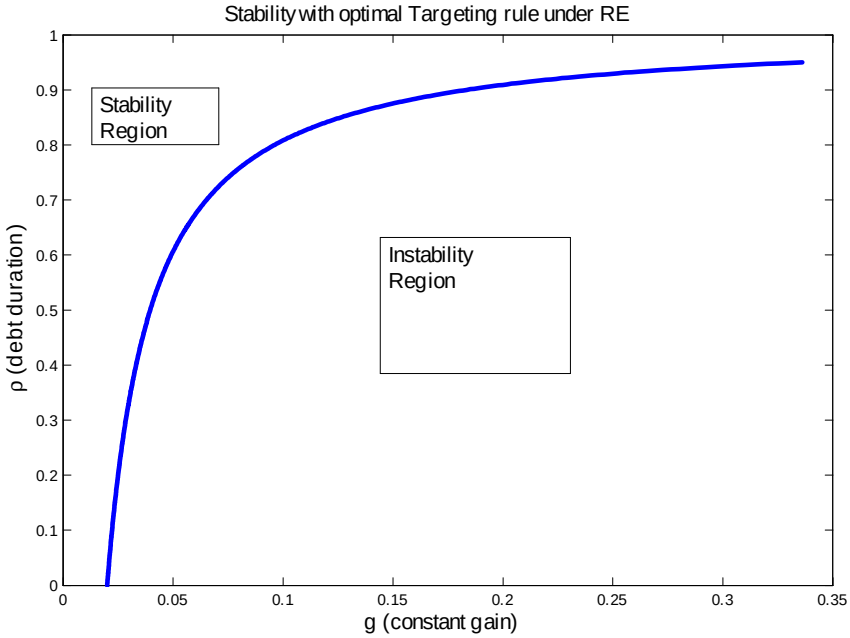

Figure 3: Stability regions in gain-maturity space for the optimal rational expectations target criterion under discretion.

dynamics are inferred from the aggregate Euler equation, which necessarily delivers a unique bounded rational expectations equilibrium path, as it does not involve expectations of variables other than inflation and output. This is not true under arbitrary assumptions about beliefs: stability of output and inflation dynamics do not ensure stability of interest-rate dynamics.

As ρ increases from zero to unity, current interest-rate movements become increasingly divorced from bond-price expectations and therefore long-term interest rates. The arbitrage conditions defining the expectations hypothesis of the yield curve tend to break down. This in turn engenders weaker feedback from the evolution of expected future bond prices to aggregate demand. Hence, in contrast with the Taylor rule, large values of ρ promote stability. This permits greater latitude to adjust current interest-rate policy without inducing destabilizing movements in longer-term interest rates. In contrast to the results in Figure 2, the second destabilizing mechanism associated with rising average maturities of debt does not operate under a targeting rule. This approach to policy has the property that it implicitly defines an interest-rate rule that, by responding directly to the expected path of inflation and income, is always sufficiently aggressive to ensure satisfaction of the target criterion, regardless of agents' expectations about future prices and long-term bond prices in particular.

Comparison of Figures 2 and 3 reveals that the target criterion and Taylor rule confer stabilization advantages at different maturities of debt. The stable region for the Taylor rule is located in very short-maturity-debt structures, while the target criterion performs better at long-debt maturities and is consistent with delivering stability at all maturities for small enough values of the gain coefficient. Despite these improvements associated with implicit instrument rules that respond to asset price expectations, it remains the case that model dynamics are bounded only for fairly small gain coefficients. Gains on the interval $[0.05, 0.15]$, commonly used in the learning literature, imply that instability occurs for many average-debt maturities. Unlike a rational expectations equilibrium analysis of the target criterion, where determinacy is guaranteed (see Giannoni and Woodford, 2010), expectations stability is not assured under alternative belief assumptions.

Furthermore, even for higher values of ρ, which imply stability, monetary policy might not be able to control expectations. If policy stabilization requires an aggressive response of the policy instrument to changing economic conditions, then it is plausible that the short-term interest rate will be at the zero lower bound with sufficiently high frequency to hinder the efficacy of targeting rules.

These concerns beg the question of whether stabilization policy can be improved further when unanchored financial market expectations impair aggregate demand management. The remainder of the paper is devoted to the design of optimal monetary policies under learning.

4 Optimal Monetary Policy

This section studies a central bank that minimizes a welfare-theoretic loss function given correct knowledge of the true economic model. Included in the central bank's information set is the specification of household and firm forecasting functions. Following Woodford (2003), the period loss function is assumed to be of the form

$$L_t = \pi_t^2 + \lambda_x x_t^2 + \lambda_i i_t^2 \tag{20}$$

where $\lambda_x, \lambda_i \geq 0$ determine the relative priority given to output, interest rate and inflation stabilization. This period loss is implied by a second-order approximation to household utility and it includes an explicit concern for the constraint imposed by the zero lower bound on nominal interest rates — see the discussion in Woodford (2003) and Rotemberg and Woodford (1998).

The central bank's choice over sequences of inflation, output and nominal interest rates

is constrained by the aggregate demand and supply relations (17) and (9), the no-arbitrage condition (5) and beliefs about the evolution of inflation, dividends, wages, and bond prices. Using the belief dynamics in the aggregate demand and supply schedules permits writing inflation and output as a function of the current state. There is no distinction between commitment and discretion under learning dynamics. The central bank can only influence expectations through current and past actions — not through announced commitments to some future course of action.

A more subtle issue warrants remark. The inclusion of the aggregate demand as a constraint on feasible state-contingent choices over inflation and output is required even in the case that there is no loss from interest-rate variation in (20). This requirement is apparent from earlier discussion on the merits of rational expectations policy advice in a world with learning — recall section 3.3. Bounded dynamics for output and inflation need not imply bounded state-contingent paths for interest rates and interest-rate expectations. Whether dynamics in interest rates are stable depends critically on the size of the gain coefficient. To ensure bounded variation in interest rates the aggregate demand relation is always a constraint on central bank optimization. Failure to acknowledge this constraint implies unbounded variation in interest rates for some choice of gain, a property of policy that is clearly both undesirable and infeasible.

Subject to aggregate demand and supply, the arbitrage condition and the evolution of beliefs, the central bank solves the problem

$$\min_{\left\{x_t,\pi_t,i_t P_t^m,a_t^{\pi},a_t^{P^m},a_t^{w},a_t^{\Gamma}\right\}} (1-\beta)\,\bar{E}_t^{RE}\sum_{T=t}^{\infty}\beta^t L_T \tag{21}$$

where we assume that the central bank correctly understands the true model of the economy and constructs rational expectation forecasts. The first-order conditions are described in the appendix and discussed in detail in Eusepi, Giannoni, and Preston (2011) for a variety of related problems. As first pointed out by Molnar and Santoro (2005), an interesting feature of this decision problem is that the first-order conditions constitute a linear rational expectations model.[12] The system can be solved using standard methods. Using results from Giannoni and Woodford (2010), the following proposition can be stated.

Proposition 1 *The model comprised of (i) the aggregate demand, supply and arbitrage equations (17), (9) and (5); (ii) the law of motion for the beliefs* $a_t^{\pi}, a_t^{P^m}, a_t^{w}, a_t^{\Gamma}$, *and (iii) the*

[12] In an innovative study, Molnar and Santoro (2005) explore optimal policy under learning in a model where only one-period-ahead expectations matter to the pricing decisions of firms. Gaspar, Smets, and Vestin (2006) provide a global solution to the same optimal policy problem but under a more general class of beliefs.

first-order conditions resulting from the minimization of (20)–(21) subject the restrictions listed in (i) and (ii) admits a unique bounded rational expectations solution for all parameter values. In particular, model dynamics under optimal monetary policy are unique and bounded for all possible gains.

