Adapting the spectral vanishing viscosity method for large-eddy simulations in cylindrical configurations

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Abstract
During the last decade the spectral vanishing viscosity (SVV) method has been adopted successfully for large-eddy-type simulations (LES) with high-order discretizations in both Cartesian and cylindrical coordinate systems. For the latter case, however, previous studies were confined to annular domains. In the present work, we examine the applicability of SVV in cylindrical coordinates to flows in which the axis region is included, within the setting of an exponentially convergent spectral element–Fourier discretization. In addition to the ‘standard’ SVV viscosity kernel, three modified kernels with enhanced stabilization in the axis region are considered. Three fluid flow examples are considered, including turbulent pipe flow. The results, on the one hand, show a surprisingly small influence of the SVV kernel, while on the other, they reveal the importance of spatial resolution in the axis region.

Key words: Large-eddy simulation, Spectral vanishing viscosity, Cylindrical coordinates, Spectral element method, Pipe flow

1. Motivation
The spectral vanishing viscosity (SVV) method was introduced by Tadmor [1] as a concept for stabilising Fourier spectral approximations of hy-
perbolic conservation laws. The main idea is to define a viscosity in spectral space that reaches a maximum at high wave numbers, but which vanishes for wave numbers below a resolution-dependent threshold. In this way stabilization is achieved through damping at fine length scales without degrading the convergence properties of the underlying discretization. Maday et al. [2] applied the SVV approach to Legendre spectral methods and introduced a smooth viscosity kernel, which was commonly adopted by subsequent researchers. Karamanos & Karniadakis [3] incorporated the method into a modal spectral element discretization of the incompressible Navier–Stokes equations and employed it for LES of turbulent channel flow. Xu & Paquetti [4] adapted the SVV technique to the nodal spectral element method and investigated the importance of SVV parameters in applications to high-Reynolds number flows. Following these works, several researchers advocated SVV as a tool for LES, pointing out its conceptual simplicity and inherent spectral accuracy [5, 6, 7, 8, 9, 10]. In particular this preservation of spectral convergence when the flow is well resolved renders SVV an attractive technique for LES when turbulent, transitional or laminar zones may co-exist.

Our motivation to explore the SVV technique emerged from studies of electromagnetic stirring of metal and semiconductor melts. Depending on the applied magnetic field, the flow may be turbulent only in a part of the container, or throughout the whole volume [11]. Although the degree of turbulence can be considerable, the flow tends to preserve a transitional character. Since most container configurations in stirring applications have rotational symmetry, we are interested particularly in an SVV formulation that is suitable for cylindrical coordinates. To our knowledge, Serre and co-workers are the only group to have considered a corresponding approach to cylindrical coordinate configurations [8, 9, 10]. They incorporated the SVV technique into a Chebychev–Fourier spectral method and employed it to investigate turbulent rotor–stator flows including heat transfer. However, these studies were confined to annular domains and thus the results do not guarantee applicability to problems that include the axis. As will be shown below, the approach proposed in [8] is characterised by a loss of azimuthal SVV stabilization near the axis for any resolved mode and, as a result, instabilities may occur.

The objective of this paper is to provide a validation of SVV for axisymmetric geometric configurations that may include the core region and to investigate possible remedies for near-axis instability. To that end, we first briefly recapitulate in § 2 the principles of SVV in the setting of the one-
dimensional diffusion equation. Following this, a detailed description is provided for development of SVV in a nodal spectral element–Fourier method for three-dimensional incompressible flows in cylindrical coordinates. We note that the underlying numerical method into which SVV was incorporated has previously demonstrated spectral convergence properties for direct numerical simulation (DNS) of flows in cylindrical coordinates where the axis is included [12]. In addition to the ‘standard’ SVV viscosity kernel, three modified kernels with enhanced stabilization in the axis region are introduced. In § 3 the SVV approach is applied to three different flow problems. As a first step, the spectral convergence of the method and the different kernels is verified by means of Kovasznay flow, as in the previous treatment [12]. Further assessment is based on LES of turbulent pipe flow at $Re_D = 10000$ and the turbulent flow driven by a travelling magnetic field in a cylinder. In both these cases DNS serves as reference. In particular, we want to assess how far modifications of the SVV kernel or adaptions of the spatial resolution affect the flow characteristics near the axis. Finally, in § 4, we summarize the main results and provide concluding remarks.

We note that while we have here focussed on problems set in domains with a cylindrical geometry as well as coordinate system, the underlying discretization supports more general axisymmetric geometries. Additionally it will deal with three-dimensional Cartesian geometries which possess a homogeneous direction.

2. SVV spectral element–Fourier method

We consider the incompressible unsteady Navier–Stokes equations

$$\begin{align*}
\partial_t u + N(u) &= -\rho^{-1}\nabla p + \nu\nabla^2 u + \rho^{-1} f \\
\nabla \cdot u &= 0
\end{align*}$$

(1)

where $u$ represents the velocity, $p$ the pressure, $\rho$ and $\nu$ the density and kinematic viscosity of the fluid, and, if any exists, $f$ the volume force. The nonlinear advection term $N(u)$ can be composed in a number of ways which are equivalent for continua, but have different properties in discrete form. Here, the skew-symmetric form i.e. $N(u) = (u \cdot \nabla u + \nabla \cdot uu)/2$ is employed for robustness.

The equations, expressed above in coordinate-free form, are subsequently written in cylindrical coordinate component form and discretized using a
nodal spectral element method in the meridional \((z, r)\) semi-plane and a Fourier spectral method in the azimuthal \((\varphi)\) direction. Time integration is based on a second order velocity correction scheme with implicit treatment of diffusion. For details of the method and demonstration of its exponential convergence properties when implemented as a DNS technique we refer to [12].

