Distributed Slowly Varying Exact Coherent Structures and the Attached Eddy Hypothesis

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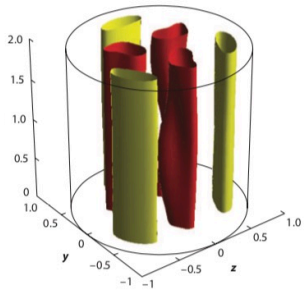
Background: exact coherent structures

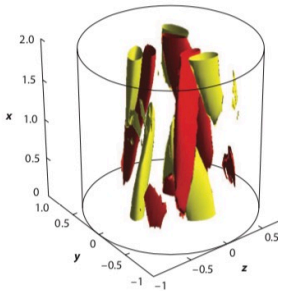
Also known as

Invariant (self-symmetric) solutions
Self-sustained processes (SSP)
Vortex-wave interactions (VWI)
Nonlinear invariant solutions
Nonlinear equilibria

all are 3D solutions of the Navier–Stokes equations.

Kerswell/Tutty/Wedin (2004–2007)

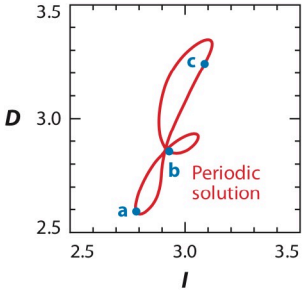


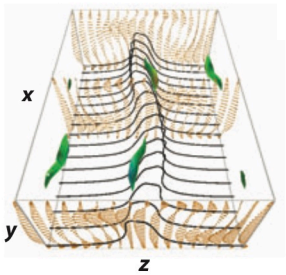


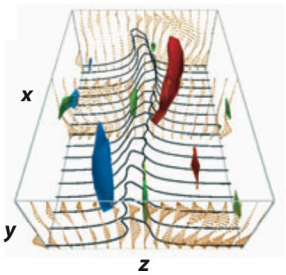
Usually obtained in simple/canonical geometries, e.g. plane Couette, plane/circular Poiseuille.

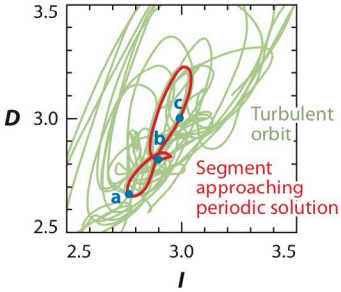
Can be **steady** (standing wave, SW), trivially steady (travelling wave, TW), or (space-)time-periodic.

Observationally related to repetitive signatures of chaotic/transitional/turbulent flows.



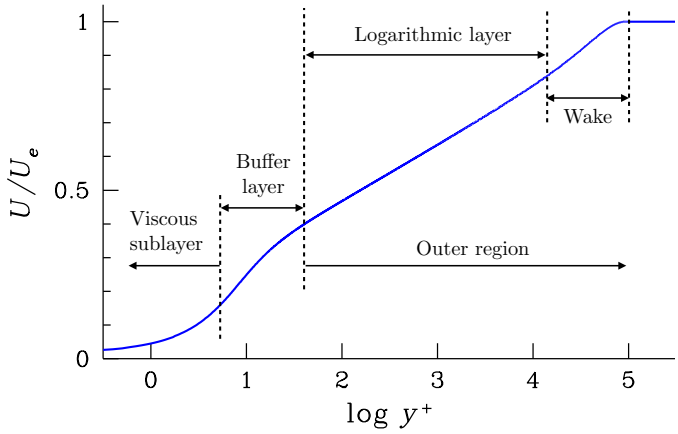






Kawahara & Kida (2001)

The law of the wall



What role might exact coherent structures play here?

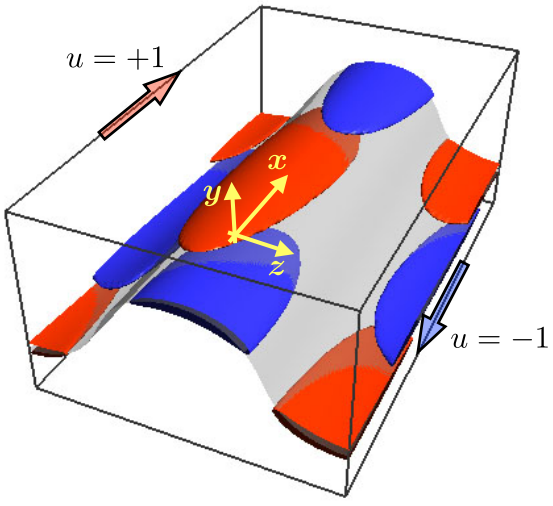
1. Lower-branch states known to play a role in bypass transition (Gibson, Wang, Waleffe …).
2. Turbulent solutions re-visit upper-branch states and periodic exact coherent structures capture many properties of turbulence (Kawahara & Kida).
3. But for a log layer, dominant eddy sizes must vary (linearly) with distance from the wall – basis of attached eddy hypothesis (Townsend).
4. Simulations (Cossu & Hwang) suggest that the VWI/SSP mechanism is operative in log layer.

The challenge

1. Can we generalise vortex-wave interaction (VWI) theory to find flows with spatially varying dominant wavelengths?
2. How would these generalised VWI states hold together?
3. How might such states relate to the attached eddy hypothesis, log layer folklore?
4. How would we compute such flows?

Part 1. VWI theory

Definitions



Consider plane Couette flow (PCF) between two moving walls:

'Streamwise' = *x*, velocity component *u*.

'Vertical' = *y*, velocity component *v*.

'Spanwise' = *z*, velocity component *w*.

$\boldsymbol{u} \equiv (u, v, w)$

Reynolds number $Re = V_{\text{wall}} \times \frac{1}{2} L_y / \nu \equiv \nu^{-1}$

Streamwise wavenumber $\alpha = 2\pi / L_x$

Spanwise wavenumber $\beta = 2\pi / L_z$

Unsteady incompressible Navier–Stokes:

$$\partial_t \boldsymbol{u} + \boldsymbol{\nabla} \cdot (\boldsymbol{u}\boldsymbol{u}) = -\boldsymbol{\nabla} p + Re^{-1} \nabla^2 \boldsymbol{u}, \quad \text{with} \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0,$$

Apply 1D Fourier transform in *x* with associated fundamental wavenumber α $\qquad \partial_x(\,) \equiv \mathrm{i}\alpha k(\,)$

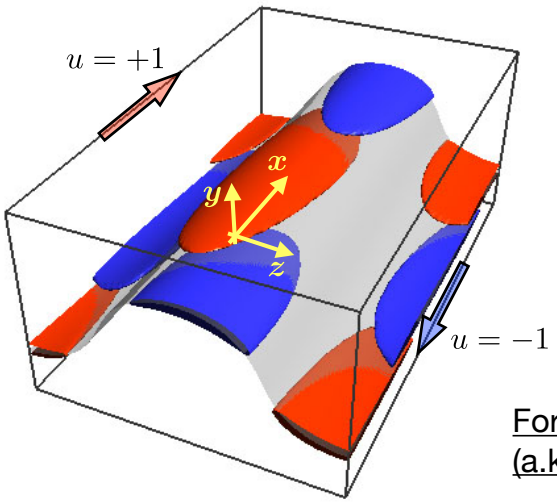
Fourier space $$\widehat{\boldsymbol{u}}_k(y,z) = \frac{1}{2\pi} \int_0^{L_x} \boldsymbol{u}(x,y,z) \exp(-\mathrm{i}k\alpha x)\, \mathrm{d}x,$$