Proof. See Appendix. ■

This model nests both anchored and unanchored financial expectations as a function of the parameter ρ. Equilibrium dynamics under optimal policy are stable for all gain values, in contrast to the dynamics induced by policy rules that emerge from rational expectations analyses. Optimal monetary policy has the property that the evolution of beliefs is managed in exactly the right way to ensure a bounded rational expectations equilibrium consistent with minimization of the loss (21). In this sense the economy is stable: it has unique bounded state-contingent evolution for all endogenous variables given bounded stochastic disturbance processes. But this does not necessarily imply that departures from the expectations hypothesis of the yield curve are not problematic for the transmission of monetary policy. The result only implies that regardless of the nature of financial market expectations, an optimal policy can be characterized which has the property of being stable for all admissible gains.

What remains to be determined are the dynamic properties implied by optimal policies under anchored and unanchored financial market expectations. Three exercises are conducted. First, we compute impulse response functions in response to technology and cost-push shocks to elucidate the dynamic interrelations between interest rates and the objectives of stabilization policy. Second, efficient policy frontiers are computed to study the trade-offs inherent in models of learning dynamics vis-a-vis rational expectations. Specifically, we seek to understand how interest-rate volatility depends on specific shocks and also the maturity structure of the public debt. Third, for each model we compute the unconditional probability of being at the zero lower bound on nominal interest rates.

To presage subsequent results, aggregate demand management is more difficult regardless of how asset prices are determined — though the underlying mechanisms in each case are fundamentally distinct. In interpreting these findings, note that they constitute a best-case scenario. Should the central bank possess less accurate information about agents' decisions and beliefs, monetary policy can only become more difficult.

5 Impulse Response Functions

This section develops an understanding of the underlying dynamics induced by optimal monetary policy by plotting model impulse response functions to each disturbance. The cost-push and technology shocks are assumed i.i.d. with no serial correlation. The plots give dynamics from a unit increase in each disturbance for the three models under consideration: optimal policy under rational expectations; learning with anchored financial expectations ($\rho = 0$); and learning with unanchored financial expectations ($\rho = 0.96$). In the case of rational expectations the optimal policy is given by the target criterion

$$\pi_t = -\bar{\theta}^{-1}\left(x_t - x_{t-1}\right)$$

requiring inflation to be proportional to the change in the output gap. The presence of the lagged output gap reflects the history dependence of optimal commitment policy. Policy under commitment is considered here, rather than policy under discretion, to compare the inertial character of optimal policy under alternative belief assumptions. The four panels give inflation, the output gap, the short-term interest rate and the interest-rate spread — the difference between the long interest rate and the short interest rate. There is no weight on interest-rate stabilization to permit comparison to well-known results under rational expectations.

Figure 4 gives model dynamics in response to a cost-push shock with a gain equal to 0.15.[13] Considering inflation and the output gap the impact effects for the rational expectations model and the learning model with unanchored financial expectations are broadly similar. Subsequent dynamics differ for these models, with rational expectations predicting a persistent negative output gap which reduces inflation from positive to negative values, before converging back to steady state. Anticipated negative output gaps restrain current inflation. For the unanchored financial expectations model, the initial negative output gap is followed by a boom, before slow convergence to steady state. For this reason inflation falls to essentially its steady-state value in the period after the shock, with little variation thereafter.

Relative to these two models, the anchored financial expectations model has different impact effects. Inflation rises on impact by a magnitude twice that observed in the rational expectations model. Consistent with this, the output gap falls by substantially less than other models. Inflation converges to its steady state roughly in the period after the shock while output remains below the steady state for a few more periods as the nominal interest rate

[13] This larger gain is assumed for aesthetics and clarity — smaller gains tend to obscure the same basic patterns due to the presence of a negative eigenvalue in the interest-rate dynamics.

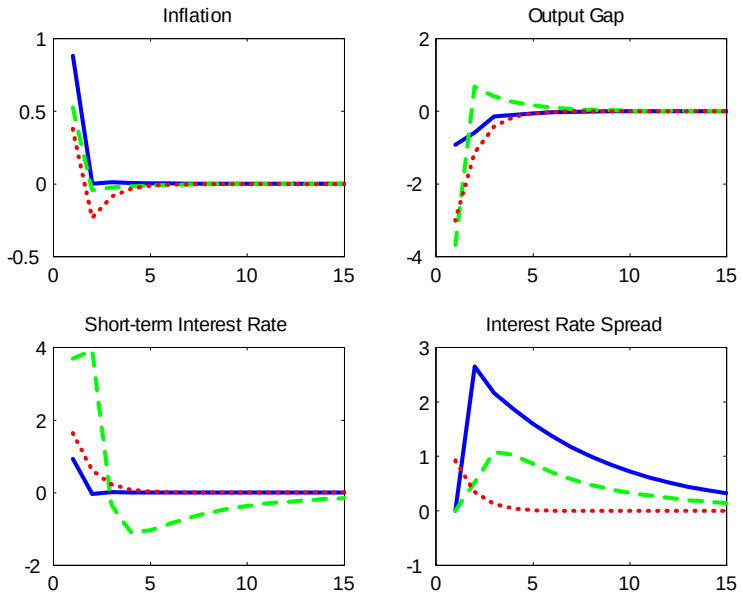

Figure 4: Impulse response functions in response to a cost-push shock. Gain = 0.15. Rational expectations: red dotted line; learning with anchored expectations: blue solid line; and learning with unanchored expectations: green dashed line.

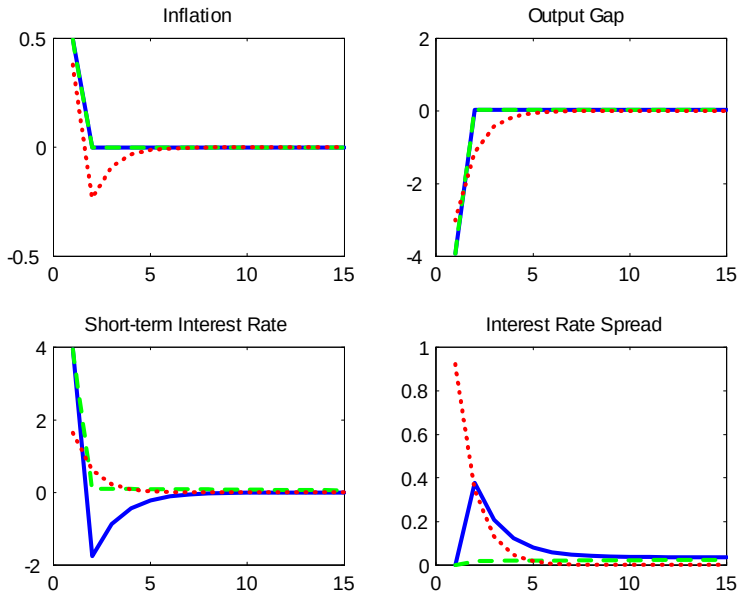

Figure 5: Impulse response functions in response to a cost-push shock. Gain = 0.005. Rational expectations: red dotted line; learning with anchored expectations: blue solid line; and learning with unanchored expectations: green dashed line.