2.1. Principles of the spectral vanishing viscosity method

To recap the principal ideas of the SVV method we start with the one-dimensional diffusion equation

\[ \partial_t u = \nu \partial_x \partial_x u. \]  

(2)

When the Fourier method is used for spatial discretization, the scalar \(u\) is decomposed in its complex Fourier modes \(\hat{u}_k\) by

\[ u(x, t) = \sum_{k=-\infty}^{\infty} \hat{u}_k(t) \exp(ikx). \]  

(3)

Taking into account that \(\partial_x(\cdot)_k \equiv ik(\cdot)_k\), the Fourier transform of the diffusion equation (2) can be written as

\[ \partial_t \hat{u}_k = -\nu k^2 \hat{u}_k \text{ with } -\infty \leq k \leq \infty. \]  

(4)

As mentioned in § 1, the idea underlying SVV is to add a spectral viscosity term that reaches its maximum at the highest wave numbers, but vanishes for wave numbers below a resolution-dependent threshold. This is achieved by augmenting the diffusivity by the term \(\varepsilon \hat{Q}_k\) on the right hand side of (4):

\[ \partial_t \hat{u}_k = -(\nu + \varepsilon \hat{Q}_k)k^2 \hat{u}_k, \]  

(5)

where constant \(\varepsilon\) is the maximum amplitude of the spectral viscosity and \(\hat{Q}_k\) is a modal shape function, referred to as the SVV kernel. The multiplication with \(\hat{Q}_k\) in spectral space corresponds to a convolution with \(Q\) in physical space. The SVV-stabilized diffusion equation thus reads in physical space

\[ \partial_t u = \nu \partial_x \partial_x u + \varepsilon \partial_x (Q \ast \partial_x u), \]  

(6)

with the convolution operator \(\ast\).
The exact forward and inverse Fourier transform requires an infinite number of modes (cf. 3), however, only a finite number of modes are retained in practice. In addition the conjugate-symmetric property of the Fourier transform of real variables may be exploited, so that only the positive-$k$ modes are required. Consequently, the modal SVV kernel $\hat{Q}_k$ has only to be defined for positive wave numbers $0 \leq k \leq N$ in which $N$ is the wave number of the highest considered mode. Following Maday et al. [2] most authors have adopted the kernel given by

$$\hat{Q}_k = \hat{Q}(k) = \exp \left( -\frac{(N - k)^2}{(M - k)^2} \right), \quad M < k \leq N$$

and $\hat{Q}_k = 0$ for $0 \leq k \leq M$, where $M$ is the mode above which SVV is applied. Figure 1 shows this kernel for $N = 31$ and various $M$.

![Figure 1: Standard SVV kernel magnitude (7) as a function of wave number $k$.](image)

When a nodal spectral element method is applied to the one-dimensional diffusion equation (2), the computational domain is split into elements and a polynomial expansion of the scalar $u$ with a nodal function basis is employed within each element. Without loss of generality, we restrict the following derivation to the one-element case in the standard interval $\Omega = [-1, 1]$. Furthermore, homogeneous boundary conditions — either Dirichlet or Neumann conditions — are presumed. Using the Lagrange cardinal functions $h_i$, $0 \leq i \leq P$ with the Gauß–Lobatto–Legendre (GLL) quadrature points $x_i$ as an expansion basis, the projection of the scalar is given by

$$u(x, t) \approx \sum_{i=0}^{P} u_i(t)h_i(x),$$

(8)
where \( u_i \) is the scalar value at point \( x_i \) and \( P \) the polynomial order of the GLL-Lagrange basis. With \( \dot{u}_i(t) = \frac{du_i(t)}{dt} \) and \( h'_i(x) = \frac{dh_i(x)}{dx} \) the Galerkin weak formulation of (2) then reads

\[
\int_{\Omega^*} \left( \sum_{i=0}^{P} \dot{u}_i h_i \right) h_j \, d\Omega = -\int_{\Omega^*} \left( \sum_{i=0}^{P} u_i h'_i \right) h'_j \, d\Omega 
\]  

(9)

or in matrix-vector notation

\[
\sum_{i=0}^{P} \dot{u}_i M_{ij} = -\sum_{i=0}^{P} u_i L_{ij} 
\]  

(10)

where \( M \) and \( L \) are respectively the mass and Laplacian matrices with

\[
M_{ij} = \int_{\Omega^*} h_i h_j \, d\Omega \quad \text{and} \quad L_{ij} = \int_{\Omega^*} h'_i h'_j \, d\Omega \quad \text{for} \quad 0 \leq i, j \leq P. 
\]  

(11)

Owing to use of GLL quadrature on the underlying nodal points, the mass matrix is diagonal, with the leading diagonal containing the quadrature weights \( \omega_i \) and the Laplace matrix can be calculated by

\[
L_{ij} = \sum_{l=0}^{P} \omega_l D_{li} D_{lj} 
\]  

(12)

where \( D \) is the GLL differentiation matrix. See [13] for more detail.

When incorporating SVV into the nodal spectral element method, two aspects must be considered. First, the GLL–Lagrange basis is not hierarchical, i.e. it does not have the property that smaller length scales are associated with increasing mode index. Projection into a suitable modal space is required before a spectrally vanishing viscosity can be adjoined to (9). This projection can be realized in various ways, e.g. by a mapping from nodal space to Legendre spectral space, or to the space of Jacobi-modal basis functions as employed for the \( hp \) form of the spectral element method (another hierarchical basis). In the present work we have restricted consideration to the Legendre polynomial space. The associated passage matrix \( T \) from physical to modal space is variously referred to as a discrete polynomial transform (DPT) matrix or a generalized Vandermonde matrix. See [14] Appendix A for details of how \( T \) may be computed. The second consideration is that the
Laplacian matrix $L$ is symmetric, positive semi-definite, and it is required that SVV stabilization preserves these properties.