Physical space $$\boldsymbol{u}(x,y,z) = \sum_{k=-\infty}^{\infty} \widehat{\boldsymbol{u}}_k(y,z) \exp(\mathrm{i}k\alpha x),$$

with $\widehat{\boldsymbol{u}}_{-k} = \widehat{\boldsymbol{u}}^*_{+k}$

(in physical space, fields are real)

The `self-sustained process'



Momentum equations of Navier–Stokes:

Physical space

$$\partial_t \boldsymbol{u} + \boldsymbol{\nabla} \cdot (\boldsymbol{u}\boldsymbol{u}) = -\boldsymbol{\nabla} p + Re^{-1}\nabla^2 \boldsymbol{u}$$

Fourier space, apply convolution theorem on NL terms

$$\partial_t \widehat{\boldsymbol{u}}_k + \sum_{p+q=k} \boldsymbol{\nabla} \cdot (\widehat{\boldsymbol{u}}_p \widehat{\boldsymbol{u}}_q) = -\boldsymbol{\nabla} \widehat{p}_k + Re^{-1}\nabla^2 \widehat{\boldsymbol{u}}_k$$

For simplicity we consider just a pair of Fourier modes (a.k.a. the 'Reduced' Navier–Stokes equations):

In (streamwise-average) mode *k*=0:

$$\begin{aligned}\partial_t \widehat{\boldsymbol{u}}_0 + \boldsymbol{\nabla} \cdot (\widehat{\boldsymbol{u}}_0 \widehat{\boldsymbol{u}}_0) &= -\boldsymbol{\nabla} \widehat{p}_0 + Re^{-1}\nabla^2 \widehat{\boldsymbol{u}}_0 - \boldsymbol{\nabla} \cdot (\widehat{\boldsymbol{u}}_1 \widehat{\boldsymbol{u}}_{-1} + \widehat{\boldsymbol{u}}_{-1} \widehat{\boldsymbol{u}}_1) \\ &= -\boldsymbol{\nabla} \widehat{p}_0 + Re^{-1}\nabla^2 \widehat{\boldsymbol{u}}_0 - \boldsymbol{\nabla} \cdot (\widehat{\boldsymbol{u}}_1 \widehat{\boldsymbol{u}}_1^* + \widehat{\boldsymbol{u}}_1^* \widehat{\boldsymbol{u}}_1)\end{aligned}$$

Interpretation:

Wave Reynolds stresses **sustain**/drive mode 0 away from simple behaviour determined just by BCs.

In (wavy) mode *k*=1:

$$\partial_t \widehat{\boldsymbol{u}}_1 + \boldsymbol{\nabla} \cdot (\widehat{\boldsymbol{u}}_0 \widehat{\boldsymbol{u}}_1 + \widehat{\boldsymbol{u}}_1 \widehat{\boldsymbol{u}}_0) = -\boldsymbol{\nabla} \widehat{p}_1 + Re^{-1}\nabla^2 \widehat{\boldsymbol{u}}_1$$

No higher-order interactions on RHS.

Notation change alert, upcoming in the following: $\widehat{\boldsymbol{u}}_0 \equiv \boldsymbol{U}, \quad \widehat{\boldsymbol{u}}_1 \equiv \boldsymbol{u}$

Streamwise-average flow – the 'vortex' in VWI

$$\partial_t \widehat{\boldsymbol{u}}_0 + \boldsymbol{\nabla} \cdot (\widehat{\boldsymbol{u}}_0 \widehat{\boldsymbol{u}}_0) = -\boldsymbol{\nabla} \widehat{p}_0 + Re^{-1}\nabla^2 \widehat{\boldsymbol{u}}_0 - \boldsymbol{\nabla} \cdot (\widehat{\boldsymbol{u}}_1 \widehat{\boldsymbol{u}}_1^* + \widehat{\boldsymbol{u}}_1^* \widehat{\boldsymbol{u}}_1)$$

NB: *x*-derivatives are all zero, $\widehat{\boldsymbol{u}}_0$ is real (so are all the forcing terms).

Component form:

Constant – non-zero only if flow is driven by a streamwise pressure gradient.

$$(\partial_t + \widehat{v}_0\partial_y + \widehat{w}_0\partial_z) \begin{bmatrix}\widehat{u}_0 \\ \widehat{v}_0 \\ \widehat{w}_0\end{bmatrix} = -\begin{bmatrix}G \\ \partial_y \widehat{p}_0 \\ \partial_z \widehat{p}_0\end{bmatrix} + Re^{-1}(\partial_{yy} + \partial_{zz}) \begin{bmatrix}\widehat{u}_0 \\ \widehat{v}_0 \\ \widehat{w}_0\end{bmatrix} - \begin{bmatrix}\partial_y(\widehat{v}_1\widehat{u}_1^* + \widehat{u}_1^*\widehat{v}_1) + \partial_z(\widehat{u}_1\widehat{w}_1^* + \widehat{w}_1^*\widehat{u}_1) \\ \partial_y(\widehat{v}_1\widehat{v}_1^* + \widehat{v}_1^*\widehat{v}_1) + \partial_z(\widehat{v}_1\widehat{w}_1^* + \widehat{w}_1^*\widehat{v}_1) \\ \partial_y(\widehat{v}_1\widehat{w}_1^* + \widehat{w}_1^*\widehat{v}_1) + \partial_z(\widehat{w}_1\widehat{w}_1^* + \widehat{w}_1^*\widehat{w}_1)\end{bmatrix}$$

Temporarily ignoring the final set of driving terms:

1. The equation for streamwise component $\widehat{u}_0$ is simple advection-diffusion driven by roll flow field.

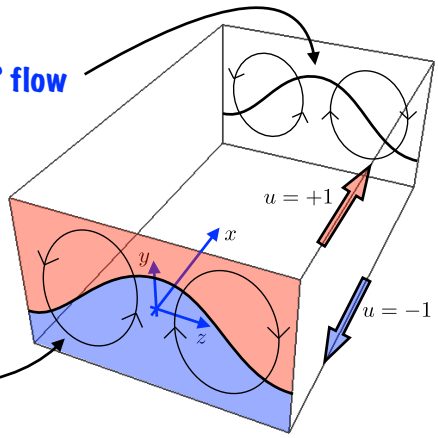
$$\partial_t \widehat{u}_0 + \widehat{v}_0\partial_y \widehat{u}_0 + \widehat{w}_0\partial_z \widehat{u}_0 = G + Re^{-1}(\partial_{yy}\widehat{u}_0 + \partial_{zz}\widehat{u}_0)$$

the `streak' flow

2. The equations for cross-flow components $\widehat{v}_0, \ \widehat{w}_0$ are uncoupled from streamwise component.

$$\partial_t \widehat{v}_0 + \widehat{v}_0\partial_y \widehat{v}_0 + \widehat{w}_0\partial_z \widehat{v}_0 = -\partial_y \widehat{p}_0 + Re^{-1}(\partial_{yy}\widehat{v}_0 + \partial_{zz}\widehat{v}_0)$$