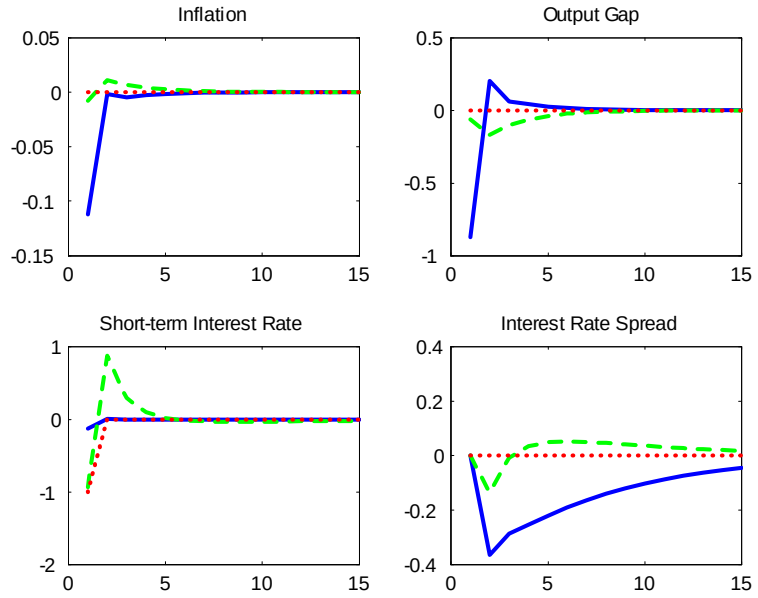

Figure 6: Impulse response functions in response to a technology shock. Gain = 0.15. Rational expectations: red dotted line; learning with anchored expectations: blue solid line; and learning with unanchored expectations: green dashed line.

remains slightly above its steady state.

The cause of these differing dynamics across learning models is seen clearly in the paths for the short-term interest rate. In the anchored financial expectations model, interest rates rise much less than in the unanchored financial expectations model, leading to both higher inflation and output gaps. The intuition established in the study of simple rules applies here. Through revisions to interest-rate expectations, aggressive movements in current interest rates can generate macroeconomic instability. This limits scope to adjust current interest rates in response to evolving macroeconomic conditions. In the unanchored financial expectations model, interest rates move aggressively to restrain inflation leading to a significant contraction in real activity. Current interest-rate policy is divorced from interest-rate expectations. Because optimal policy cannot rely to the same degree on the restraining influence of high anticipated interest rates that occurs under rational expectations, short-term interest rates increase further in the period after impact. In subsequent periods interest rates decline sharply.

These patterns are reflected in interest-rate spreads. Under anchored financial expectations, long-term bond prices rise slowly as expectations about future short-term interest rates

rise with current interest rates. This ultimately restrains inflation and aggregate demand. Note that the slow adjustment of interest-rate expectations limits the degree to which short-term interest rates rise at the time of the shock — else long rates eventually rise too much, overly restricting demand. This constrains the central bank's ability to restrain initial inflation. Beliefs represent an additional constraint on monetary policy. Once inflation pressures abate, the spread slowly declines to steady state. In the case of unanchored financial expectations, policy relies on aggressive adjustment of short rates. Despite the aggressive two-period rise in short-term interest rates, the spread only adjusts slowly. This is because bond-price expectations are not influenced directly by interest-rates: they only adjust because of past changes in their own price — i.e. general equilibrium considerations. This makes clear that the restraining influence of future short-term interest-rate expectations renders policy less potent. This is the source of instability in short rates.

Figure 5 gives the impulse response to a cost-push shock but for a gain equal to 0.005. Here the inflation and output gap dynamics are identical across learning models. In fact, it can be shown that these paths are identical to model dynamics under optimal discretion with rational expectations. As the gain becomes small, learning models replicate outcomes from optimal discretion. This was first demonstrated by Molnar and Santoro (2005) in the case of models in which only one-period-ahead expectations matter. Eusepi, Giannoni, and Preston (2011) extend these results in various dimensions and provide a proof of this limiting result. Note, however, that the interest-rate paths supporting these discretion-induced dynamics are quite different. The case of anchored financial expectations most closely resembles rational expectations, while unanchored financial expectations require a period of negative interest rates with slow convergence to steady state after the period of the shock.

Figure 6 plots model dynamics in response to a technology shock. Under rational expectations, the optimal commitment policy completely accommodates the technology shock. There are no consequences for the output gap or inflation. Because short-term and long-term interest rates move in tandem for i.i.d. shocks, there are no interest-rate spread dynamics. The learning models give strikingly different stories. With unanchored financial expectations monetary policy largely neutralizes the impact effect of the technology shock, with large swings in interest rates in subsequent periods to manage evolving beliefs. In contrast, the short-term interest rate adjusts little with anchored financial expectations leading to a substantial contraction in inflation and real economic activity. However as long-rate expectations fall, the output gap becomes positive which restores inflation close to steady state values.

6 Policy Frontiers

To examine the consequences of imperfect monetary control, we explore the trade-off between the stabilization of inflation and output gap on the one hand, and stabilization of the short-term interest rate on the other hand. For each of the variables of interest, we compute the unconditional variance $V[z]$ of the respective variable $z = \{\pi, x, i\}$. Because inflation, the output gap and the short-term interest rate have mean values equal to zero under the optimal policy being considered, the discounted value of the losses (20)–(21) is equivalent to

$$\bar{L} = V[\pi] + \lambda_x V[x] + \lambda_i V[i],$$

when the operator $\bar{E}^{RE}$ denotes the rational expectation taken over the unconditional distribution of exogenous disturbance processes $\hat{u}_t$ and $\hat{A}_t$.

The analysis considers policies minimizing the deadweight loss associated with variation in inflation and the output gap $V[\pi] + \lambda_x V[x]$ subject to the constraint that the variability in short-term interest rates not exceed some finite value. Variation in this finite value traces out the efficient frontier describing the trade-off between inflation/output gap stabilization and interest-rate stabilization. In practice this is achieved by minimizing the expected loss $\bar{L}$ over different values of λ_i.

The gain is assumed to be 0.05. The standard deviations of the technology and cost-push shocks are chosen to be equal to one — there is no attempt here to build a serious quantitative model and it is only the relative volatilities, which are independent of the scale of disturbances, that matter.[14] The intention is to elucidate the central trade-offs confronting policy makers under learning dynamics when subject to various kinds of disturbances.

Cost-push Disturbances. Figure 7 plots the efficient policy frontier for various economies. The thick black line denotes the familiar efficiency policy frontier under rational expectations with the earlier described optimal commitment policy. As the tolerance for interest-rate variability rises, optimal policy focuses more on inflation and output gap stabilization so that $V[\pi] + \lambda_x V[x]$ falls along the frontier as $V[i]$ increases. When interest-rate variation reaches its maximum, which is equivalent to $\lambda_i = 0$ in the loss function (20), inflation and output variation reach their minimum value. In the presence of cost-push shocks it is not possible to simultaneously stabilize inflation and the output gap, leading to positive deadweight losses — see Clarida, Gali, and Gertler (1999) and Woodford (2003) for further discussion. Instead,

[14] The variances of inflation, output and nominal interest rates are themselves linear functions of the shock variances. The ratios are therefore independent of the assumed standard deviations.