In order to satisfy both requirements, Xu & Pasquetti [4] proposed a symmetric form, where the SVV kernel is split by root extraction. Subsequently the kernel is applied to the standard differentiation matrix applied to diffusion terms by counterposing it between the DPT matrix $T$ and its inverse $T^{-1}$. Thus, the modified differentiation matrix is given by

$$D_{SVV} = T^{-1} \text{diag} \left(1 + \frac{\varepsilon}{\nu} \hat{Q}_k \right)^{1/2} T D,$$

where $\hat{Q}_k$ are the coefficients of the SVV kernel in (Legendre–) modal space which are defined in the same manner as presented in (7). The only difference is that, here, the polynomial order $P$ replaces the highest Fourier wave number $N$.

As outlined above, the incorporation of SVV into the nodal spectral element is not quite so straight-forward as for the Fourier method, and has no simple direct counterpart in physical space equivalent to a global convolution. However, the principles are quite similar in each setting. Since below we will discuss the application of SVV to the combined spectral element–Fourier method for three-dimensional geometries (in which the spectral element discretization is orthogonal to the Fourier coordinate), we introduce for brevity the following symbolic notation to indicate SVV stabilization:

$$\varepsilon \partial_{x} Q \partial_{x} u,$$

where $Q$ is regarded as an operator as opposed to an ordinary factor. In both cases — for the Fourier method as well as for the nodal spectral element method — augmentation of the diffusion equation with SVV involves only slight changes to the solution algorithm. In the first case, only the value $\nu k^2$ has to be modified for each mode, while in the second, the standard differentiation matrix employed for diffusion operators has only to be manipulated once, during preprocessing. Consequently, the computational cost of either a direct solve or one PCG-type iteration step remains unchanged while additional damping is applied at small length scales, and this, together with the fact that the formal convergence properties may be unaltered, makes SVV an attractive method.

2.2. SVV for three-dimensional flow equations in cylindrical coordinates

As seen above, SVV influences exclusively the diffusion term of a differential equation. In the incompressible unsteady Navier–Stokes equations (1),
this term reads in cylindrical coordinates

\[ \nu \nabla^2 u = \nu \Delta u - \frac{\nu}{r^2} \begin{bmatrix} 0 \\ v + 2 \partial_z w \\ w - 2 \partial_v v \end{bmatrix} \]  \tag{15} \]

where

\[ \Delta \equiv \partial_z \partial_z + \frac{1}{r} \partial_r r \partial_r + \frac{1}{r^2} \partial_\varphi \partial_\varphi \]  \tag{16} \]

and \( u = (u, v, w)^T \) are the velocity components in axial, radial and azimuthal direction (\( z, r, \varphi \)), respectively. After Fourier transform, diagonalization with mapped variables \( \tilde{v}_k = \hat{v}_k + i \hat{w}_k \) and \( \tilde{w}_k = \hat{v}_k - i \hat{w}_k \) and symmetrization by multiplication with \( r \) (see [12]) we arrive at

\[ r \left[ \nabla^2 u \right]_k \sim \begin{bmatrix} (\partial_z r \partial_z + \partial_r r \partial_r - \frac{k^2}{r^2}) \hat{u}_k \\ (\partial_z r \partial_z + \partial_r r \partial_r - \frac{(k+1)^2}{r^2}) \hat{v}_k \\ (\partial_z r \partial_z + \partial_r r \partial_r - \frac{(k-1)^2}{r^2}) \hat{w}_k \end{bmatrix}. \]  \tag{17} \]

Following [4, 8] we employ a diagonal SVV tensor, i.e. each coordinate direction is provided with its own intensity \( \varepsilon \) and kernel \( Q \):

\[ \varepsilon Q = \text{diag}(\varepsilon_z Q_z, \varepsilon_r Q_r, \varepsilon_\varphi Q_\varphi) \]  \tag{18} \]

Thus, the spectral viscosity term reads in symbolic notation

\[ \nabla \cdot \varepsilon Q \nabla u = \Delta_{svv} u - \frac{\varepsilon \varphi}{r^2} \begin{bmatrix} 0 \\ Q_\varphi(v + 2 \partial_z w) \\ Q_\varphi(w - 2 \partial_v v) \end{bmatrix} \]  \tag{19} \]

with

\[ \Delta_{svv} \equiv \varepsilon_z \partial_z Q_z \partial_z + \frac{\varepsilon_r}{r} \partial_r r Q_r \partial_r + \frac{\varepsilon_\varphi}{r^2} \partial_\varphi Q_\varphi \partial_\varphi. \]  \tag{20} \]

Since diagonalization and symmetrization are linear operations, the SVV term (19) can be transformed in Fourier direction in the same way as the physical diffusion term (15). The composition of both terms yields

\[ r \left[ \nu \nabla^2 u + \nabla \cdot \varepsilon Q \nabla u \right]_k \sim \nu \begin{bmatrix} \left( \partial_z r \partial_z^{svv} + \partial_r r \partial_r^{svv} - \frac{k^2}{r^2}(1 + \frac{\varepsilon \varphi}{\nu} Q_\varphi,k) \right) \hat{u}_k \\ \left( \partial_z r \partial_z^{svv} + \partial_r r \partial_r^{svv} - \frac{(k+1)^2}{r^2}(1 + \frac{\varepsilon \varphi}{\nu} Q_\varphi,k) \right) \hat{v}_k \\ \left( \partial_z r \partial_z^{svv} + \partial_r r \partial_r^{svv} - \frac{(k-1)^2}{r^2}(1 + \frac{\varepsilon \varphi}{\nu} Q_\varphi,k) \right) \hat{w}_k \end{bmatrix}. \]  \tag{21} \]
where \( \hat{Q}_{\phi,k} \) represents the Fourier transformed, azimuthal SVV kernel and

\[
\partial_z r \partial_z \hat{svv} = \partial_z r (1 + \frac{\varepsilon}{\nu} \hat{Q}_z) \partial_z \quad \text{and} \quad \partial_r r \partial_r \hat{svv} = \partial_r r (1 + \frac{\varepsilon}{\nu} \hat{Q}_r) \partial_r
\]

(22)

stands for SVV modifications in the spectral element directions, as described in the previous section. Owing to the use of quadrilateral elements, the tensorial structure of the expansion basis can be exploited for an efficient evaluation of the derivatives, which can be expressed as a series of the one-dimensional operations. In particular, the two-dimensional differentiation matrices can be constructed using the one-dimensional standard differentiation matrix, which can be augmented by SVV as shown in (13).