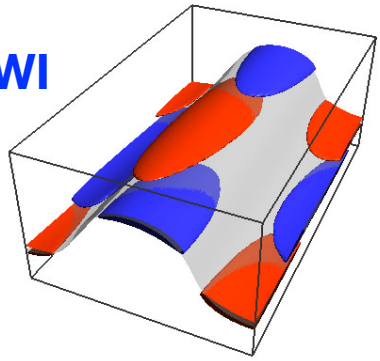
$$\partial_t \widehat{w}_0 + \widehat{v}_0\partial_y \widehat{w}_0 + \widehat{w}_0\partial_z \widehat{w}_0 = -\partial_z \widehat{p}_0 + Re^{-1}(\partial_{yy}\widehat{w}_0 + \partial_{zz}\widehat{w}_0)$$



the `roll' flow

streak + roll = vortex

Wavy mode – the 'wave' in VWI



$$\partial_t \widehat{\boldsymbol{u}}_1 + \boldsymbol{\nabla}\cdot(\widehat{\boldsymbol{u}}_0\widehat{\boldsymbol{u}}_1 + \widehat{\boldsymbol{u}}_1\widehat{\boldsymbol{u}}_0) = -\boldsymbol{\nabla}\widehat{p}_1 + Re^{-1}\nabla^2\widehat{\boldsymbol{u}}_1$$

NB: if the vortex flow $\widehat{\boldsymbol{u}}_0$ is given, this equation is linear in (complex) $\widehat{\boldsymbol{u}}_1$.

Summary

To get a steady solution, we could solve with the time-derivative terms set to zero:

$$\boldsymbol{\nabla}\cdot(\widehat{\boldsymbol{u}}_0\widehat{\boldsymbol{u}}_0) = -\boldsymbol{\nabla}\widehat{p}_0 + Re^{-1}\nabla^2\widehat{\boldsymbol{u}}_0 - \boldsymbol{\nabla}\cdot(\widehat{\boldsymbol{u}}_1\widehat{\boldsymbol{u}}^* + \widehat{\boldsymbol{u}}^*\widehat{\boldsymbol{u}}_1) \qquad \textit{Streak + roll = vortex}$$

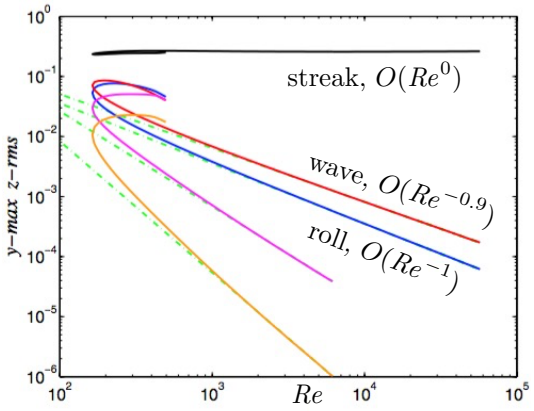
$$\boldsymbol{\nabla}\cdot(\widehat{\boldsymbol{u}}_0\widehat{\boldsymbol{u}}_1 + \widehat{\boldsymbol{u}}_1\widehat{\boldsymbol{u}}_0) = -\boldsymbol{\nabla}\widehat{p}_1 + Re^{-1}\nabla^2\widehat{\boldsymbol{u}}_1 \qquad \textit{Wave}$$

We could do this e.g. via Newton–Raphson. We don't have to do it in Fourier-transformed variables (or, actually, to truncate the Fourier series).

But first, we observe some scaling for plane Couette flow with SSP:

We note that the magnitudes of the roll and wave both fall with *Re* in well-defined power laws.

Wang, Gibson & Waleffe (2007)



Asymptotics for high *Re*

Streak + roll equation $\partial_t \widehat{\boldsymbol{u}}_0 + \boldsymbol{\nabla}\cdot(\widehat{\boldsymbol{u}}_0\widehat{\boldsymbol{u}}_0) = -\boldsymbol{\nabla}\widehat{p}_0 + Re^{-1}\nabla^2\widehat{\boldsymbol{u}}_0 - \boldsymbol{\nabla}\cdot(\widehat{\boldsymbol{u}}_1\widehat{\boldsymbol{u}}_1^* + \widehat{\boldsymbol{u}}_1^*\widehat{\boldsymbol{u}}_1)$

$$(\partial_t + \widehat{v}_0\partial_y + \widehat{w}_0\partial_z)\begin{bmatrix}\widehat{u}_0\\ \widehat{v}_0\\ \widehat{w}_0\end{bmatrix} = -\begin{bmatrix}G\\ \partial_y\widehat{p}_0\\ \partial_z\widehat{p}_0\end{bmatrix} + Re^{-1}(\partial_{yy}+\partial_{zz})\begin{bmatrix}\widehat{u}_0\\ \widehat{v}_0\\ \widehat{w}_0\end{bmatrix} - \begin{bmatrix}\partial_y(\widehat{v}_1\widehat{u}_1^* + \widehat{u}_1^*\widehat{v}_1) + \partial_z(\widehat{u}_1\widehat{w}_1^* + \widehat{w}_1^*\widehat{u}_1)\\ \partial_y(\widehat{v}_1\widehat{v}_1^* + \widehat{v}_1^*\widehat{v}_1) + \partial_z(\widehat{v}_1\widehat{w}_1^* + \widehat{w}_1^*\widehat{v}_1)\\ \partial_y(\widehat{v}_1\widehat{w}_1^* + \widehat{w}_1^*\widehat{v}_1) + \partial_z(\widehat{w}_1\widehat{w}_1^* + \widehat{w}_1^*\widehat{w}_1)\end{bmatrix}$$

First consider that the driving terms become asymptotically small and we have only

$$(\partial_t + \widehat{v}_0\partial_y + \widehat{w}_0\partial_z)\begin{bmatrix}\widehat{u}_0\\ \widehat{v}_0\\ \widehat{w}_0\end{bmatrix} = -\begin{bmatrix}G\\ \partial_y\widehat{p}_0\\ \partial_z\widehat{p}_0\end{bmatrix} + Re^{-1}(\partial_{yy}+\partial_{zz})\begin{bmatrix}\widehat{u}_0\\ \widehat{v}_0\\ \widehat{w}_0\end{bmatrix}$$

Then it is 'easy' to show that the roll components are smaller than the streak component by 1/*Re*.