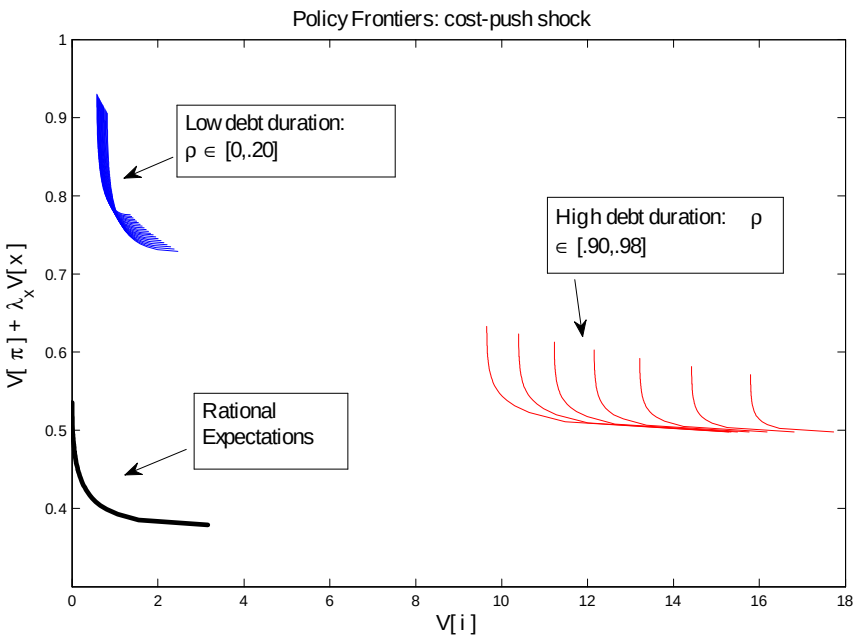

Figure 7: Policy frontiers as weight on interest rate stability is increased. Exogenous disturbance is a cost-push shock.

when the tolerated variation in the short-term interest rates approaches zero the losses are about 50 percent higher.

The blue lines in Figure 7 represent policy frontiers with optimal policy under learning and unanchored financial expectations for various low values of debt duration $\rho \in [0, 0.2]$.[15] As ρ increases the frontiers progressively shift down and to the right in an overlapping manner. (Recall that $\rho = 0$ is isomorphic to the case of anchored financial expectations. This economy is given by the upper left-most frontier.) Again, the policy frontiers are downward sloping reflecting the fact that a higher tolerated variability of the short-term interest rate is consistent with increased stabilization of inflation and the output gap. The deadweight losses associated with output and inflation are substantially greater than optimal policy under rational expectations — even if no weight is given to stabilization of the short-term interest rate.

As learned from the stability properties of targeting rules derived under rational expectations, the central bank cannot allow for too volatile interest rates even with $\lambda_i = 0$. In fact, the volatility of interest rates under learning and anchored financial expectations is about half that observed with optimal policy under rational expectations, when $\rho = 0$. This is because

[15] The relevant equilibrium conditions are described in the appendix.

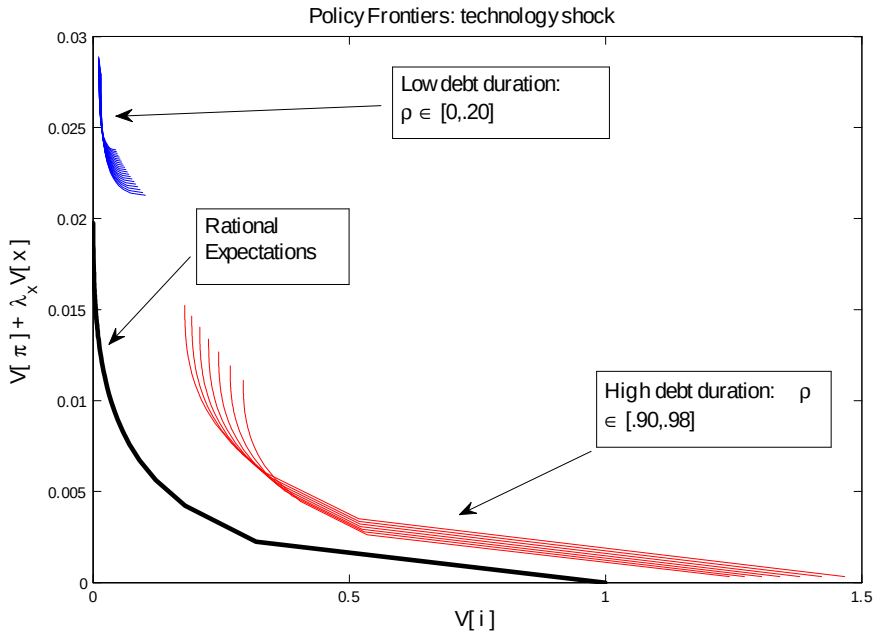

Figure 8: Taylor frontiers as weight on interest rate stability is increased. Exogenous disturbance is a technology shock.

a volatile interest rate would lead to unstable learning dynamics in interest-rate beliefs. As a result, cost-push shocks are allowed to increase the volatility of inflation and inflation expectations, which in turn increases the volatility in the output gap. Learning dynamics represent a non-trivial constraint on what can be achieved by the central bank. Having to manage the distortions induced from beliefs compromises the stabilization of inflation and the output gap.

The red lines show the optimal policy frontiers under learning and unanchored financial expectations for the model with longer-term bonds, with maturities indexed by $\rho \in [0.9, 0.96]$. These lines present a striking result: Under unanchored financial expectations output and inflation losses are substantially smaller than under anchored financial expectations, or unanchored expectations with short-duration debt. The optimal monetary policy under learning can almost deliver a variability of inflation and output gap comparable to the optimal commitment policy under rational expectations. The intuition is the same as presented for the case of the targeting criterion under rational expectations. With high values of ρ, the expected future path of bond prices — equivalently long-term interest rates — do not have strong effects on aggregate demand. In other words, the dynamics of expectations about bond prices do not feed back to aggregate demand, preventing unstable outcomes. At the same time,

the central bank has to move the short-term interest rate aggressively to control aggregate demand. Stabilizing output and inflation comes at the cost of substantial variability in the short-term interest rate. Indeed, for these longer-duration-debt economies, the variance of short-term interest rates varies from around 10 to just under 18. Such large numbers suggest that implementation of the optimal policy would in fact be infeasible when unanchored financial expectations impair aggregate demand management — a point to which discussion will return. The result is best interpreted as an example of the difficulties that arise for monetary policy design when aggregate demand management is impaired because of unanchored financial market expectations. If it is the case that expectations fail to be consistent with the expectations hypothesis of the yield curve, inflation control is reduced in so far as it requires much more aggressive adjustments in interest-rate policy.

Technology Shocks. Figure 8 provides analogous policy frontiers in the face of technology shocks under rational expectations; learning with anchored financial expectations; and learning with unanchored financial expectations. In the case of rational expectations (thick black line) if the variance of short-term interest rates is unconstrained, then inflation and the output gap can be completely stabilized. Technology shocks are the only source of variation in the natural rate of interest in this economy. Optimal policy calls for price stability, with the nominal interest rate tracking the natural rate of interest. As the penalty on interest-rate volatility rises, the ability to completely stabilize prices declines, with higher associated volatility of inflation and the output gap.

