

On three-dimensional quasiperiodic Floquet instabilities of two-dimensional bluff body wakes

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Previous studies dealing with Floquet secondary stability analysis of the wakes of circular and square cross-section cylinders have shown that there are two synchronous instability modes, with long (mode A) and short (mode B) spanwise wavelengths. At intermediate wavelengths another mode arises, which reaches criticality at Reynolds numbers higher than modes A or B. Here we concentrate on these intermediate-wave number modes for the wakes of circular and square cylinders. It is found that in both cases these modes possess complex-conjugate pair Floquet multipliers, and can be combined to produce either standing or traveling waves. Both these states are quasiperiodic. © 2003 American Institute of Physics. [DOI: 10.1063/1.1591771]

Three-dimensional secondary stability analysis of bluff body flows has been a topic of special interest since the classic experimental investigations of Williamson,¹ where it was demonstrated that two distinct types of secondary instability, each with a particular characteristic frequency and spanwise wavelength, were dominant in the secondary instability of the circular cylinder wake. Following the stimulus of these experimental results, numerical Floquet stability analysis of the time-periodic two-dimensional base flows was carried out,^{2,3} culminating in the equally classic study of Barkley and Henderson,⁴ hereafter referred to as BH96. In that work, the three-dimensional Fourier Floquet instability modes of the two-dimensional cylinder wake were computed as functions of both spanwise wave number $\beta = 2\pi D/\lambda$, where D is cylinder diameter and λ is spanwise wavelength, and the mean flow Reynolds number $Re = UD/\nu$, where U is the freestream speed and ν is the fluid's kinematic viscosity. As Reynolds number is increased, the long wavelength mode A ($\lambda_A/D = 3.96$ at onset) is the first to become unstable, via a subcritical bifurcation with a linear onset Reynolds number $Re_{cA} = 188$.² Mode B bifurcates supercritically⁵ from the two-dimensional flow at $Re_{cB} = 256$, with the much shorter spanwise wavelength $\lambda_B/D = 0.822$. In the case of mode B, the two-dimensional flow is already unstable to mode A, and for intermediate Reynolds numbers (between Re_{cA} and Re_{cB}) it appears that mixed-mode states exist, while ultimately for $Re \gtrsim 265$, only mode B-type states are observed. A low-dimensional model for the mixed-mode scenario, and the subsequent dominance of mode B at higher Reynolds numbers, has appeared.⁶

Floquet stability analysis of the two-dimensional wake of cylinders with square cross section has been presented by Robichaux, Balachandar, and Vanka,⁷ subsequently referred to as RBV99. The dominant three-dimensional instabilities are similar to those for the circular cylinder: mode A, with onset $Re \approx 166$, has a spanwise wavelength $\lambda = 5.22D$ (where

D is now the length of a side of the square); mode B, onset $Re \approx 190$, has wavelength $\lambda = 1.2D$.

All the three-dimensional modes by definition break the $O(2)$ spanwise symmetry of the two-dimensional base flows. Modes A and B are synchronous with the base flow, i.e., at onset they have a temporal periodicity which is very close to that of the underlying two-dimensional flow. The two-dimensional flow has a spatiotemporal symmetry such that (in primitive variables) the flow at any time is identical to that displaced half a shedding period in time and reflected about the wake centerline. It is found that in both flows under discussion here, mode A preserves this symmetry, while mode B breaks it.^{4,6,7}

In this Letter, the focus is not on mode A and mode B instabilities, but rather on a third instability mode, that has wavelengths intermediate between modes A and B, and which bifurcates from the two-dimensional base flows at Reynolds numbers above those for either mode A or B. Such modes were first reported by BH96 [in passing, see Fig. 7 of their paper, data reproduced in Fig. 2(a) here] as possessing complex-conjugate pair Floquet multipliers. An apparently similar mode (intermediate wavelengths, higher onset Re) was reported for the square cylinder by RBV99, where it was said not to have complex-conjugate pair multipliers but to be a subharmonic mode, i.e., a mode with the real multiplier $\mu = -1$ at onset. By repeating RBV99's analysis for the square cylinder, but using the same numerical technique as employed by BH96, we demonstrate that in fact these intermediate modes in both cases have complex-conjugate pair multipliers. The complex-conjugate modes can lead either to quasiperiodic standing wave or traveling-wave type solutions.