(This matches/explains what we observed in the asymptotics for roll vs. streak. ✔)

Let $U = \widehat{u}_0,\ V = Re^{-1}\widehat{v}_0,\ W = Re^{-1}\widehat{w}_0,\ P = Re^2\widehat{p}_0,\ T = Re^{-1}t$ and the last equation becomes

$$(\partial_T + V\partial_y + W\partial_z)\begin{bmatrix}U\\ V\\ W\end{bmatrix} = -\begin{bmatrix}Re\,G\\ \partial_y P\\ \partial_z P\end{bmatrix} + (\partial_{yy}+\partial_{zz})\begin{bmatrix}U\\ V\\ W\end{bmatrix}$$

Which is computed at *Re* = 1. We have ignored the Reynolds stress forcing terms here – next we focus on an appropriate asymptotic form for these.

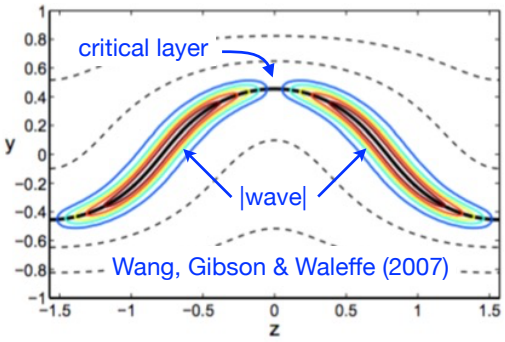
Asymptotics for high *Re*

Wave equation $\partial_t \widehat{\boldsymbol{u}}_1 + \boldsymbol{\nabla}\cdot(\widehat{\boldsymbol{u}}_0\widehat{\boldsymbol{u}}_1 + \widehat{\boldsymbol{u}}_1\widehat{\boldsymbol{u}}_0) = -\boldsymbol{\nabla}\widehat{p}_1 + Re^{-1}\nabla^2\widehat{\boldsymbol{u}}_1$ $\quad \boldsymbol{U} \equiv \widehat{\boldsymbol{u}}_0$ = vortex

i.e. $\partial_t \boldsymbol{u} + \boldsymbol{\nabla}\cdot(\boldsymbol{U}\boldsymbol{u} + \boldsymbol{u}\boldsymbol{U}) = -\boldsymbol{\nabla}p + Re^{-1}\nabla^2\boldsymbol{u}$ $\quad \boldsymbol{u} \equiv \widehat{\boldsymbol{u}}_1$ = wave

The wave is observed to be large at the critical layer where $U(y,z) = 0$.

Why?



Say the wave travels in *x* at speed *c* (= 0 for SW). Then

$$\partial_t(\,) = -c\partial_x(\,) \equiv -\mathrm{i}\alpha c(\,)$$

Since $\boldsymbol{U} \to [U(y,z), 0, 0]$ as $Re \to \infty$ the asymptotic component form of the wave equation is

These terms disappear in critical layer,

$$\mathrm{i}\alpha(U-c)\begin{bmatrix}u\\v\\w\end{bmatrix} + \begin{bmatrix}v\partial_y U + w\partial_z U\\0\\0\end{bmatrix} = -\begin{bmatrix}\mathrm{i}\alpha p\\ \partial_y p\\ \partial_z p\end{bmatrix} + Re^{-1}(-\alpha^2 + \partial_{yy} + \partial_{zz})\begin{bmatrix}u\\v\\w\end{bmatrix}$$

✔

which allows $\boldsymbol{u}(y,z)$ to obtain its largest values near where $U(y,z) = 0$ (the critical layer for *c* = 0).

But: how large (what is scaling) of the wave around the critical layer?

How large is the wave?

From the Orr–Sommerfeld equation one finds the thickness of a critical layer scales like $Re^{-1/3}$.

Reynolds stresses generated by the wave:
are what drives the roll flow V, W.

Scale factor for wave relative to roll.

$$\gamma \begin{bmatrix}0\\ \partial_y(vv^* + v^*v) + \partial_z(vw^* + w^*v)\\ \partial_y(vw^* + w^*v) + \partial_z(ww^* + w^*w)\end{bmatrix}$$

Negligible for streak component.

On condition that these Reynolds stresses are *O*(1) relative to the roll flow, the amplitude of the wave must scale like $Re^{-7/6}$ outside the critical layer (and like $Re^{-5/6}$ within it).

(See Hall & Smith 1991, Hall & Sherwin 2010 for details.)

Consequently the wave is asymptotically smaller than the roll by a factor $Re^{-1/6}$ (since the roll is smaller than the streak by factor Re^{-1}). Then the multiplier above, $\gamma \propto Re^{-1/3}$.

Finally we need a constant of proportionality, ρ, which fixes the (asymptotic) size of the wave.

Hence $\gamma = \rho^2 Re^{-1/3}$. The constant ρ is to be identified during solution.

This leads us back to the (Fourier mode 0) vortex equations, now with their wave forcing terms:

$$(\partial_T + V\partial_y + W\partial_z)\begin{bmatrix}U\\V\\W\end{bmatrix} = -\begin{bmatrix}Re\,G\\ \partial_y P\\ \partial_z P\end{bmatrix} + (\partial_{yy} + \partial_{zz})\begin{bmatrix}U\\V\\W\end{bmatrix} - \rho^2 Re^{-1/3}\begin{bmatrix}0\\ \partial_y(vv^* + v^*v) + \partial_z(vw^* + w^*v)\\ \partial_y(vw^* + w^*v) + \partial_z(ww^* + w^*w)\end{bmatrix}$$

$$-\rho^2 Re^{-1/3}\boldsymbol{\nabla}\cdot(\boldsymbol{u}^*\boldsymbol{u} + \boldsymbol{u}\boldsymbol{u}^*)$$

★ (In the measure of WGW07, this is $O(Re^{-11/12}) = O(Re^{-0.91})$ ✔.)

Summary

Streak + roll (vortex) equation

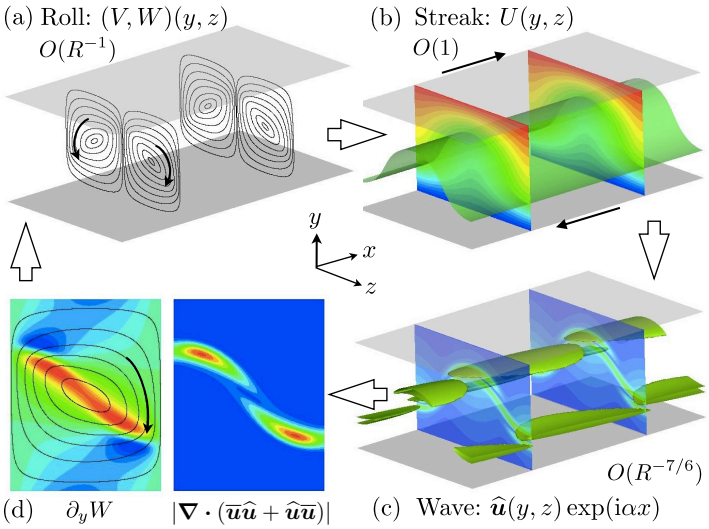
$$(\partial_T + V\partial_y + W\partial_z)\begin{bmatrix}U\\V\\W\end{bmatrix} = -\begin{bmatrix}Re\,G\\ \partial_y P\\ \partial_z P\end{bmatrix} + (\partial_{yy}+\partial_{zz})\begin{bmatrix}U\\V\\W\end{bmatrix} - \rho^2 Re^{-1/3}\begin{bmatrix}0\\ \partial_y(vv^*+v^*v)+\partial_z(vw^*+w^*v)\\ \partial_y(vw^*+w^*v)+\partial_z(ww^*+w^*w)\end{bmatrix}$$