The top blue line shows policy frontiers associated with the model under learning with unanchored financial expectations and short duration debt $\rho \in [0, 0.2]$. Again, learning dynamics render stabilization policy more difficult, giving higher losses at any level of interest-rate variation, when compared with the efficient frontier under rational expectations. A central difference between rational expectations and learning is that complete stabilization of inflation and the output gap is no longer feasible under learning. The logic is precisely the same as under cost-push shocks. Learning presents an additional distortion, placing constraints on monetary policy. Attempts to reign in inflation expectations by moving the nominal interest rate aggressively induces a destabilizing feedback and is infeasible. This dynamic limits the ability of interest-rate policy to stabilize inflation and the output gap.

The red lines show the policy frontiers for the model with unanchored financial expectations and a higher duration of bonds indexed by $\rho \in [0.9, 0.96]$. As in the case of anchored financial expectations, full stabilization of output and inflation is not feasible. As in the case of cost-

push shocks, the central bank achieves better inflation and output gap stabilization at the cost of higher interest-rate volatility when compared to anchored expectations. Of note, the volatility in the interest rate under unanchored financial expectations is not as dramatic as in the case of cost push shock (it is mildly higher than rational expectations). However, interest-rate volatility does not fall to zero with higher λ_i: stabilization policy consistent with a zero lower bound might not be feasible, as discussed in the last section.

7 The Zero Lower Bound on Nominal Interest Rates

The previous sections make clear that under unanchored financial expectations policy stabilization requires volatile short-term nominal interest rates. In this section we discuss whether such policies are in fact feasible, given that we are ignoring a crucial constraint to policy stabilization: the zero lower bound on nominal interest rates.

Here we compute the unconditional probability of being at the zero lower bound in each model. This calculation requires two additional parameter assumptions. The average level of the short-term nominal interest rate (which reflects both the average real rate and average inflation) and the volatility of disturbances. We assume the annualized steady-state nominal interest rate equals 5.4%, which corresponds to the average rate of the US 3-month Treasury-bill for the period 1954Q3-2011Q3. We permit only technology shocks, but similar results are obtained with cost-push shocks.[16] Given the simplicity of the model, it is difficult to choose a realistic calibration for the standard deviation of the technology shock. We use a calibration that better serves in illustrating the differences between the models under consideration. Using the rational expectations model with optimal policy and, following Woodford (2003, chap. 7), an interest-rate stabilization parameter $\lambda_i = 0.08$, we calibrate the volatility of the technology shocks to deliver a standard deviation of output of 1.5% (in log-deviations from its steady state). This roughly corresponds to the standard deviation of HP-detrended US real GDP in our sample. The choice of λ_i implies that the unconditional probability of being at the zero lower bound on nominal interest rates under the optimal policy is about 3.5%, which roughly corresponds to the historical frequency of being at the zero lower bound for the US reported in Coibon, Gorodnichenko, and Wieland (2010).

Figure 9 plots the unconditional probability of being at the zero lower bound as a function of λ_i for the three models: rational expectations, anchored financial expectations ($\rho = 0$) and

[16] In fact, under cost-push shock the interest rate even more volatile, especially in the model with unanchored financial expectations.

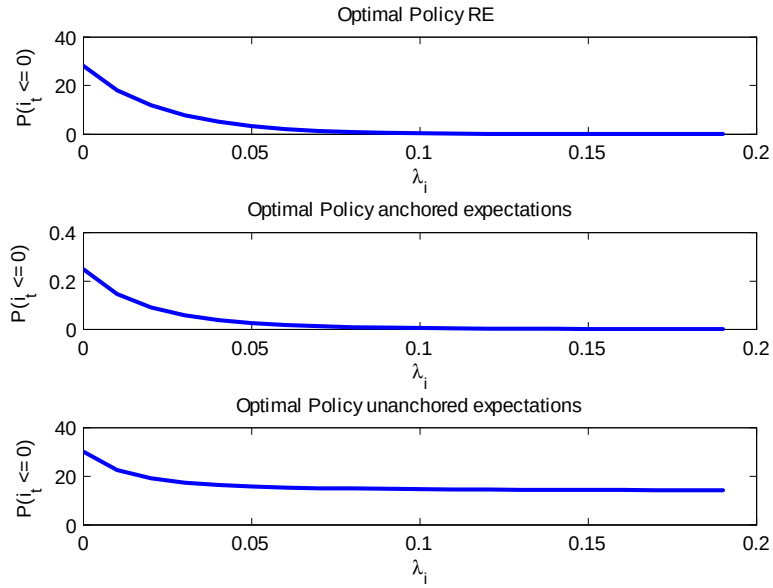

Figure 9: The figure shows the unconditional probability of being at the ZLB as a function of λ_i for the three different models.

unanchored financial expectations ($\rho = 0.96$). Under rational expectations, an interest-rate stabilization motive in the loss function delivers an optimal monetary policy consistent with a low probability of being at the zero lower bound. As expected from the earlier discussion, optimal policy under anchored financial expectations induces very little volatility in the nominal interest rate, to prevent unstable learning dynamics. Concomitantly, the model implies a very low unconditional probability of being at the zero bound. Finally, under unanchored financial expectations the unconditional probability of being at the zero bound remains above 14% regardless of λ_i. This suggests that in situations where monetary policy is less effective in managing expectations, the historical frequency of being at the zero lower bound might not be a useful statistic.

8 Conclusion

This paper explores the effects of monetary policy under imperfect knowledge and incomplete markets. In this environment the expectations hypothesis of the yield curve need not hold, a situation called unanchored financial market expectations. Whether or not financial market expectations are anchored, private sector imperfect knowledge mitigates the efficacy of optimal

monetary policy. Under anchored expectations, slow adjustment of interest-rate beliefs limits scope to adjust current interest-rate policy in response to evolving macroeconomic conditions. Imperfect knowledge represents an additional distortion confronting policy, leading to greater inflation and output volatility relative to rational expectations. Under unanchored expectations, current interest-rate policy is divorced from interest-rate expectations. This permits aggressive adjustment in current interest-rate policy to stabilize inflation and output. However, unanchored expectations are shown to raise significantly the probability of encountering the zero lower bound constraint on nominal interest rates. This constraint is more severe the longer is the average maturity structure of the public debt.

A Appendix

A.1 Optimal Policy: Anchored Expectations

To derive the first-order conditions under optimal policy, we write the model in following compact notation.

1. Inflation

$$\hat{\pi}_t = \bar{c}_{\pi,x} x_t + \bar{c}_{\pi,aw} a^w_{t-1} + \bar{c}_{\pi,a\pi} a^\pi_{t-1} + \bar{c}_{\pi,x} \hat{u}_t \tag{22}$$

where

$$\begin{aligned} \bar{c}_{\pi,x} &= \psi(\gamma+1) \\ \bar{c}_{\pi,aw} &= \psi \frac{\alpha\beta}{1-\alpha\beta} \\ \bar{c}_{\pi,a\pi} &= \frac{(1-\alpha)\beta}{1-\alpha\beta} \end{aligned}$$

2. Aggregate demand

$$x_t = -i_t - \hat{A}_t + \bar{c}_{i,ai} a^i_{t-1} + \bar{c}_{i,a\pi} a^\pi_{t-1} + \bar{c}_{i,aw} a^w_{t-1} + \bar{c}_{i,a\Gamma} a^\Gamma_{t-1} \tag{23}$$

where

$$\begin{aligned} \bar{c}_{i,ai} &= -\frac{\beta}{1-\beta} \\ \bar{c}_{i,a\pi} &= \frac{1}{1-\beta} \\ \bar{c}_{i,aw} &= \bar{s}_C^{-1} \left(\frac{\bar{\theta}-1}{\bar{\theta}} \right) \left(1+\gamma^{-1}\right) \\ \bar{c}_{i,a\Gamma} &= \bar{s}_C^{-1} \bar{\theta}^{-1} \end{aligned}$$