2.3. Kernel modifications in azimuthal Fourier direction

At this point it should be noted that in Cartesian coordinates each wave number \( k \) is associated with a unique wave length, which is given by \( \lambda = \frac{2\pi L}{k} \), where \( L \) represents a geometric length scale. Hence, waves of length \( \lambda \) are uniformly damped according to the spectral viscosity \( \varepsilon \hat{Q}(k) \). In cylindrical coordinates, this relation is still valid for the radial and axial directions. For the azimuthal direction, however, the wave length is given by \( \lambda = \frac{2\pi r}{k} \), i.e. for a fixed wave number the length varies linearly with the radial position. Consequently, the effect of the azimuthal spectral viscosity \( \varepsilon_{\phi} \hat{Q}(k) \) to the fluid flow depends on the radial position. To clarify this we introduce \( \hat{Q}_{\phi,\lambda} \) with

\[
\hat{Q}_{\phi,\lambda} = \hat{Q}_{\phi}(\lambda, r) = \hat{Q}_{\phi}\left(\frac{\lambda}{2\pi r}\right).
\]

(23)

Since only the ratio of \( \lambda \) and \( r \) is required, (23) is equivalent to

\[
\hat{Q}_{\phi,\lambda} = \hat{Q}_{\phi}\left(\frac{\lambda^*}{2\pi r^*}\right),
\]

(24)

with nondimensional wave length \( \lambda^* = \lambda/R \) and radius \( r^* = r/R \), where \( R \) is any reference radius. In the present work, the maximum radius of the spatial domain is always chosen as reference. Fig. 2 shows a comparison of the \( \hat{Q}_{\phi,k} \) and \( \hat{Q}_{\phi,\lambda} \) for \( N = 31 \) and \( M = 16 \). Clearly the threshold wave length \( \lambda_M^* = 2\pi r^*/M \), above which \( \hat{Q}_{\phi,\lambda} \) vanishes, becomes infinitely small for \( r \) approaching zero. More precisely, it can be shown that for \( r^* < M/N \) the value of \( \lambda_M^* \) falls below the shortest wave length \( \lambda_{min}^* = 2\pi/N \), that can be resolved at the outermost radial position. Thus, if we consider e.g.
$M = N/2$, this means that the azimuthal SVV stabilization becomes ineffective for $r^* < 1/2$. Hence, small structures with length scales of order $\lambda_{\text{min}}$ may accumulate in the core region and, possibly, destabilize the numerical method. Instabilities of this type were actually observed in the numerical studies and will be further detailed below.

Figure 2: Standard SVV kernel as a function of wave number $k$ and nondimensional wave length $\lambda^*$ with $N = 31, M = 16$.

In order to avoid the axis-region instability associated with the standard approach, we consider the following modifications of the SVV kernel:

- **Q1:** Down-scaling of the spectral viscosity activation mode $M$.
  \[
  \hat{Q}^{(1)}_{\varphi,k} = \exp \left( -\frac{(N - k)^2}{(Mr^* - k)^2} \right)
  \]  
  for $Mr^* \leq k \leq N$. Here, the nondimensional radius $r^*$ is used, because $N, M$ and $k$ are integer values and possess no dimension.

- **Q2:** Shifting of the shape function.
  \[
  \hat{Q}^{(2)}_{\varphi,k} = \exp \left( -\frac{(N - M(1 - r^*) - k)^2}{(Mr^* - k)^2} \right)
  \]  
  for $Mr^* \leq k \leq N - M(1 - r^*)$ and $\hat{Q}^{(2)}_{\varphi,k} = 1$ for $k > N - M(1 - r^*)$.

- **Q3:** Formulation based on wave length.
  \[
  \hat{Q}^{(3)}_{\varphi,k} = \exp \left( -\frac{(Nr^* - k)^2}{(Mr^* - k)^2} \right)
  \]  
  for $Mr^* \leq k \leq Nr^*$ and $\hat{Q}^{(3)}_{\varphi,k} = 1$ for $k > Nr^*$.
Figure 3 shows the modified kernels as a function of wave number $k$ and wave length $\lambda$.

In comparison to the standard kernel, Q1 and Q2 introduce a gradual enhancement of the spectral viscosity near the axis. However, in both cases the viscosity still disappears for any finite wave length if $r$ goes to zero. In contrast, Q3 depends on the azimuthal wave length only. From a physical point of view this suggests that waves of equal length experience the same attenuation independent from their radial position. However, this superficially attractive feature is accompanied by the presence near the axis of steep gradients in wave number space.

3. Numerical experiments

In this section we present three incompressible flow problems to highlight different aspects of SVV in cylindrical configurations: a three-dimensional laminar flow to study the convergence properties of the SVV kernels, turbulent pipe flow, which is a classical test case to examine LES configurations in cylindrical coordinates, and, finally, a turbulent flow in a closed cavity driven by an electromagnetic volume force.

Prior to commencement, we relate the following restriction: since the same polynomial order is used in both spectral element directions, an identical SVV parameterization is applied in the axial and the radial direction. Therefore, only two parameters control the spectral vanishing viscosity in the spectral element directions, which we call henceforth $M_{rz}$ and $\varepsilon_{rz}$.

