# COMPONENTS OF MEAN STREAMING FLOW IN A PRECESSING CYLINDER AT SMALL NUTATION ANGLES

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*Summary* Mean streaming is often observed in precessing cylinder flows, yet the exact mechanism of its creation remains obscure. By Reynolds-decomposition of the Navier–Stokes equation for a rotating frame of reference, we identify two forcing terms which, when applied to an axisymmetric simulation, generate a streaming flow identical to that obtained from a full three-dimensional simulation. These terms are (i), the usual Reynolds-stress term, and (ii), a term originating from the Coriolis term. The latter accounts for most of the energy of the streaming flow at small nutation angles.

Mean streaming flows (MSFs) are due to the quadratic nature of the nonlinear term in the momentum equation. When amplitudes are small, the nonlinear term is a second-order correction evaluated by substituting the linear solution into it. The time-average of a linear wave solution is zero, simply because the average of a sinusoidal function of time is zero. However, the nonlinear term is the square of a sinusoidal function, so its time average is nonzero. This provides a stress, effectively the Reynolds stress, that can drive time-mean flows at second order. The nonlinear term in contained rotating flows is known to lead to a variety of time-dependent phenomena explicable by triadic resonances [1, 2, 3, 4, 5]. However, the origin of the mean streaming remains unclear.

In rotating flows, inertial waves can exist which drive the streaming flow, as is regularly observed in experiments and simulations. Kelvin modes, the inviscid eigenmodes of solid body rotation flow, are usually considered a good approximation for inertial waves contained in a cylinder, however, prior theory [6] has been interpreted to suggest that Kelvin modes cannot create mean streaming. Our ultimate aim is to explain the observed streaming flow.

We use precession to excite inertial waves in a set-up shown in figure 1(a). Characterised by the Poincaré number $Po = \Omega_2/\Omega_1$, precession directly injects energy into a mode referred to as the forced mode. Certain combinations of forcing frequency and aspect ratio $H/R$ enable energy transfer via triadic resonance from the forced mode to a pair of free modes. Triadic resonance represents the system's first bifurcation, found to be ubiquitous at low nutation angles $\alpha$ [5].

In order to synthesise the streaming flow, we first integrate the full 3D problem. Once forced and axisymmetric modes are saturated, we extract the reference streaming flow and correlations from which we compute a number of forcing terms, details of which will be given shortly. These forcing terms are then used to drive an axisymmetric DNS, and we expect the resulting flow to match the reference MSF. Computing the isolated response to individual forcing terms provides further insight.

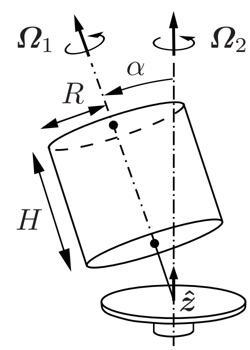


Figure 1: Schematic of the precessing flow.

We compute the flow in the cylinder frame of reference, which has time-dependent angular velocity $\boldsymbol{\Omega} = [\Omega_z, \Omega_r, \Omega_\varphi] = [\Omega_1 + \Omega_2 \cos\alpha,\ \Omega_2 \sin\alpha \cos(-\Omega_1 t),\ \Omega_2 \sin\alpha \sin(-\Omega_1 t)]^T$. The rotating frame of reference is accounted for by Coriolis terms in the incompressible Navier–Stokes equation

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} + \partial_t \boldsymbol{\Omega} \times \boldsymbol{x} = -\boldsymbol{\nabla} p + Re^{-1} \nabla^2 \boldsymbol{u}, \qquad \boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}} = 0, \tag{1}$$

where potential terms have been absorbed into the reduced pressure $p$, and the Reynolds number $Re = R\Omega_1/\nu$. The walls are at rest and the forced mode rotates at $-\Omega_1$. The streaming flow is defined as the time-averaged, axisymmetric component of the flow. A comparison of numerical and experimental results showed excellent agreement [7].