3. Wage beliefs: noting that the wage and is related to the output gap according to

$$w_t = (\gamma + 1)\, x_t + \hat{A}_t$$

permits

$$a_t^w = \bar{c}_{aw,aw} a_{t-1}^w + \bar{c}_{aw,x} x_t + \bar{c}_{aw,A} \hat{A}_t \tag{24}$$

where

$$\begin{aligned} \bar{c}_{aw,aw} &= 1 - g \\ \bar{c}_{aw,x} &= g\,(\gamma + 1) \\ \bar{c}_{aw,A} &= g, \end{aligned}$$

4. Dividend beliefs: noting that the dividend can be written in terms of the output gap as

$$\hat{\Gamma}_t = \hat{Y}_t - \left(\bar{\theta} - 1\right)(\gamma + 1)\, x_t$$

permits

$$a_t^\Gamma = \bar{c}_{a\Gamma,a\Gamma} a_{t-1}^\Gamma + \bar{c}_{a\Gamma,x} x_t + \bar{c}_{a\Gamma,A} \hat{A}_t \tag{25}$$

where

$$\begin{aligned} \bar{c}_{a\Gamma,a\Gamma} &= 1 - g \\ \bar{c}_{a\Gamma,x} &= g\left(1 - \left(\bar{\theta} - 1\right)(\gamma + 1)\right) \\ \bar{c}_{a\Gamma,A} &= g\frac{1}{(\gamma + 1)}, \end{aligned}$$

5. Inflation beliefs:

$$a_t^\pi = (1 - g) a_{t-1}^\pi + g\hat{\pi}_t \tag{26}$$

6. Interest rate beliefs:

$$a_t^i = (1 - g) a_{t-1}^i + g i_t. \tag{27}$$

A.2 Optimal Policy: Anchored Expectations

The Central Bank chooses $\left\{\pi_t, x_t, i_t, a_t^\pi, a_t^w, a_t^\Gamma, a_t^i\right\}$ to minimize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \frac{1}{2}\left[\pi_t^2 + \lambda_x x_t^2 + \lambda_i i_t^2\right] + \lambda_{1,t}\left(-\hat{\pi}_t + \bar{c}_{\pi,x} x_t + \bar{c}_{\pi,aw} a_{t-1}^w + \bar{c}_{\pi,a\pi} a_{t-1}^\pi + \bar{c}_{\pi,x} \hat{u}_t\right) \\ +\lambda_{2,t}\left(-a_t^\pi + (1-g) a_{t-1}^\pi + g\hat{\pi}_t\right) + \lambda_{3,t}\left(-a_t^w + \bar{c}_{aw,aw} a_{t-1}^w + \bar{c}_{aw,x} x_t + \bar{c}_{aw,A} \hat{A}_t\right) \\ +\lambda_{4,t}\left(-x_t - i_t - \hat{A}_t + \bar{c}_{i,ai} a_{t-1}^i + \bar{c}_{i,a\pi} a_{t-1}^\pi + \bar{c}_{i,aw} a_{t-1}^w + \bar{c}_{i,a\Gamma} a_{t-1}^\Gamma\right) \\ +\lambda_{5,t}\left(-a_t^\Gamma + \bar{c}_{a\Gamma,a\Gamma} a_{t-1}^\Gamma + \bar{c}_{a\Gamma,x} x_t + \bar{c}_{a\Gamma,A} \hat{A}_t\right) + \lambda_{6,t}\left(-a_t^i + (1-g) a_{t-1}^i + g i_t\right) \end{array} \right\}.$$

The first order conditions are:

Inflation:

$$\pi_t - \lambda_{1,t} + g\lambda_{2,t} = 0$$

The output gap:

$$\lambda_x x_t + \bar{c}_{\pi,x}\lambda_{1,t} + \bar{c}_{aw,x}\lambda_{3,t} - \lambda_{4,t} + \bar{c}_{a\Gamma,x}\lambda_{5,t} = 0$$

Interest rate:

$$\lambda_i i_t - \lambda_{4,t} + g\lambda_{6,t} = 0$$

Inflation beliefs:

$$-\lambda_{2,t} + \beta\bar{c}_{\pi,a\pi}E_t\lambda_{1,t+1} + \beta(1-g)E_t\lambda_{2,t+1} + \beta\bar{c}_{i,a\pi}E_t\lambda_{4,t+1} = 0$$

Wage beliefs:

$$\beta\bar{c}_{\pi,aw}E_t\lambda_{1,t+1} - \lambda_{3,t} + \beta\bar{c}_{aw,aw}E_t\lambda_{3,t+1} + \beta\bar{c}_{i,aw}E_t\lambda_{4,t+1} = 0$$

Dividend beliefs:

$$\beta\bar{c}_{i,a\Gamma}E_t\lambda_{4,t+1} - \lambda_{5,t} + \beta\bar{c}_{a\Gamma,a\Gamma}E_t\lambda_{5,t+1} = 0$$

Interest-rate beliefs:

$$\beta\bar{c}_{i,ai}E_t\lambda_{4,t+1} - \lambda_{6,t} + \beta(1-g)E_t\lambda_{6,t+1} = 0.$$

A.3 Optimal Policy: Unanchored Expectations

Under unanchored expectations the aggregate demand relation can written in terms of model state variables as

$$x_t = \hat{P}_t^m - \hat{A}_t + \bar{c}_{i,aP^m}a_{t-1}^{P^m} + \bar{c}_{i,a\pi}a_{t-1}^{\pi} + \bar{c}_{i,aw}a_{t-1}^{w} + \bar{c}_{i,a\Gamma}a_{t-1}^{\Gamma} \tag{28}$$

where

$$\begin{aligned}
\bar{c}_{i,aP^m} &= \frac{\beta(1-\rho)}{1-\beta} \\
\bar{c}_{i,a\pi} &= \frac{1}{1-\beta} \\
\bar{c}_{i,aw} &= \bar{s}_C^{-1}\left(\frac{\bar{\theta}-1}{\bar{\theta}}\right)\left(1+\gamma^{-1}\right) \\
\bar{c}_{i,a\Gamma} &= \bar{s}_C^{-1}\bar{\theta}^{-1}.
\end{aligned}$$

The arbitrage condition

$$\hat{\imath}_t = -\hat{E}_t\left(\hat{P}_t^m - \rho\beta\hat{P}_{t+1}^m\right)$$

comprises an additional constraint under unanchored expectations. Written in terms of state variables gives