3.1. Convergence study

To examine the spatial convergence properties of the SVV kernels we consider the (steady) Kovasznay flow. Originally defined for two-dimensional Cartesian coordinates [15], the exact solution of this flow can be adopted as a test case for cylindrical coordinates [12] using

$$u = 1 - \exp(\lambda z) \cos(2\pi r \cos(\varphi + \theta + \Delta))$$

$$v = \frac{1}{2\pi} \lambda \exp(\lambda z) \sin(2\pi r \cos(\varphi + \theta + \Delta)) \cos(\varphi + \theta)$$

$$w = -\frac{1}{2\pi} \lambda \exp(\lambda z) \sin(2\pi r \cos(\varphi + \theta + \Delta)) \sin(\varphi + \theta)$$

$$p = \frac{1}{2} \left(1 - \exp(\lambda z)\right)$$

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Figure 3: Modified SVV kernels Q1–Q3 as functions of wave number $k$ or nondimensional wave length $\lambda^*$ with $N = 31$, $M = 16$. 
where \( \lambda = \text{Re}/2 - \sqrt{\text{Re}/4^2 + 4\pi^2} \) with the Reynolds number \( \text{Re} \). The introduction of the parameters \( \Delta \) and \( \theta \) enables an arbitrary offset and an arbitrary rotation about the cylinder axis, as illustrated in Fig. 4. An offset \( \Delta \neq 0 \) leads to a flow that crosses the axis and a rotation \( \theta \), which is a non-rotational multiple of \( \pi \), ensures that both the real and the imaginary parts of all modes are exercised.

For the convergence tests, similar to those of [12] but here with SVV kernels also included, we use \( \Delta = 0.1, \theta = 0.75 \) and Reynolds number \( \text{Re} = 40 \). The computational domain is divided into \( 2N + 2 \) meridional semi-planes such that \( N \) denotes the highest resolved wave number. Each semi-plane is discretized with a non-orthogonal grid consisting of seven spectral elements; Fig. 5 shows the spectral element grid and the internal mesh nodes for polynomial order \( P = 6 \). Furthermore, Dirichlet boundary conditions are applied on all exterior (non-axial) domain nodes. For each interpolation order \((P, N)\), the exact velocity field is supplied as initial condition. The problem is then integrated forwards in time to steady state. Finally, the error is determined by computing the maximum deviation of the steady state from the exact solution. Here, the axial velocity \( u \) component is used as reference; the other two components behave similarly.

We have performed two different studies in order to investigate the con-
vergence properties of the SVV stabilization separately for the spectral element method and the Fourier method. In the first, we have validated the SVV implementation in the meridional directions: the polynomial order of the spectral elements is varied between \( P = 2 \) and \( P = 18 \), while a constant number of Fourier modes \( N = 24 \) is maintained in azimuthal direction. This ensures that the error is dominated by the spectral element discretization at low \( P \), while the azimuthal direction is fully resolved. Fig. 6 (a) shows error plotted against \( P \) both with and without SVV. As demonstrated by previous authors, SVV somewhat reduces the rate of convergence, but exponential/spectral convergence is preserved. SVV parameters are computed [as

\[ M_{\varphi} = \frac{P}{2} \quad \text{and} \quad \epsilon_{\varphi} = \frac{1}{P}. \]

in 4] as follows: \( M_{\varphi} = P/2 \) and \( \epsilon_{\varphi} = 1/P \). Alternative parameterizations,
e.g. with $M = 2\sqrt{P+1}$ and $\varepsilon = 1/(P+1)$ [as in 7], gave similar results.

In the second study, a fixed spectral element polynomial order $P = 14$ is used, and the number of azimuthal Fourier modes $N$ is varied. We have examined the standard SVV kernel, referred to as Qs in the following, and its variations Q1, Q2 and Q3, which were introduced in the previous section. The SVV parameters are scaled as above, i.e. $M = N/2$ and $\varepsilon = 1/N$. Fig. 6 (b) presents a summary of these results: the standard kernel Qs behaves as for the previous study where $P$ was varied, i.e. the adoption of SVV with this kernel reduces the rate of convergence, but convergence remains exponential. In comparison to Qs, the kernels Q1, Q2 or Q3 further reduce the convergence rate. This can be explained by the fact that in these cases the number of stabilized modes increases with decreasing radial position. Thus, the overall effect of SVV is stronger than in the standard case. When Q1 is applied, this effect is marginal, because $\hat{Q}^{(1)}_{\varphi,k}$ is similar to $\hat{Q}_{\varphi,\lambda}$ even for small radii, as illustrated in Figures 2 and 3 of the previous section. The second kernel, Q2, still achieves spectral convergence, but at a significantly lower rate. This is not surprising when we consider that Q2 introduces considerably more diffusion in the core region. In contrast, spectral convergence is removed for kernel Q3, even though its physical foundation may naively seem more consistent than for Qs, Q1 and Q2. It may be conjectured that the reason for this behaviour lies in the discretization of the steep gradients near the axis and the fact that, directly at the axis, all modes are affected by SVV independent of the choice of $M$ and $\varepsilon$. Owing to these findings, only Qs, Q1 and Q2 are considered in the following.

3.2. Turbulent pipe flow

The second example deals with turbulent pipe flow at Reynolds number $Re = U_{bulk}D/\nu = 10000$. We wish to assess whether the SVV technique enables stable LES in cylindrical configurations that include the axis, and which aspects are relevant to obtaining reliable results. For all simulations we consider a pipe with diameter $D = 2R = 1$ and length $L = 2\pi D$, which is adequate at this Reynolds number [16]. No-slip boundary conditions are imposed at the wall while periodicity is employed in the streamwise direction. Our assessment is based on first and second-order statistics.

DNS data serves as reference. The DNS grid consists of 320 semi-planes (i.e. $N = 159$) in the azimuth and $30 \times 8$ spectral elements with $P = 10$ in the axial and the radial direction (Fig. 7). When designing the mesh, we have employed rules of thumb for for near-wall resolution in DNS supplied by...
Piomelli [17]. We have retained a strategy which has been found successful in previous spectral-element-based DNS and wall-resolving LES [14, 16]: the first element covers the laminar sublayer and another one the buffer layer, where turbulent energy production is greatest. A geometric progression with a stretching factor of 1.2 is then used to reach the centre of the pipe.