We restrict the discussion to flows in which geometries and boundary conditions have a homogeneous spatial coordinate, here taken to be the z coordinate. In these cases, three-dimensional instabilities can be described with a Fou-

rier series expansion in the homogeneous coordinate. In a linear stability analysis these Fourier modes are independent. The spatial forms which Floquet Fourier modes can be assumed to take depend on three factors: (a) the number of independent spatial dimensions required to represent the base flow (its “dimensionality,” here two); (b) the number of independent velocity components required to represent the base flow (its “componentality,” here also two); (c) the complexity of the Floquet multipliers. At each spanwise wavelength λ , the velocity component and pressure Floquet Fourier modes are represented by two-dimensional complex fields which can be equivalently considered at each point (x, y) as a complex scalar with, in general, both real and imaginary parts nonzero, a wave in the z -direction with arbitrary amplitude and phase, or a pair of cosine and sine waves, each of arbitrary magnitude.

For three-component base flows, it is found by examination of the linearized Navier–Stokes equations that the fully general complex form of the perturbation Fourier modes must be retained throughout the analysis.⁸ For two-dimensional, two-component base flows (present cases) there are two sets of spanwise expansions for perturbation velocity (u', v', w') and pressure p' that each pass uncoupled through the linearized Navier–Stokes equations,

$$\begin{aligned} \{u', v', w', p'\}(x, y, z, t) \\ = \{u' \cos \beta z, v' \cos \beta z, w' \sin \beta z, p' \cos \beta z\}(x, y, t), \end{aligned} \quad (1)$$

or

$$\{u' \sin \beta z, v' \sin \beta z, w' \cos \beta z, p' \sin \beta z\}(x, y, t). \quad (2)$$

The independence of these two sets of solutions means that we are free to choose eigenfunctions based on either set, or an arbitrary (but temporally constant) linear combination of the two—which corresponds to an arbitrary z -coordinate shift of the eigenfunction. The above holds regardless of the temporal structure of the base flows.

We now turn to consider time-periodic base flows (period T), whose stability may be characterized by Floquet multipliers μ , which can be real or occur in complex-conjugate pairs. Marginal stability for a Floquet mode occurs as its multiplier crosses the unit circle, i.e., at $|\mu| = 1$. If the Floquet multiplier is real ($\mu = \pm 1$ at marginal stability), then in effect the corresponding Floquet mode is a standing wave type solution, since it evolves through each period by a multiplication with this real value, which (to within a constant) leaves the spatial shape invariant. In these cases, it is sufficient to use either (1) or (2) for the spatial expansions, and this was the approach adopted by both BH96 and RBV99. However, in the case of complex-conjugate pair multipliers (where marginal stability, with $\mu = e^{\pm i\theta}$, corresponds to a Neimark–Sacker bifurcation) then *while it is still possible* to choose expansions (1) or (2), restriction to either of these cases again corresponds to choosing a standing wave type solution, whereas either traveling or standing waves are possible outcomes. While in linear stability analysis the eigenfunctions for the standing wave are just symmetric combinations of those for traveling waves, the Floquet multipliers are

the same in either case, and marginal stability occurs at the same parameter values, the nonlinear evolution of the two cases produces two distinct solution branches. As with $O(2)$ -equivariant Hopf bifurcations,⁹ at most one of these branches will have stable solutions. Neimark–Sacker bifurcations introduce a new temporal frequency (related to the value of the imaginary parts of the Floquet multipliers), in general incommensurate with that of the base flow, and the three-dimensional solutions are temporally quasiperiodic.