$$\hat{\imath}_t = -\hat{P}_t^m + \rho\beta a_{t-1}^{P^m} \tag{29}$$

where beliefs about the bond price are updated according to

$$a_t^{P^m} = (1-g)a_{t-1}^{P^m} + g\hat{P}_t^m. \tag{30}$$

The optimal policy problem of the Central Bank is to choose $\left\{\pi_t, x_t, i_t, \hat{P}_t^m, a_t^\pi, a_t^w, a_t^\Gamma, a_t^{P^m}\right\}$ to minimize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \frac{1}{2}\left[\pi_t^2 + \lambda_x x_t^2 + \lambda_i i_t^2\right] + \lambda_{1,t}\left(-\hat{\pi}_t + \bar{c}_{\pi,x}x_t + \bar{c}_{\pi,aw}a_{t-1}^w + \bar{c}_{\pi,a\pi}a_{t-1}^\pi + \bar{c}_{\pi,x}\hat{u}_t\right) \\ +\lambda_{2,t}\left(-a_t^\pi + (1-g)a_{t-1}^\pi + g\hat{\pi}_t\right) + \lambda_{3,t}\left(-a_t^w + \bar{c}_{aw,aw}a_{t-1}^w + \bar{c}_{aw,x}x_t + \bar{c}_{aw,A}\hat{A}_t\right) \\ +\lambda_{4,t}\left(-x_t + \hat{P}_t^m - \hat{A}_t + \bar{c}_{i,aP^m}a_{t-1}^{P^m} + \bar{c}_{i,a\pi}a_{t-1}^\pi + \bar{c}_{i,aw}a_{t-1}^w + \bar{c}_{i,a\Gamma}a_{t-1}^\Gamma\right) \\ +\lambda_{5,t}\left(-a_t^\Gamma + \bar{c}_{a\Gamma,a\Gamma}a_{t-1}^\Gamma + \bar{c}_{a\Gamma,x}x_t + \bar{c}_{a\Gamma,A}\hat{A}_t\right) + \lambda_{6,t}\left(-a_t^{P^m} + (1-g)a_{t-1}^{P^m} + g\hat{P}_t^m\right) \\ +\lambda_{7,t}\left(-\hat{\imath}_t - \hat{P}_t^m + \rho\beta a_{t-1}^{P^m}\right) \end{array} \right\}.$$

The first order-conditions are

Inflation:

$$\pi_t - \lambda_{1,t} + g\lambda_{2,t} = 0 \tag{31}$$

The output gap:

$$\lambda_x x_t + \bar{c}_{\pi,x}\lambda_{1,t} + \bar{c}_{aw,x}\lambda_{3,t} - \lambda_{4,t} + \bar{c}_{a\Gamma,x}\lambda_{5,t} = 0 \tag{32}$$

Inflation beliefs:

$$-\lambda_{2,t} + \beta\bar{c}_{\pi,a\pi}E_t\lambda_{1,t+1} + \beta(1-g)E_t\lambda_{2,t+1} + \beta\bar{c}_{i,a\pi}E_t\lambda_{4,t+1} = 0 \tag{33}$$

Wage beliefs:

$$\beta\bar{c}_{\pi,aw}E_t\lambda_{1,t+1} - \lambda_{3,t} + \beta\bar{c}_{aw,aw}E_t\lambda_{3,t+1} + \beta\bar{c}_{i,aw}E_t\lambda_{4,t+1} = 0 \tag{34}$$

Dividend beliefs

$$\beta\bar{c}_{i,a\Gamma}E_t\lambda_{4,t+1} - \lambda_{5,t} + \beta\bar{c}_{a\Gamma,a\Gamma}E_t\lambda_{5,t+1} = 0 \tag{35}$$

Bond price beliefs:

$$\beta\bar{c}_{i,aP^m}E_t\lambda_{4,t+1} - \lambda_{6,t} + \beta(1-g)E_t\lambda_{6,t+1} + \beta^2\rho E_t\lambda_{7,t+1} = 0. \tag{36}$$

Interest rates:

$$\lambda_i\hat{\imath}_t - \lambda_{7,t} = 0. \tag{37}$$

Bond prices:

$$\lambda_{4,t} + g\lambda_{6,t} - \lambda_{7,t} = 0 \tag{38}$$

Notice that optimal policy under unanchored financial expectations with $\rho = 0$ corresponds to optimal policy under anchored expectations.

A.4 Proof of Proposition 1

We here establish a sketch of the proof. More details are provided in a more general case in Giannoni and Woodford (2010). Consider the vector of $m = 8$ endogenous variables $y_t = \left[\pi_t, x_t, i_t, \hat{P}_t^m, a_t^\pi, a_t^w, a_t^\Gamma, a_t^{P^m}\right]'$ and the vector of exogenous variable $\xi_t = \left[\hat{A}_t, \hat{u}_t\right]'$. As in Giannoni and Woodford (2010), the $n = 7$ structural equations (22), (24)–(26), (28)–(30) can be written compactly in the form

$$\bar{I} y_t = \bar{A} y_{t-1} + \bar{C}\xi_t \tag{39}$$

for all $t \geq 0$ where $\bar{A}$ and $\bar{I}$ are $n \times m$ matrices of coefficients and $\bar{C}$ is $n \times 2$. This system implies

$$\bar{I} E_t y_{t+1} = \bar{A} y_t + \bar{C} E_t \xi_{t+1} \tag{40}$$

for all $t \geq 0$ and

$$\bar{I}\left(y_t - E_{t-1} y_t\right) = \bar{C}\left(\xi_t - E_{t-1}\xi_t\right) \tag{41}$$

for all $t > 0$. Conversely, (40), (41) and the initial condition $\bar{I} y_0 = \bar{A} y_{-1} + \bar{C}\xi_0$ imply (39) for all $t \geq 0$. The system (40), (41) and the initial condition is thus equivalent to (39) for all $t \geq 0$. It follows from the n restrictions (41) that y_t contains n "predetermined" endogenous variables.

The m first-order conditions (31)–(38) can be written as

$$\bar{A}' E_t \lambda_{t+1} = \beta^{-1} \bar{I}' \lambda_t - S y_t \tag{42}$$

where $\lambda_t = [\lambda_{1,t} ..., \lambda_{7,t}]'$ is a vector of n non-predetermined Lagrange multipliers and S is a diagonal matrix with $[1, \lambda_x, \lambda_i, 0, 0, 0, 0, 0]$ on the diagonal. To show that the system composed of the structural equations (40) and the first-order conditions (42) yields a unique bounded solution, the identities

$$y_t = y_t, \tag{43}$$

$$E_t \lambda_{t+1} = E_t \lambda_{t+1}, \tag{44}$$

are adjoined. This yields the complete dynamic

$$\begin{bmatrix} 0 & \bar{A} & 0 & -\bar{I} \\ \bar{A}' & S & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \end{bmatrix} E_t \begin{bmatrix} \lambda_{t+1} \\ y_t \\ E_{t+1}\lambda_{t+2} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \beta^{-1}\bar{I}' & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} \begin{bmatrix} \lambda_t \\ y_{t-1} \\ E_t\lambda_{t+1} \\ y_t \end{bmatrix} - \begin{bmatrix} \bar{C}E_t\tilde{\xi}_{t+1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\bar{M}E_t d_{t+1} = \bar{N}d_t - \bar{N}_s\bar{s}_t, \tag{45}$$

where d_t is the $2(m+n)$-dimensional (here 30-dimensional) vector

$$d_t \equiv \begin{bmatrix} \lambda_t \\ y_{t-1} \\ E_t\lambda_{t+1} \\ y_t, \end{bmatrix},$$

with $m+n$ predetermined variables, $\bar{s}_t$ is a vector of exogenous disturbances that includes the elements of $\tilde{\xi}_t$ and $\tilde{\xi}_{t-1}$, and

$$\bar{M} \equiv \begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} \\ I_{m+n} & 0 \end{bmatrix}, \qquad \bar{N} \equiv \begin{bmatrix} -\beta^{-1}\bar{M}_{12}' & 0 \\ 0 & I_{m+n} \end{bmatrix} \tag{46}$$

where

$$\bar{M}_{11} \equiv \begin{bmatrix} 0 & \bar{A} \\ \bar{A}' & S \end{bmatrix} = \bar{M}_{11}', \quad \text{and} \quad \bar{M}_{12} \equiv \begin{bmatrix} 0 & -\bar{I} \\ 0 & 0 \end{bmatrix}.$$