The forcing terms are obtained by Reynolds-averaging the Navier–Stokes equation (1): decomposition of linear and angular velocities and pressure into mean and fluctuating parts, $\boldsymbol{a} = \bar{\boldsymbol{a}} + \boldsymbol{a}'$, $\boldsymbol{a} = (\boldsymbol{u}, \boldsymbol{\Omega}, p)$, followed by averaging in time, produces two additional terms through which higher azimuthal modes can generate the $m = 0$ mean streaming flow, where $m$ is the azimuthal wave number. First, the usual Reynolds-stress term $\boldsymbol{R} = \nabla \cdot (\overline{\boldsymbol{u}'\boldsymbol{u}'})$, and, second, a term resulting from the Coriolis force $\boldsymbol{C} = 2\overline{\boldsymbol{\Omega}' \times \boldsymbol{u}'}$. Their respective axisymmetric part is used to force a steady, axisymmetric flow that satisfies

$$\bar{\boldsymbol{u}}_0 \cdot \boldsymbol{\nabla} \bar{\boldsymbol{u}}_0 + 2\bar{\boldsymbol{\Omega}}_0 \times \bar{\boldsymbol{u}}_0 + [\nabla \cdot (\overline{\boldsymbol{u}'\boldsymbol{u}'})]_0 + [2\overline{\boldsymbol{\Omega}' \times \boldsymbol{u}'}]_0 = -\boldsymbol{\nabla} \bar{p}_0 + Re^{-1} \nabla^2 \bar{\boldsymbol{u}}_0, \qquad \boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}}_0 = 0, \tag{2}$$

and boundary conditions $\bar{\boldsymbol{u}}_0 = 0$, where subscript 0 indicates $m = 0$. The Reynolds-stress term can be computed using all azimuthal modes (referred to as $\boldsymbol{R}$), just the (leading) forced $m = 1$ mode ($\boldsymbol{R}_1$), or either of the free modes ($\boldsymbol{R}_5$, $\boldsymbol{R}_6$), which are known to participate in triadic resonances for these parameters [5].

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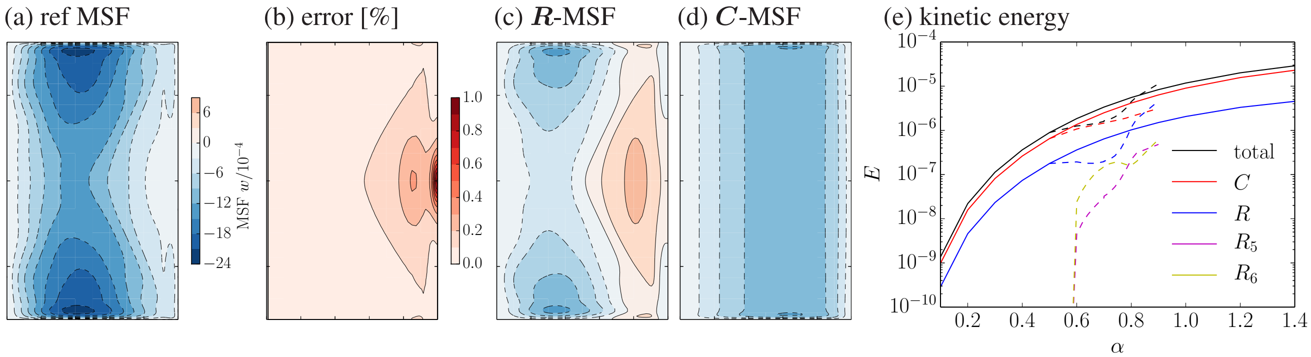


Figure 2: Contours of the streaming flow in a meridional semi-plane with the axis on the left, for a case below the threshold of triadic resonance: (a) reference MSF, (b) relative error $(w_{\mathrm{syn}}/w_{\mathrm{ref}} - 1)$ of the total synthetic MSF, (c) synthetic MSF due to $\boldsymbol{R}$-forcing, only, and (d) ditto due to $\boldsymbol{C}$-forcing, only. All MSF plots use the contour levels given in (a). $H/R = 1.62, Re = 7670, Po = -0.153, \alpha = 0.4°$. (e) Kinetic energy of the total streaming flow, and of the response to individual forcing parts, vs. nutation angle.