There are a number of possible numerical techniques that may be applied to Floquet analysis. Unless it is known *a priori* that the Floquet multipliers will be (say) real, a principal requirement of the method is that it must be sufficiently general to be capable of computing either real or complex-conjugate pair Floquet multipliers. Noack and Eckelmann³ found the eigensystem of the monodromy matrix obtained using a low-dimensional projection of the velocity field, while BH96 used a Krylov subspace iterative method based on the linearized Poincaré return map of the perturbation to estimate Floquet multipliers and eigenfunctions.^{4,10} Both these methods are sufficiently general to compute complex-conjugate pair eigensystems, provided of course that the dimension of the subspace in which the eigensystem is solved is sufficiently large (two, minimum, and typically many more in practice). On the other hand, RBV99 used a one-dimensional power-type method, taking $|\mu|$ to be the limiting value of

$$|\mu_{\max}| = N(t+T)/N(t) \quad (3)$$

[Eq. (19) of RBV99], where $N(t)$ is the square-root of the kinetic energy in the perturbation velocity at time t , and then giving μ a sign, positive or negative, depending on the observed temporal behavior of the perturbation velocity from period to period. This means that for their analysis the multipliers could either be positive, or negative, but *real only*. And indeed RBV99 reported that all the modes they found had real Floquet multipliers.

The synchronous (A and B) Floquet modes for the wakes of circular and square cylinders described by BH96 and RBV99 are, as outlined above, evidently quite similar in nature, which is unsurprising given the similarities of the two-dimensional base flows. Given this, it is somewhat surprising that the third mode that can potentially bifurcate from the two-dimensional wake as Reynolds number is increased (a mode with spanwise wavelength intermediate between those for modes A and B) was reported by BH96 to have complex-conjugate pair Floquet multipliers, but by RBV99 to have a real, negative multiplier (a subharmonic mode, which they referred to as mode S). That discrepancy has prompted the present comparative re-examination of the Floquet analysis, using the more general numerical method employed by BH96.

The underlying spatial discretization in each case employed spectral elements, and this has been maintained in the present investigation. Figure 1 shows the outlines of the spectral elements used for the wakes of the circular and square cylinders. The circular cylinder mesh, while different in detail, is very similar in overall resolution and domain extent to M_2 , the principal analysis domain used by BH96.

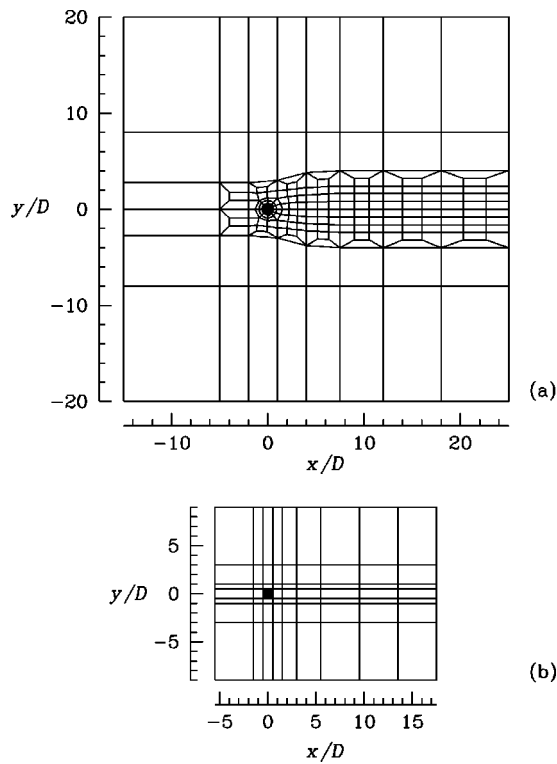


FIG. 1. Two-dimensional spectral element meshes for (a) circular cylinder, 218 elements, and (b) square cylinder, 62 elements, reproduced to the same scale.

The one-dimensional Lagrange interpolants used as the tensor-product basis for shape functions in each element had polynomial order $N=7$, while BH96 used $N=8$. The square cylinder mesh used is identical to that employed by RBV99, in extent, number and placement of elements, and element interpolation order, $N=15$.