Wave equation

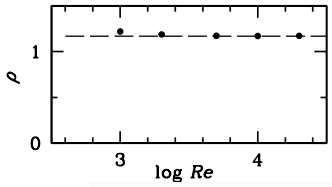
$$\mathrm{i}\alpha(U-c)\begin{bmatrix}u\\v\\w\end{bmatrix} + \begin{bmatrix}v\partial_y U + w\partial_z U\\0\\0\end{bmatrix} = -\begin{bmatrix}\mathrm{i}\alpha p\\ \partial_y p\\ \partial_z p\end{bmatrix} + Re^{-1}(-\alpha^2+\partial_{yy}+\partial_{zz})\begin{bmatrix}u\\v\\w\end{bmatrix}$$

1. The equilibria (flows) we seek are fixed points ($\partial_t(\,) = 0$) of the system of PDEs, which derive from the Navier–Stokes equations.
2. But, they also must have at least one non-growing eigenvalue – the equilibria are non-hyperbolic fixed points. (Actually this is automatic since the equilibria are wavy in x.)
3. We have included Reynolds number as a parameter but may seek asymptotic solutions – in which case we have to check that asymptotic behaviour is approached as $Re \to \infty$ (e.g. $\rho \to$ const.).
4. Solution methods:
   (a) Fixed-point iteration with different methods for vortex and wave (HS, BHS, etc).
   (b) Fully coupled Newton-Raphson: include wave speed c as variable (Nagata, DH).

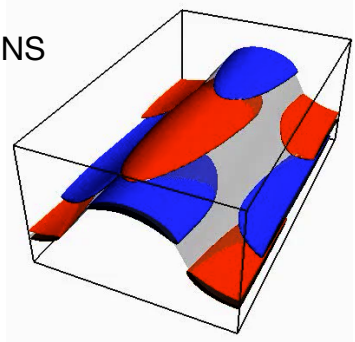
VWI by fixed-point method of BHS (2013)



5. Check ρ becomes independent of wave *Re*.



6. DNS



Fixed point iteration

1. Solve 2D/2C roll NSE at *Re* = 1 driven by 2D/2C components of wave Reynolds stress $-\rho^2 Re^{-1/3}\nabla\cdot(\boldsymbol{u}^*\boldsymbol{u}+\boldsymbol{u}\boldsymbol{u}^*)$
2. Solve 2D/1C streak advection at *Re* = 1 driven by roll velocity field.
3. Solve 3D/3C wave eigensystem at large *Re* using 2D/1C streak as base flow.
4. Adjust either α or ρ to achieve neutrally stable wave eigensystem.

Part 2. Distributed Vortex-Wave Interactions

Triply periodic VWI for an infinite shear flow

In the context of an infinite shear flow with background variation $\boldsymbol{U} = (y, 0, 0)$ we can superimpose an array of triply-periodic streak/roll/wave variations.

This is incorporated into the same framework as we used for PCF, simply by changing base flow, and allowing vertical periodic variations in streak and roll structure:

$$\boldsymbol{U}(x,y,z) = (y + U, \frac{V}{Re}, \frac{W}{Re})(y,z) + \frac{\rho \mathrm{e}^{\mathrm{i}\alpha^\dagger x}}{Re^{7/6}} \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}n\alpha^\dagger t}(u,v,w)(y-n,z) + \mathrm{c.c.}$$

$y \equiv \partial\overline{u}/\partial y$

(A vertically stacked array of waves.)

where streak and roll $(U, V, W)(y, z)$, are periodic in y and z over lengths 1 and $2\pi/\beta^\dagger$

$$(U,V,W)(y,z) = (U,V,W)(y \pm n, z \pm 2\pi m/\beta^\dagger), \quad n, m \in \mathbb{I}$$

The wave speed in each vertical module is $\pm n\alpha^\dagger$, i.e. is zero relative to a critical layer in each module. In the base module, we have a SW; in all other modules, TWs.

Each wave, here periodic in *x* and *z*, is unlimited in vertical extent *y.*

But owing to the concentration of wave amplitude around one critical layer, each roll module principally feels a driving force from the wave at rest inside that module.

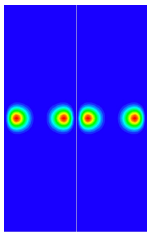
We now have space-filling structure that is periodic in *x, y,* and *z.*

Building blocks 1. – Flat critical layer

A.k.a. mirror-symmetric solutions.

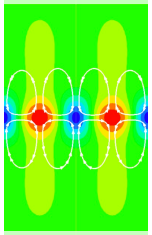
These can be maintained by enforcing *z*-reflection symmetry on the wave.

Wave forcing of roll



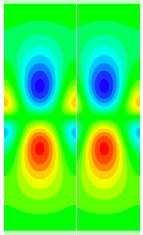
$|\nabla \cdot (\boldsymbol{u}^*\boldsymbol{u} + \boldsymbol{u}\boldsymbol{u}^*)|^2$

Roll



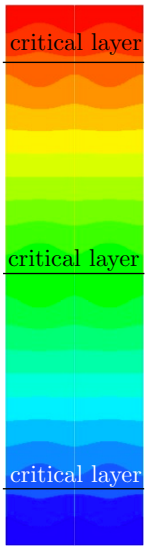
(V, W, P)