As in Giannoni and Woodford (2010), the matrix pencil $\bar{M} - \mu\bar{N}$ is *regular*; that is, its determinant is non-zero for at least some complex μ. Hence the matrix pencil $\bar{M} - \mu\hat{N}$ where $\hat{N} \equiv \beta^{1/2}\bar{N}$ is also regular. Let us define the $2(n+m) \times 2(n+m)$ matrix

$$J \equiv \begin{bmatrix} 0 & I_{n+m} \\ -I_{n+m} & 0 \end{bmatrix},$$

and observe that

$$\bar{M}'J\bar{M} = \hat{N}'J\hat{N},$$

so that the transposed matrix pencil $(\bar{M} - \mu\hat{N})'$ is symplectic. It follows that the generalized eigenvalues of the transposed pencil $(\bar{M} - \mu\hat{N})'$ are symmetric with respect to the unit circle: if $\mu \in \mathbb{C}$ is a generalized eigenvalue of the real matrix pencil $(\bar{M} - \mu\hat{N})'$, then so are μ^{-1} and the complex conjugates $\bar{\mu}$, $\overline{\mu^{-1}}$. In particular, if $\mu = 0$ is an eigenvalue of $(\bar{M} - \mu\hat{N})'$, so is

$\mu = \infty$. Since $\det[\bar{M} - \mu\hat{N}] = \det[\bar{M}' - \mu\hat{N}']$ for all μ, it follows that if $\mu \in \mathbb{C}$ is an eigenvalue of $(\bar{M} - \mu\hat{N})$, then so are μ^{-1} and the complex conjugates $\bar{\mu}$, $\overline{\mu^{-1}}$. Moreover, $\det[\bar{M} - \mu\hat{N}] = 0$ if and only if $\det[\bar{M} - \beta^{1/2}\mu\bar{N}] = 0$. Hence μ is a generalized eigenvalue of $(\bar{M} - \mu\bar{N})$ if and only if $\beta^{-1/2}\mu$ is a generalized eigenvalue of the transformed pencil $(\bar{M} - \mu\hat{N})$. It then follows that $\beta\mu^{-1}$, $\bar{\mu}$, and $\overline{\beta\mu^{-1}}$ must also be generalized eigenvalues of $(\bar{M} - \mu\bar{N})$.

As a result, the system (45) admits $n + m$ eigenvalues with modulus smaller than $\beta^{1/2}$ and the remaining $n + m$ eigenvalues with modulus greater than $\beta^{1/2}$. Since this system has exactly $m + n$ predetermined variables, it admits a unique bounded solution. ■

References

Adam, K., and A. Marcet (2011): "Internal Rationality, Imperfect Knowledge and Asset Prices," *Journal of Economic Theory*, 146, 1224–1252.

Chung, H., J.-P. Laforte, D. Reifschneider, and J. C. Williams (2011): "Have we Underestimated the Likelihood and Severity of Zero Lower Bound Events?," unpublished, Board of Governors of the Federal Reserve.

Clarida, R., J. Gali, and M. Gertler (1999): "The Science of Monetary Policy: A New Keynesian Perspective," *Journal of Economic Literature*, 37, 1661–1707.

Coibon, O., Y. Gorodnichenko, and J. F. Wieland (2010): "The Optimal Rate of Inflation in New Keynesian Models," NBER Working Paper 16093.

Eusepi, S., M. Giannoni, and B. Preston (2011): "The Theory of Optimal Monetary Policy Under Learning Dynamics," unpublished, Columbia University.

Eusepi, S., and B. Preston (2010): "Debt, Policy Uncertainty and Expectations Stabilization," *Journal of the European Economics Association*, forthcoming.

——— (2011a): "Expectations, Learning and Business Cycle Fluctuations," *American Economic Review*, 101(6), 2844–2872.

——— (2011b): "The Maturity Structure of Debt,Monetary Policy and Expectations Stabilization," unpublished, Columbia University.

Evans, G. W., and S. Honkapohja (2001): *Learning and Expectations in Economics*. Princeton, Princeton University Press.

——— (2003): "Expectations and the Stability Problem for Optimal Monetary Policies," *Review of Economic Studies*, 70(4), 807–824.

——— (2006): "Monetary Policy, Expectations and Commitment," *Scandanavian Journal of Economics*, 108, 15–38.

——— (2008): "Robust Learning Stability with Operational Moentary Policy Rules," unpublished, University of Oregon.

Gaspar, V., F. Smets, and D. Vestin (2006): "Adaptive Learning, Persistence and Optimal Monetary Policy," *Journal of the European Economics Association*, 4, 376–385.

Giannoni, M. P., and M. Woodford (2010): "Optimal Target Criteria for Stabilization Policy," unpublished, Columbia University.

Kreps, D. (1998): "Anticipated Utility and Dynamic Choice," in *Frontiers of Research in Economic Theory*, ed. by D. Jacobs, E. Kalai, and M. Kamien, pp. 242–274. Cambridge: Cambridge University Press.

Lorenzoni, G. (2008): "A Theory of Demand Shocks," unpublished, MIT.

Marcet, A., and T. J. Sargent (1989): "Convergence of Least-Squares Learning in Environments with Hidden State Variables and Private Information," *Journal of Political Economy*, pp. 1306–1322.

Milani, F. (2007): "Expectations, Learning and Macroeconomic Persistence," *Journal of Monetary Economics*, 54, 2065–2082.

Molnar, K., and S. Santoro (2005): "Optimal Monetary Policy When Agents Are Learning," unpublished, Iniversitat Pomeu Fabra.

Preston, B. (2005a): "Adaptive Learning in Infinite Horizon Decision Problems," unpublished, Columbia University.

——— (2005b): "Learning About Monetary Policy Rules when Long-Horizon Expectations Matter," *International Journal of Central Banking*, 1(2), 81–126.

——— (2006): "Adaptive Learning, Forecast-Based Instrument Rules and Monetary Policy," *Journal of Monetary Economics*, 53, 507–535.

——— (2008): "Adaptive Learning and the Use of Forecasts in Monetary Policy," *Journal of Economic Dynamics and Control*, 32(4), 2661–3681.

Rotemberg, J. J., and M. Woodford (1998): "Interest-Rate Rules in an Estimated Sticky Price Model," NBER Working Paper, 6618.

Sargent, T. J. (1999): *The Conquest of American Inflation*. Princeton University Press.

Schmitt-Grohe, S., and M. Uribe (2007): "Optimal Inflation Stabilization in a Medium-Scale Model," in *Monetary Policy Under Inflation Targeting*, ed. by K. Schmidt-Hebbel, and R. Mishkin, pp. 125–186.

Slobodyan, S., and R. Wouters (2009): "Estimating a medium-scale DSGE Model with expectations based on small forecasting models," unpublished, National Bank of Belgium.

Woodford, M. (1998): "Public Debt and the Price Level," unpublished manuscript, Columbia University.

——— (2001): "Fiscal Requirements of Price Stability," *Journal of Money, Credit and Banking*, 33, 669–728.

——— (2003): *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press.

——— (2007): "Forecast Targeting as a Monetary Policy Strategy: Policy Rules in Practice," unpublished, Columbia University.

Yun, T. (1996): "Nominal Price Rigidity, Money Supply Endogeneity, and Business Cycles," *Journal of Monetary Economics*, 37, 345–370.