Figure 7: Spectral element boundaries for DNS and LES of turbulent pipe flow.

In order to help build confidence in the veracity of the DNS results, we compare the mean flow and the fluctuations in stream-wise (axial) and wall-normal (radial) direction with experimental data from den Toonder & Nieuwstadt [18] who carried out laser Doppler anemometer (LDA) measurements in a recirculatory pipe flow facility. Fig. 8 presents the time-averaged stream-wise velocity $U$ and the rms values $u'$ and $v'$ of the axial and radial velocity fluctuations in viscous units, i.e. the velocity components are scaled with the wall friction velocity $u_\tau$: $U^+ = U/u_\tau$, $u'^+ = u'/u_\tau$ and $v'^+ = v'/u_\tau$. The corresponding coordinate in wall-normal direction is given by $r^+ = (R-r)u_\tau/\nu$. Generally, the computed results are in good agreement with the experimental results, and with similar DNS results presented in [16]. We note that the experimental data for rms values seem unreliable close to the wall.

To generate a wall-resolving LES grid, the DNS grid is coarsened approximately by a factor of three in each direction: In the azimuth 96 semi-planes ($N = 47$) are used. The domain is divided into 10 spectral elements in the streamwise direction. The discretization in the radial direction was not so straightforward as for the other two directions, since this direction has been discretized using only three elements. We have proceeded as follows: the size of the element at the wall is trebled compared to the DNS, giving a near-wall wall-normal resolution of $r^+ \approx 0.9$, sufficient to capture the viscous sublayer. The remaining space is occupied by two elements with a stretching factor of
1.5. In summary, the element extents in the wall-normal direction, starting from the wall, are $\Delta r_1 = 0.045$, $\Delta r_2 = 0.182$ and $\Delta r_3 = 0.273$. Fig. 7 shows the resulting spectral element grid compared to the DNS grid. The polynomial order in the axial and the radial direction was retained from the DNS, i.e. $P = 10$. Further, the size of the time step could be doubled so that the overall computational cost for the LES was approximately 0.5% that of the DNS\(^1\).

First we examine the influence of SVV parameters when the standard kernel $Q_s$ is applied. Starting with the spectral viscosity activation modes $M_z = P/2 = 5$ and $M_\varphi = (N + 1)/2 = 24$, the SVV amplitudes were increased simultaneously. Fig. 9 illustrates the outcome of this variation for $\varepsilon_z = \varepsilon_\varphi = 5\nu$, $10\nu$, and $20\nu$. One observes that the LES results are generally in very good agreement with the DNS reference data. Moreover, the mean flow as well as the rms profiles are almost unaffected by the parameter variation. Only the streamwise fluctuations increase slightly for higher amplitudes. The best agreement is received with the smallest stable amplitude, namely $\varepsilon_z = \varepsilon_\varphi = 5\nu$. Smaller amplitudes were also considered, but these simulations were unstable.

Subsequently, the activation wave numbers $M_z$ and $M_\varphi$ were varied, with fixed SVV amplitudes $\varepsilon_z = \varepsilon_\varphi = 5\nu$. These results are presented in Fig. 10. Here, the effect of parameter variation is more evident. With increasing acti-

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\(^1\)Simulations were performed on a SGI Altix 4700 at ZIH/TU Dresden.
Figure 9: Mean (left) and rms velocity fluctuation profiles (right) for turbulent pipe flow, when $\varepsilon_z$ and $\varepsilon_\phi$ are varied. The spectral viscosity activation wave numbers are held constant at $(M_z = 5, M_\phi = 24)$ and the DNS results serve as reference.

Figure 10: Mean (left) and rms velocity fluctuation profiles (right) for turbulent pipe flow at various activation wave numbers $M_z$ and $M_\phi (\varepsilon_z = \varepsilon_\phi = 5\nu)$.

With $M_z = 7, M_\phi = 36$ showed first signs of numerical instability in terms of temporarily increasing divergence energy of the velocity field. Therefore, the ‘optimal’ parameter configuration for this test case was provided by

$$M_z = 5, M_\phi = 24, \varepsilon_z = \varepsilon_\phi = 5\nu.$$  \hspace{1cm} (29)

We are aware that the present parameter study is not universally valid, but a trend is apparent: the best results were obtained for high activation
modes and small SVV amplitudes, as long as these parameters ensured a stable calculation. Similar observations were made in previous SVV studies [see e.g. 4, 8]).

When we take a closer look at developments near the pipe axis (see Fig. 11) a non-physical increase of fluctuation becomes evident in this region, especially for high activation wave numbers. A possible explanation for this is given by the mutual disappearance of SVV stabilization with decreasing radius, allowing small-scale structures to accumulate in the core region. To investigate this effect in more detail, azimuthal wave number spectra for various radii were computed for the DNS as well as the LES calculations. The azimuthal spectra were determined by first computing the kinetic energy of Fourier transformed, instantaneous velocity fields. These instantaneous fields were averaged in axial direction, followed by an averaging in time, for which up to 40 independent snapshots were used. Finally, local spectra were extracted at different radial positions.

Fig. 12 presents a comparison between DNS and LES spectra for \( r = 0.01, 0.1, 0.25 \) and 0.45, corresponding to 2, 20, 50 and 90 percent of \( R \), respectively. At the outer two radial positions, the DNS and LES spectra agree well with each other. The stronger decay of the LES spectrum for modes \( k \gtrsim 30 \) is attributed to the augmented viscosity on these modes. In case of \( r = 0.1 \), this effect of SVV is still apparent. However, in comparison to DNS, a higher energy level is observed at higher wave numbers. In the lower range, \( k \lesssim 10 \), the correspondence of both spectra is still good. In contrast, at the smallest radial position \( r = 0.01 \), the energy level of the
LES is approximately one order of magnitude larger than the DNS results at all wave numbers. This observation suggests that there may be insufficient dissipation of turbulent kinetic energy in the centre of the pipe. At this point it is worth noting that the azimuthal wave length of the largest structures near the axis, e.g.

\[ \lambda_{\text{max}}(r = 0.01) = \frac{2\pi r}{1} = \frac{\pi}{50}, \]

approximately corresponds to the smallest resolved scales in the outer region

\[ \lambda_{\text{min}}(r = R) = \frac{2\pi R}{N} = \frac{\pi}{47}. \]

This implies that turbulent structures transported from the outer regions to the centre are not adequately dissipated and hence appear to accumulate in the core region.