Floquet analyses for both wakes have been computed over ranges of Reynolds numbers and spanwise wave numbers consistent with the original studies. For both the circular and square cylinder wakes, the Krylov dimension (size of the subspace in which eigenpairs are computed) was $K=25$, but the results for the leading modes were nearly identical when computed at $K=13$. To summarize the outcome, Fig. 2 presents the values of $|\mu|$ found at different values of spanwise wave number, β , for Reynolds numbers $Re=280$ for the circular cylinder wake (the highest Reynolds number employed by BH96, where the intermediate mode near $\beta \approx 4$ is closest to criticality) and $Re=205$ for the square cylinder, where the intermediate wavelength mode $\beta \approx 2.5$ has just exceeded criticality. In both Figs. 2(a) and 2(b), values for $|\mu|$, digitized from the plots published in BH96 (Fig. 7) and RBV99 (Fig. 8), are shown as open squares. Clearly the agreement between the present work and the original papers for the computed values of $|\mu|$ is very good. The only substantive difference is that Floquet multipliers for the intermediate wavelength mode (labeled QP in both plots) were found here to occur in complex-conjugate pairs for both wakes. That is, the present results are in agreement with the finding of BH96 for the circular cylinder wake but in disagreement with that of RBV99 for the square cylinder wake in that the interme-

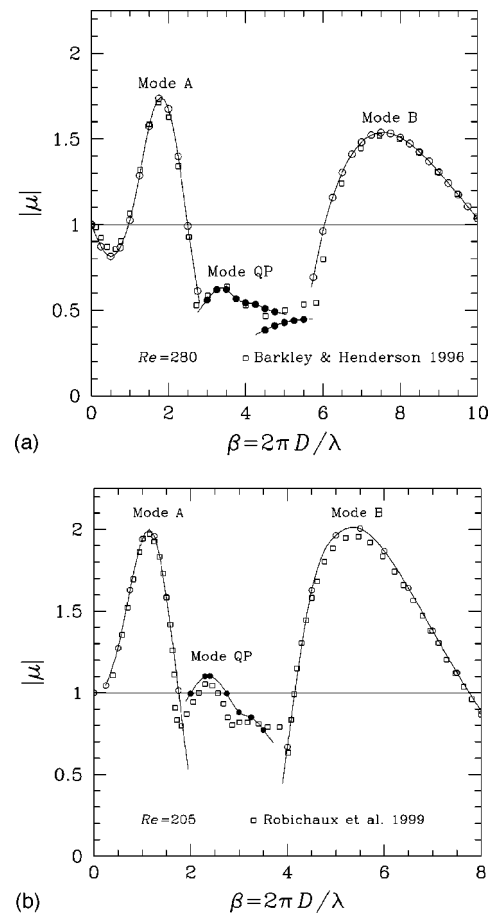


FIG. 2. Comparison of magnitudes of Floquet multipliers computed in the present study with those previously presented (Refs. 4 and 7) for the (a) circular-section cylinder at $Re=280$ and (b) square section cylinder at $Re=205$. Previous results: \square . Current results: \circ , real Floquet multipliers, synchronous modes; \bullet , complex-conjugate multipliers, quasiperiodic modes.

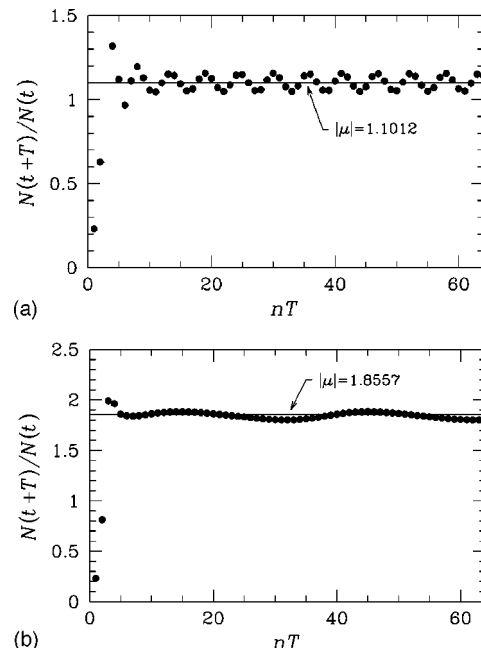


FIG. 3. Plot showing estimates (\bullet) for $|\mu|$ for the square-section cylinder, computed using power method (3), starting from random initial conditions, compared to values computed using a Krylov subspace method (solid line) at $\beta=2.3$: (a) $Re=205$; (b) $Re=225$.