Streak



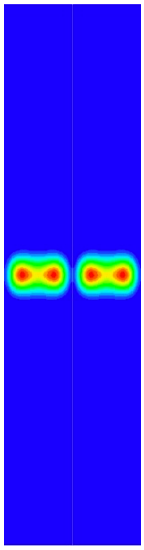
U

Wave base



$y + U$

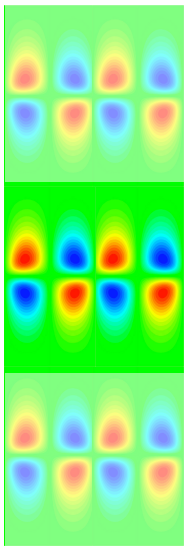
Wave mode KE



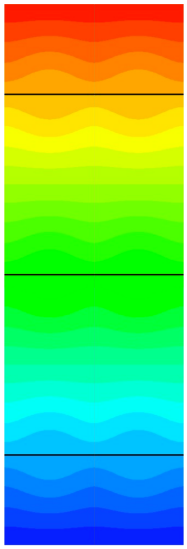
$|\boldsymbol{u}|^2$

Building blocks 1. – Flat critical layer

Roll streamfunction

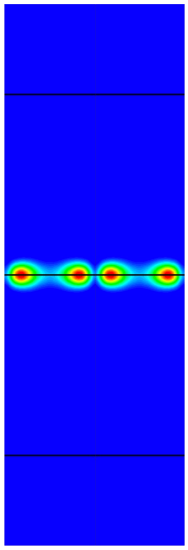


Wave base



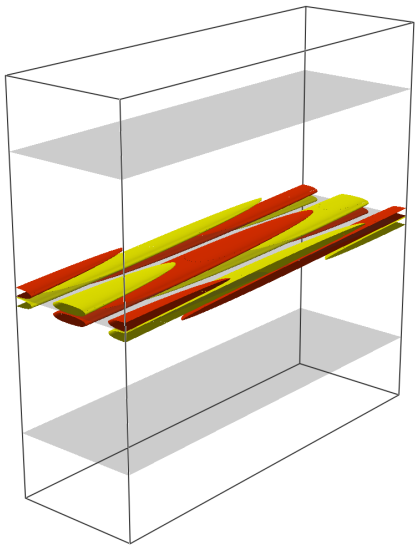
$y + U$

Wave mode KE



$|\boldsymbol{u}|^2$

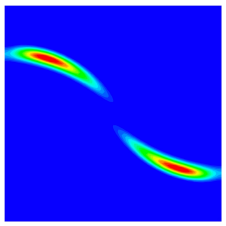
Wave streamwise vorticity



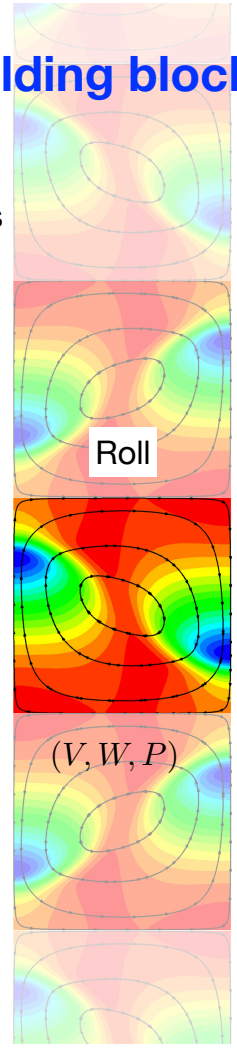
Building blocks 2. – Sinuous critical layer

This set of solutions has an infinite array of counter-rotating rolls.

Wave forcing of roll



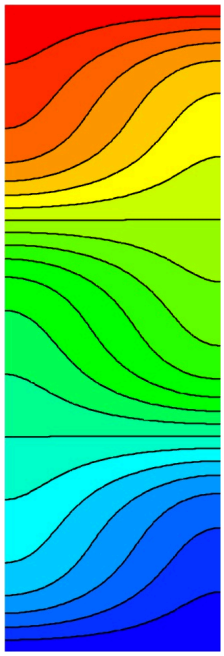
$|\nabla \cdot (\boldsymbol{u}^*\boldsymbol{u} + \boldsymbol{u}\boldsymbol{u}^*)|^2$



(V, W, P)

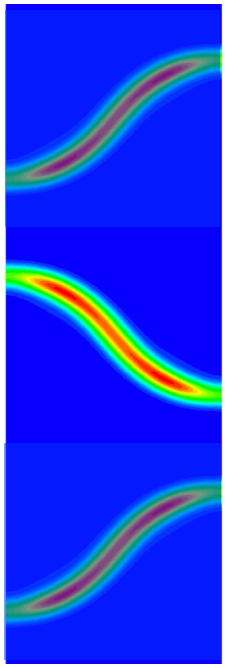
U

Wave base



$y + U$

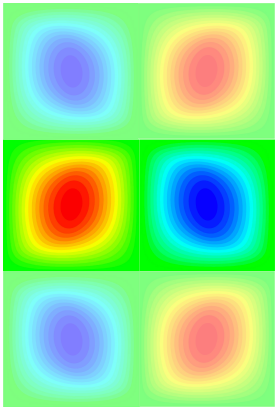
Wave mode KE



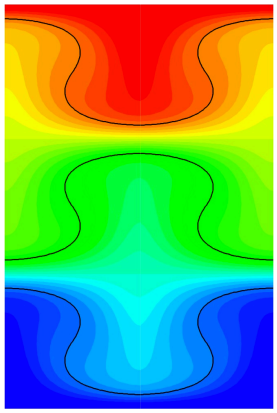
$|\boldsymbol{u}|^2$

Building blocks 2. – Sinuous critical layer

Roll streamfunction

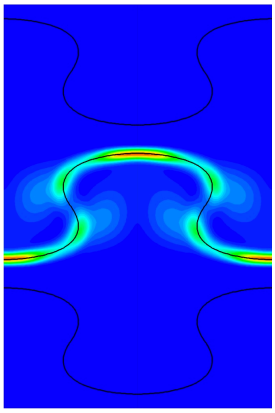


Wave base



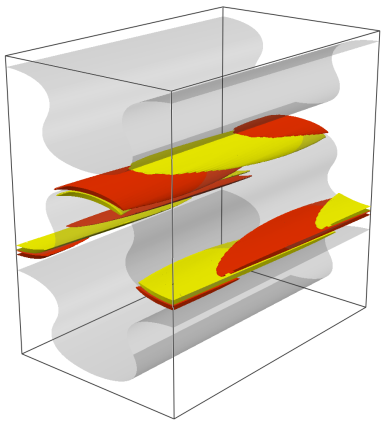
$y + U$

Wave mode KE



$|\boldsymbol{u}|^2$

Wave streamwise vorticity

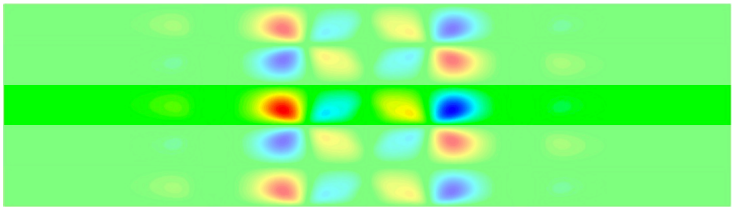


Building blocks 3. – Spatial intermittency

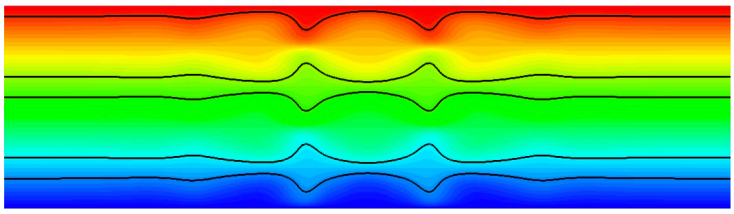
Supposing we let the domain become very wide, i.e. $\beta^\dagger \to 0$.