Next we consider the modified azimuthal kernels Q1 and Q2. We have again chosen the parameter set according to (29). Generally, the results are almost identical to those obtained with Qs. In particular, neither Q1 nor Q2 manage to diminish the overshoot of turbulent kinetic energy near the axis as displayed in Fig. 13. Closer examination of the azimuthal spectra for Q2 reveals an improvement for smaller radii such as \( r = 0.1 \) (Fig. 14). However, in the immediate vicinity of the axis, at \( r = 0.01 \), kernel Q2 (as well as Q1) fails to control an accumulation of energy low wave numbers. As argued above, for small radii all azimuthal modes correspond to small length
scales. On the other hand, those scales cannot be represented in the spectral element directions \((z, r)\), whose discretization is relatively coarse at the axis. Hence, one may presume that the nonlinear redistribution of energy between the different coordinate directions and corresponding velocity components cannot be captured properly. This suggests that a local refinement of the spectral element grid near the axis would improve redistribution.

In order to verify this hypothesis, we have performed additional LES calculations with two further spectral element grids that are modified accordingly. An illustration of these grids is given in Fig. 15. In the first case, grid 2, the radial size of the two inner layers of elements is changed to \(\Delta r_2 = \Delta r_3 = 0.2225\). For the second variation, grid 3, the axis elements of the original grid are divided into two elements, where the size of the innermost layer corresponds to the DNS resolution at the axis such that \(\Delta r_3 = 0.160\) and \(\Delta r_4 = 0.113\). Other grid parameters remained unchanged.

Figure 16 presents the influence of the radial resolution on the mean flow and the rms values of axial and radial velocity fluctuations. The LES were performed with the standard kernel Qs and, as above, the SVV parameter configuration (29) is used. One initially observes that the grid modifications do not degrade the overall good agreement with the DNS data. Closer examination of the near-axis region (Fig. 17) reveals that the use of grid 2 diminishes the accumulation of turbulent kinetic energy in this region. The introduction with grid 3 of near-axis radial element resolution equivalent to that used in DNS prevents the non-physical increase of fluctuations. In order
to study the influence of the local refinement in more detail, energy spectra of the latter case are displayed along with the DNS data in Fig. 18. In comparison to the results obtained with Qs and the original grid (cf. Fig. 12), the radial refinement in the core region removes the energy overshoot at smaller radii \( r \leq 0.1 \). For the outer radii of the pipe, the spectra remain unchanged, since the grid modification does not affect this region. Hence, the hypothesis stated above is validated in the sense that a proper choice of the spectral element resolution is able to suppress near-axis axis instability, even though the standard SVV kernel Qs is used.

### 3.3. Flow driven by a travelling magnetic field

As third test case we consider the flow driven by a travelling magnetic field (TMF) in a closed, non-conductive cylindrical cavity with aspect ratio
one, i.e. height $H$ is twice the radius $R$. The body force is given by

$$f = F \frac{\rho \nu^2}{2R^3} \left( \frac{r}{R} \right)^2 e_z,$$

where

$$F = \frac{\sigma \omega B^2 \kappa R^5}{4 \nu \nu^2}$$

is the nondimensional forcing parameter, $\sigma$ is electric conductivity, $\omega$ is angular frequency, $\kappa$ is axial wave number and $B$ is the induction of the magnetic field. DNS of the resulting turbulent flow were presented in [11] for $F = 1 \times 10^6$ and $4 \times 10^6$. This corresponds approximately to $8F_c$ and $32F_c$, where $F_c = 120400$ is the critical force parameter [19]. Here we use the DNS for $F = 1 \times 10^6$ as the reference case. Fig. 19 depicts the components of the axial and radial mean flow $U$ and $V$, the distribution of turbulent kinetic energy $K = (u'^2 + v'^2 + w'^2)/2$ and a typical snapshot of the vortex structures using the $\lambda_2$-criterion according to [20]. All quantities presented in this Section have been nondimensionalized using the radius and the kinematic viscosity as primary scales such that velocity, time, and coordinates are scaled with $\nu/R$, $R^2/\nu$, and $R$, respectively.

As can be deduced from Fig. 19, the base flow is dominated by a toroidal roll that moves fluid upwards at the rim and returns it along the centre of the cylinder. However, due to strong fluctuations, the mean flow is of little relevance for the instantaneous picture. The vortices cover the major
Figure 19: DNS results for the TMF test case. From left to right: Contours of axial mean velocity from $-750$ to $750$ with step $75$, contours of radial mean velocity from $-500$ to $500$ with step $50$, contours of turbulent kinetic energy from $0$ to $110000$ with $10000$, and instantaneous vortices visualized with second invariant of the velocity gradient tensor ($\lambda_2 = -1.5 \times 10^7$). Dashed contour lines indicate negative values.

part of the domain and show a rather random orientation. However, the evolution over a time span reveals a preference of axial orientation in the centre, whereas azimuthal vortices tend to prevail near the rim. On average, turbulence amounts to nearly 50\% of the total kinetic energy [11]. Hence, this flow provides an interesting supplemental test case in support of the results obtained for turbulent pipe flow.