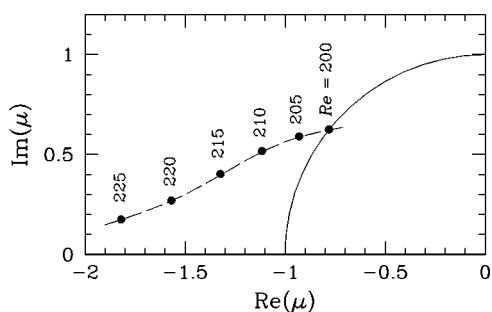


FIG. 4. The locus with increasing Reynolds number of (one half of pairs of) complex-conjugate Floquet multipliers for the quasiperiodic mode of the square-section cylinder, at $\beta=2.3$. Marginal stability occurs at $\text{Re} \approx 200$ when the locus crosses the unit circle.

diate wavelength mode had negative real multipliers. The agreement (not illustrated) between the presently computed values of $|\mu|$ and those of RBV99 at $\text{Re}=225$ is as good as seen in Fig. 2(b), but again the multipliers for the intermediate mode are complex, not real.

In seeking to resolve the discrepancy, we used the method of RBV99 to compute estimates of $|\mu|$, using (3), and compare to the results obtained using the Krylov subspace method with $K=25$ —these computations were performed at $\beta=2.3$, which is near the most-amplified wave number for the quasiperiodic mode, and at two Reynolds numbers, 205 and 225. The outcome is shown in Fig. 3. It can be seen that after starting transients have died, the estimates of $|\mu|$ computed using (3) do not reach steady states, but instead oscillate around the values computed using the Krylov subspace method. This is exactly the behavior that would be expected for standing-wave modes with complex-conjugate pair Floquet multipliers.

Further, it can be seen that the amplitude of the oscillation decreases with increasing Reynolds number, while the period of oscillation increases. This is consistent with the complex values of the multipliers—in fact, the location of the multipliers in the complex plane can be inferred from the oscillatory behavior seen in Fig. 3. In Fig. 4 is plotted the locus with Re of (one of) the complex-conjugate pair multipliers (computed using the Krylov method) in the second quadrant of the complex plane. It can be seen that the critical Reynolds number for this mode is close to 200, and that *with increasing Reynolds number the multipliers approach the negative real axis*. With increasing Reynolds number the associated Floquet modes will behave more and more like subharmonic, rather than quasiperiodic modes.

Now we have an explanation for the “discrepancy.” If RBV99 had examined the behavior of normalized Floquet

modes at Reynolds numbers well above onset they could have confused what they saw with a subharmonic mode, since the corresponding multipliers have comparatively small imaginary parts. (The Reynolds numbers for their examinations of modal behavior in their Figs. 11 and 14 were not supplied.) In addition, confining the space of shape functions employed to (1) would, as explained earlier, have precluded the possibility for spanwise travel of the mode shape, depriving them of another visual cue.

Thus, the apparent discrepancy between the reported behaviors for the three-dimensional instability modes of these two bluff body wakes appears to be resolved, and the descriptions can be unified. Both wakes possess synchronous long and short wavelength instability modes (modes A and B), with identical symmetries in each flow, and in addition they have another, intermediate wavelength mode, which is quasiperiodic, and which can manifest either as standing or traveling waves. It is possible that this is a general scenario for three-dimensional instabilities of the time-periodic wakes generated by two-dimensional bluff bodies which possess reflection symmetry about the centerline of the wake.

ACKNOWLEDGMENTS

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