Then the spanwise variation localizes near the centre of the domain:

Roll streamfunction

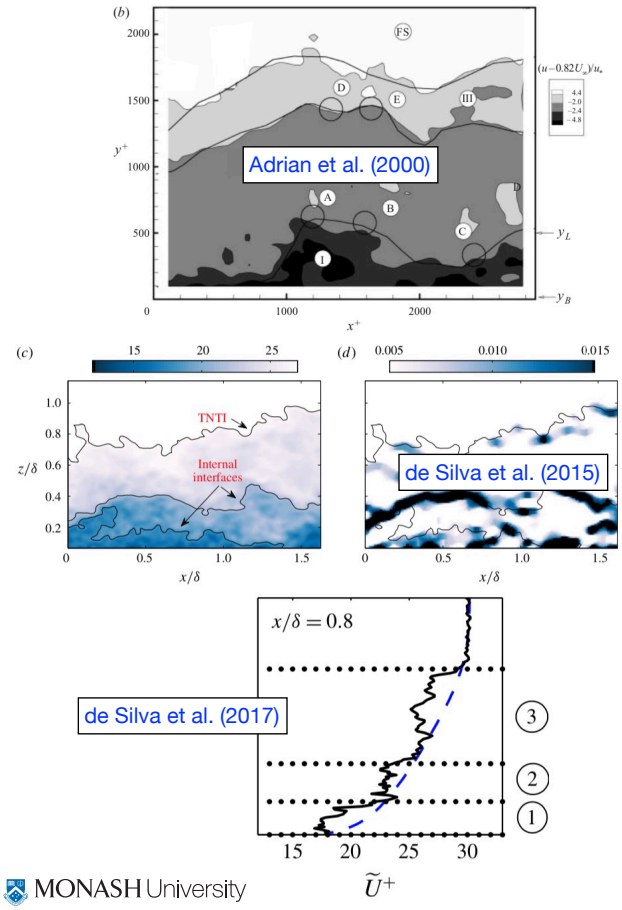


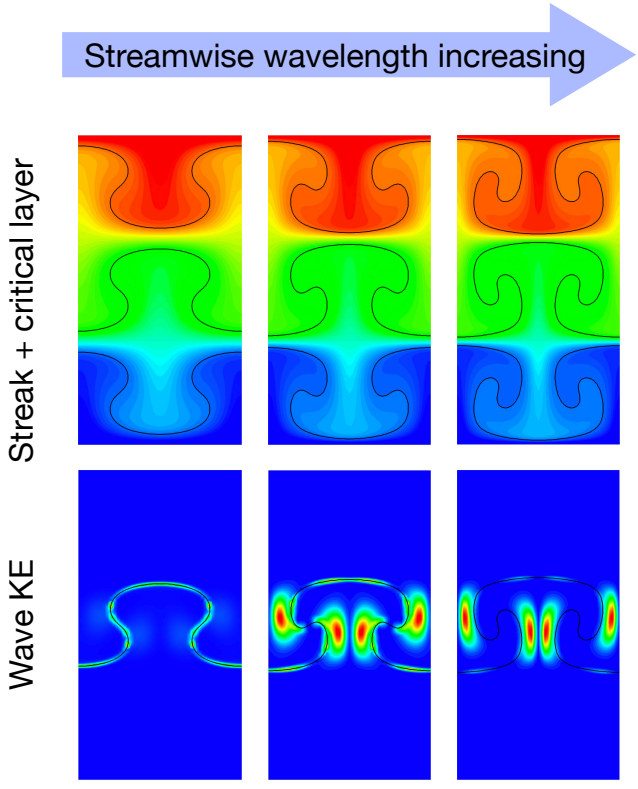
Streak and CLs



Such solutions will play a key role in the model we will develop below for wall-normal self-similarity.

Development of uniform momentum zones





Part 3. Slow wall-normal variation, self-similarity, and the log law

How to generalise to slowly-varying states

Navier–Stokes equations written in wall units based on friction velocity $u_\tau = \sqrt{\tau_w/\rho}$

i.e. with length scale ν/u_τ and time scale ν/u_τ^2 .

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = -\boldsymbol{\nabla}p + \nabla^2\boldsymbol{u} \qquad \text{N.B.} \quad Re = 1$$

Need some large parameter – an obvious one is the distance from the wall.

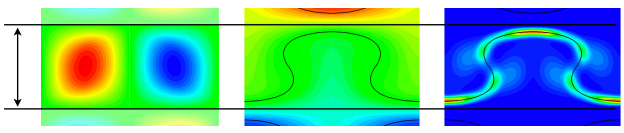
Let the (large) distance from the wall scale like $1/\epsilon$ where $\epsilon \ll 1$

and define slow variables $Y = \epsilon y$ and $T = \epsilon^2 t$

Plan is to stretch our VWI array in these slow variables such that each local cell depth is smaller than $O(1/\epsilon)$ in wall units.

The VWI cell depth in the rescaled length is δ

(in wall units it was δ/ϵ).



Introduce a WKB phase function $\Phi = \Theta(Y,T)/\delta$.

The VWI array is now of unit period in Φ, variation at slower scale accommodated by Θ.

This also implies $\dfrac{\partial}{\partial y}(\ ) \approx \left(\dfrac{\epsilon}{\delta}\dfrac{\partial\Theta}{\partial Y}\right)\dfrac{\partial}{\partial\Phi}(\ )$ with local `(wall-) normal wavenumber' $\dfrac{\partial\Theta}{\partial Y} \equiv \Theta_Y$.

How to generalise to slowly-varying states

Letting $(X, Z) = (\epsilon/\delta)(\alpha x, \beta z)$ where α, β are constants for a set of solutions,

$$u = \frac{\overline{u}(Y)}{\epsilon} + \frac{\delta \overline{u}_Y}{\epsilon \Theta_Y}\{U_1(\Phi, Y, Z) + \delta U_2(\Phi, Y, Z) + \cdots\} + \sum_{n=-\infty}^{\infty} u_w^{(n)},$$

$$v = \frac{\epsilon \Theta_Y}{\delta}\{V_1(\Phi, Y, Z) + \delta V_2(\Phi, Y, Z) + \cdots\} + \sum_{n=-\infty}^{\infty} v_w^{(n)},$$

$$w = \frac{\epsilon \Theta_Y}{\delta}\{W_1(\Phi, Y, Z) + \delta W_2(\Phi, Y, Z) + \cdots\} + \sum_{n=-\infty}^{\infty} w_w^{(n)}$$

Which we see is similar in form to the unscaled equivalent

Array of *n* waves

$$\boldsymbol{U}(x, y, z) = (y + U, \frac{V}{Re}, \frac{W}{Re})(y, z) + \frac{\rho e^{i\alpha^\dagger x}}{Re^{7/6}} \sum_{n=-\infty}^{+\infty} e^{-in\alpha^\dagger t}(u, v, w)(y - n, z) + \text{c.c.}$$

(Similar considerations apply to the asymptotic scaling of waves relative to the streak and roll as in the original unscaled equivalent.)