Figure 2 shows results for a case where $\alpha = 0.4°$ is below the threshold $\alpha_c$ of triadic resonance instability (for the present aspect ratio, weakly nonlinear theory [3] predicts $\alpha_c = 0.63°$). Hence, the asymptotic state is quasi-steady and essentially consists of a rotating forced mode and the streaming flow. The Fourier kinetic energy $E_m$ of any azimuthal mode is steady, with $E_1/E_0 \approx 141$ and $E_2/E_0 \approx 0.044$. Figure 2(a) shows the azimuthal velocity component $w$ of the reference streaming flow. It is negative throughout the domain (i.e., it opposes the solid body rotation), has an hourglass-like structure with a maximum in the end wall layers, and is about an order of magnitude larger than the axial and radial components. Since a contour plot of the synthetic streaming flow would be indistinguishable from the reference in figure 2(a), we plotted the relative error $w_{\mathrm{syn}}/w_{\mathrm{ref}} - 1$ in figure 2(b). The error is well below 1% for most of the domain, confirming we identified all terms required for the production of the streaming flow. If we decompose the total forcing into $\boldsymbol{R}$- and $\boldsymbol{C}$-parts, we can compute the corresponding flow response individually, which is shown in (c) and (d). The hourglass-like structure is created by the $\boldsymbol{R}$-forcing, whereas $\boldsymbol{C}$-forcing produces a columnar structure. Also, $\boldsymbol{R}$-forcing creates a streaming flow which is *positive* in the outer region of the cylinder.

We applied this procedure to a range of nutation angles. In figure 2(e), we plotted the kinetic energy of the total synthetic streaming flow (black). It matches the reference to within 0.1%. Also shown are energies of the individual responses to $\boldsymbol{C}$-forcing, to $\boldsymbol{R}$-forcing, or to forcings due to the free modes ($\boldsymbol{R}_5$ and $\boldsymbol{R}_6$). Solid lines show results where triadic resonance is absent and the flow appears stable, either because it in fact is ($\alpha < \alpha_c$), or because it has not evolved for long enough for the free modes to accumulate significant energy. For these "stable" cases, $\boldsymbol{C}$- and $\boldsymbol{R}$-parts account for about 75% and 20% of the total energy, respectively (these figures do not have to add up, since the underlying decomposition is not orthogonal).

Once triadic resonance is active (dashed lines), the response to $\boldsymbol{R}_5$- and $\boldsymbol{R}_6$-forcing becomes significant. However, for tilt angles just above critical ($0.6° \leq \alpha \leq 0.8°$), the total streaming flow decreases as compared to the non-resonant flow, because the free modes drain energy from the forced mode which reduces the $\boldsymbol{C}$-response, and because parts of the $\boldsymbol{R}/\boldsymbol{R}_5/\boldsymbol{R}_6$-response is positive and opposes the main streaming flow.

In an attempt to explain the origin of mean streaming in a precessing cylinder flow, we identified all relevant forcing terms. When applied to axisymmetric simulations, these forcing terms create streaming flows virtually identical to those obtained from full three-dimensional simulations. The forcing can be decomposed into contributions from Coriolis and nonlinear (Reynolds stress) terms for cases below and slightly above the threshold of triadic resonance. For all cases $0.1° \leq \alpha \leq 1.4°$ we considered, the Coriolis forcing term creates the largest response, however, the effect of the Reynolds stress term grows quickly when triadic resonance becomes active.

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