The DNS and LES grids used for this study are displayed in Fig. 20. The DNS is performed on a non-equidistant rectangular grid of $25 \times 50$ elements with a polynomial degree of $P = 7$ and 256 semi-planes in the azimuth. For LES two different grids are used. Both grids have in common, that the number of semi-planes is reduced to 48 and that the meridional grid is coarsened to $5 \times 10$ spectral elements, whereas the polynomial degree is retained at the same value as for the DNS. The size of the wall-nearest axial and radial elements is increased slightly such that the viscous sublayer is captured with $z^+ \text{ and } r^+ < 0.3$. The crucial difference between LES grids 1 and 2 lies in the near-axis radial extents of the inner spectral elements. In the first case, a constant stretching factor of 2.11 is used for all elements, including those nearest the walls. Hence, elements near the axis are extremely large. For LES grid 2, the size of the inner elements is determined independently from the
outer elements using a moderate stretching factor of 1.2 such that a higher resolution at the axis is provided. In both cases, the size of the time step is doubled. Consequently, LES costs about 0.3% of the equivalent DNS.

First, a parameter study was performed to investigate the influence of the SVV parameters on grid 1. Starting with a similar parameter set for the pipe flow, namely

\[ M_x = (P + 1)/2 = 4, \quad M_\phi = (N + 1)/2 = 12, \quad \text{and} \quad \varepsilon_x = \varepsilon_\phi = 5\nu, \]

the SVV magnitude is increased up to 50 and, independently, the activation modes \( M_x \) and \( M_\phi \) are reduced to 2 and 6, respectively. Fig. 21 summarizes the outcome of this study, presenting the profiles of \( U \) and \( K \) at \( z = -0.5 \), where the axial mean velocity approximately reaches its maximum. At first glance, the results look very similar. As for the pipe flow outcomes, LES profiles show a remarkable agreement with the corresponding DNS data and the variation of the SVV parameters does not qualitatively change the results. Closer examination near the axis again reveals an accumulation of turbulent energy in case of LES. As illustration of these artifacts, the left side of Fig. 22 presents a snapshot of the instantaneous vortices, when grid 1 and the SVV parameter set 30 is used. One observes small vortex tubes in the core region that are aligned with the axis. These structures have no counterpart in the DNS.

Figure 20: Spectral element boundaries of grids used for TMF simulations.
Figure 21: Axial mean velocity (left) and turbulent energy profiles (right) for various SVV parameter combinations. Here, LES grid 1 is used and DNS results serves as reference. For \( K \), a magnification of the core region \( r \leq 0.1 \) is displayed in the embedded diagram.

As for the pipe flow, we have carried out simulations with the modified SVV kernels as well as with the modified LES grid (grid 2) with the aim of preventing near-axis accumulation of turbulent energy. As before, the use of Q1 and Q2 does not produce improvement. However, improved grid design ensures sufficient dissipation in the core region, as one may observe in Fig. 23. The vortex tubes observed in the simulation with LES grid 1 almost completely disappear when LES grid 2 is employed (see right side of Fig. 22).

4. Conclusions

The motivation for this work was to prove the capability of the spectral vanishing viscosity method for large-eddy simulations of flow problems in cylindrical coordinates that include the axis region. We first presented the principles of SVV with focus on application to a spectral-element–Fourier method for three-dimensional incompressible fluid flows. In particular, we have shown that the scales are not damped uniformly in the azimuthal direction, when the standard SVV kernel (Qs) introduced by Maday [2] is used. To mitigate this imbalance, which is seen as a possible cause of the observed axis instabilities, we have proposed three modifications of the standard kernel. The first two variations relied on geometric considerations, where either the activation mode is down-scaled with decreasing radius or the whole shape...
function is shifted towards the lower modes. The third proposal was motivated by the relation between the wave number and the corresponding wave length at a certain radial position.

To study the accuracy and stabilization of the standard SVV and its modifications, we have considered three different flow configurations. First, we examined convergence properties of all SVV kernels using the laminar Kovasznay flow. These tests revealed that the third variation of the SVV kernel (Q3) failed to preserve the spectral convergence. Subsequently this kernel was not considered. With remaining kernels (Qs, Q1 and Q2), we performed LES of turbulent pipe flow and a supercritical flow driven by a travelling magnetic field. In general, the application of SVV stabilized the simulations and provided results in good agreement with the reference data. However, an accumulation of small-scale fluctuations near the axis was detected for both test cases. Neither variation of SVV parameters nor the use of one of the remaining kernel modifications could prevent this effect. Closer examination of azimuthal energy spectra revealed an increase of energy over the whole spectrum in the immediate vicinity of the axis independent of the SVV kernel employed. A comparison of the resolved azimuthal length scales in the outer and inner regions of the cylinder suggested that the redistribution of energy between the different spacial directions is not captured properly in the original LES grids. Hence, we have presumed that a local refinement of the spectral element grid would recover this mechanism. Indeed, proper choice of the grid prevented the accumulation of small-scale fluctuations near

Figure 22: Instantaneous vortex structures computed either with LES grid 1 (left) or with LES grid 2 (right). The vortices are displayed for the same value, $\lambda_2 = -1.5 \times 10^7$, as the DNS snapshot in Fig. 19.
the axis. However, the determination of the required resolution in the core region depends on various flow characteristics. Thus, the procedures used here are certainly not applicable to arbitrary flow problems. Furthermore, the question about the generic choice of the optimal SVV parameters still remains open and should be addressed in more detail in further research. Nevertheless, we can conclude that the spectral vanishing viscosity method is a reliable tool for the stabilization of LES using high-order discretization methods. One has to pay close attention to the spatial resolution near the axis when cylindrical configurations are considered.

Acknowledgements

The first two authors gratefully acknowledge the financial support from Deutsche Forschungsgemeinschaft in frame of the Collaborative Research Center SFB 609. Computational facilities were provided by the Center for Information Services and High Performance Computing (ZIH) of the TU Dresden. HMB acknowledges support from Australia’s National Computational Infrastructure through its Merit Allocation Scheme, Grant D77.
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