To make the match complete also requires scaling of wavenumbers $(\alpha^\dagger, \beta^\dagger) = \left(\frac{\alpha}{\Theta_Y}, \frac{\beta}{\Theta_Y}\right)$

which gives us e.g. $\boldsymbol{U}_1(\Phi, Y, Z; \alpha, \beta, \Theta_Y, \overline{u}_Y) = \boldsymbol{U}(\Phi, Z; \alpha^\dagger, \beta^\dagger)$ i.e. identical forms.

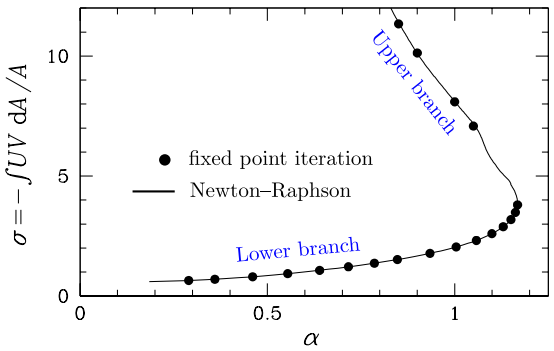
The slow wall-normal variation in mean flow

Integrating the streamwise momentum equation over an (X, Φ, Z) periodic cell we obtain

$$\frac{\partial \overline{u}}{\partial T} = \frac{\partial}{\partial Y}\left(\frac{\partial \overline{u}}{\partial Y}\sigma\right) + \frac{\partial^2 \overline{u}}{\partial Y^2}$$ (effectively a RANS equation for the mean flow),

where $\sigma = -\langle U_1 V_1 \rangle = -\frac{1}{2\pi}\int\int U_1 V_1 \, \mathrm{d}\Phi \mathrm{d}Z$ – the mean roll–streak vertical momentum flux – is a property of the VWI array solution.

A central goal here is to establish the form of the mean flow variation with distance from the wall, i.e. $\overline{u}(Y)$.



To obtain consistency at second order further requires another time-evolution equation

$$e_0 \frac{\Theta_T}{\Theta_Y} + e_1 \left(\frac{\alpha}{\Theta_Y}\right)_Y + e_2 \left(\frac{\beta}{\Theta_Y}\right)_Y + e_3 \frac{\Theta_{YY}}{\Theta_Y} = 0$$

where the constants $e_0, \ldots e_3$ can in principle be obtained (but are not directly required)…

(During development of this constraint, we find that $\delta = \epsilon^{2/5}$).

Self-similarity

In the present development we can only obtain perfect self-similarity in *y* for the spanwise-localised solutions, which leads us to

$$\sigma \to \beta^\dagger \sigma_0(\alpha^\dagger), \qquad \frac{e_0}{e_3} \to B_0(\alpha^\dagger), \qquad \frac{e_2}{e_3} \to \frac{B_2(\alpha^\dagger)}{\beta^\dagger} \quad \text{on assumption of} \quad \beta^\dagger \to 0$$

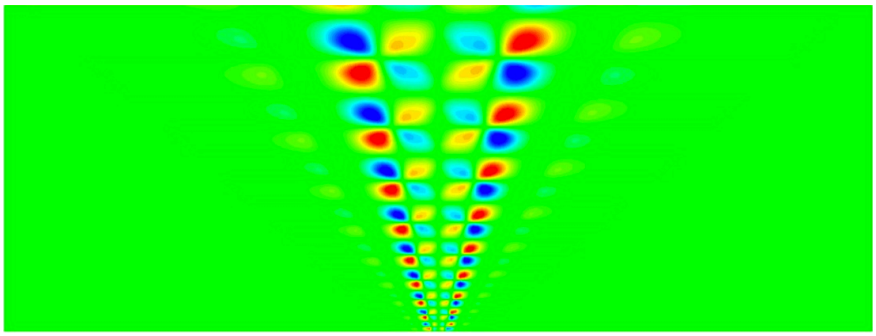
In which case the two slow-time equations become

$$\overline{u}_T = \sigma_0 \beta \left(\frac{\overline{u}_Y}{\Theta_Y} \right)_Y + \overline{u}_{YY} \quad \text{and} \quad B_0 \Theta_T + (1 - B_2) \frac{\Theta_{YY}}{\Theta_Y} = 0$$

the second requires $B_2(\alpha^\dagger) = 1$ to reach steady state.

For consistency with the attached-eddy hypothesis the cell depth has to vary linearly in *Y*, with the 'normal wavenumber' $\Theta_Y = \dfrac{1}{Y - Y_0}$ assuming self-similarity commences at $Y = Y_0$.

For the case $Y_0 = 0$ we can map our preceding spanwise-localised solution in coords $(\widehat{y}, \widehat{z})$ using $\widehat{y} = \dfrac{\ln y}{\delta}, \quad \widehat{z} = \dfrac{z}{\delta y}$



Log-law

Recall the two slow-variables equations for the mean flow $\overline{u}(Y, T)$ which are now

$$\overline{u}_T = \sigma_0 \beta \left(\frac{\overline{u}_Y}{\Theta_Y} \right)_Y + \overline{u}_{YY} \quad \text{and} \quad B_0 \Theta_T + (1 - B_2) \frac{\Theta_{YY}}{\Theta_Y} = 0$$

The second requires $B_2(\alpha^\dagger) = 1$ to reach steady state; on assuming this, we have the ODE

$$\sigma_0 \beta \left(\frac{\overline{u}_Y}{\Theta_Y} \right)_Y + \overline{u}_{YY} = 0 \qquad \text{(RANS-type momentum…)}$$

which, further assuming attached-eddy type scaling in *Y*, i.e. $\Theta_Y = (Y - Y_0)^{-1}$, we integrate to obtain

$$\overline{u}(Y) = C_1 \ln \left(Y - Y_0 + \frac{1}{\sigma_0 \beta} \right) + C_2,$$

If we shift the origin using $Y_0 = 1/(\sigma_0 \beta)$ and map back to wall units using $y = Y/\epsilon$ we obtain the Karman law of the wall

$$u_m(y) = \frac{1}{\kappa} \ln y + C$$

(But we are still left the task of evaluating constants κ and C, likely by asymptotic matching to the edges of the buffer or wake regions.)

Outcomes

1. Straightforward generalisation of bi-periodic exact solutions to the Navier–Stokes equations, thus far mostly obtained in wall-bounded flows, allows development of space-filling triply periodic arrays set in the background of infinite shear flow.
2. Examined solutions with two kinds of symmetry (other symmetries possible). Solutions can localise in z when spanwise wavelengths are large. Also it is possible to obtain uniform momentum zones.
3. The Navier–Stokes equations in wall units do not contain a large parameter – it is natural to use the distance from the wall for this.
4. Allowed for slow wall-normal variation of the periodic scales using the WKB approach. That led to the development of two slow-scales equations: a RANS-type equation for the streamwise-average momentum and another that acts as a constraint.
5. Using the properties of localised solutions, assuming a mean flow steady in time, and vertical self-similarity consistent with the attached eddy hypothesis, we can obtain a logarithmic Karman-type equation for the mean velocity profile from the RANS-type equation (but leaving the constants unknown).