Leaf conjugacies on the torus

by

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Abstract

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If a partially hyperbolic diffeomorphism on a torus of dimension $d \geq 3$ has stable and unstable foliations which are quasi-isometric on the universal cover, and its center direction is one-dimensional, then the diffeomorphism is leaf conjugate to a linear toral automorphism. In other words, the hyperbolic structure of the diffeomorphism is exactly that of a linear, and thus simple to understand, example. In particular, every partially hyperbolic diffeomorphism on the 3-torus is leaf conjugate to a linear toral automorphism.

Dedication

To my parents, who have always believed in me.

Acknowledgements

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Chapter 1

Introduction

For the latter half of the twentieth century, much of the study of dynamical systems focused on hyperbolic systems. Also called Anosov systems, these diffeomorphisms split the tangent bundle of a manifold into stable and unstable subbundles, corresponding to directions of strong contraction and expansion, respectively. They are the simplest dynamical systems which exhibit chaotic behaviour, but also occur regularly in real-world examples of dynamical systems, both in mathematics and in other disciplines.

Much of the early analysis of hyperbolic systems was advanced by S. Smale [18] and independently by D. V. Anosov [2]. Anosov showed that these systems are *structurally stable*, that is, if f is hyperbolic, then any small perturbation of f is also hyperbolic and there is a homeomorphism h conjugating the dynamics of f to the dynamics of its perturbation.

The simplest examples of hyperbolic systems are those on the torus. An invertible $d \times d$ matrix with integer entries yields an automorphism on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, and if none of the eigenvalues has modulus one, the resulting system is hyperbolic. J. Franks and A. Manning showed that, up to topological conjugacy, all hyperbolic systems on tori are of this form [12, 11, 16]. In fact, Franks and Manning give a classification for hyperbolic systems on infranilmanifolds, a class containing tori as the simplest examples.

Many examples of dynamical systems exhibit some hyperbolic behaviour, while not satisfying the definition of Anosov. In a partially hyperbolic system, the stable and unstable directions dominate a center direction. While the dynamics can expand and contract the center direction to some degree, this action is bounded by the strong expansion and contraction on the unstable and stable bundles.

This thesis develops a partially hyperbolic analogue to the results of Franks and Manning, showing that, modulo the center direction, every partially hyperbolic diffeomorphism on the three-dimensional torus is conjugate to a linear toral automorphism. The hyperbolic structure of the system is, therefore, exactly that of an easy-to-understand, linear example. This result also holds under additional assumptions for higher dimensional tori.

A diffeomorphism f of a compact Riemannian manifold M is called *partially hyperbolic*[1] if there are constants $\lambda < \hat{\gamma} < 1 < \gamma < \mu$ and $C > 1$ and a Tf-invariant splitting of TM such that for every $x \in M$, $T_xM = E^u(x) \oplus E^c(x) \oplus E^s(x)$ where

$$\begin{aligned} \frac{1}{C}\mu^n\|v\| &< \|Tf^nv\| && \text{for } v \in E^u(x) \setminus \{0\}, \\ \frac{1}{C}\hat{\gamma}^n\|v\| &< \|Tf^nv\| < C\gamma^n\|v\| && \text{for } v \in E^c(x) \setminus \{0\}, \\ &\|Tf^nv\| < C\lambda^n\|v\| && \text{for } v \in E^s(x) \setminus \{0\}. \end{aligned}$$

Roughly speaking, vectors in the stable bundle E^s are contracted by f, vectors in the unstable bundle E^u are expanded, and vectors in the center bundle E^c may be contracted or expanded, but this action is dominated by the contraction and expansion in the *strong* bundles, E^s and E^u.

In a partially hyperbolic system, it is known that the subbundles E^u, E^c, and E^s are Hölder continuous and that there are unique Hölder continuous foliations W^u and

[1] This definition of partial hyperbolicity is sometimes called *absolute* partial hyperbolicity, in contrast to *relative* partial hyperbolicity, where the values $\lambda < \hat{\gamma} < 1 < \gamma < \mu$ are not true constants, but may vary depending on the point $x \in M$.

W^s tangent to E^u and E^s respectively [7, 15, 17]. In general, E^c, $E^{cu} = E^c \oplus E^u$, and $E^{cs} = E^c \oplus E^s$ do not integrate to foliations, but when they are uniquely integrable, the system is said to be *dynamically coherent* [4, 9]. Recently, Brin, Burago, and Ivanov have shown that every partially hyperbolic system on the 3-torus is dynamically coherent [6, 8, 5].

Every diffeomorphism of the torus $f : \mathbb{T}^d \to \mathbb{T}^d$ induces an automorphism of the fundamental group $f_* : \pi_1(\mathbb{T}^d) \to \pi_1(\mathbb{T}^d)$, and there is a unique linear diffeomorphism $g : \mathbb{T}^d \to \mathbb{T}^d$ that has the same effect on $\pi_1(\mathbb{T}^d)$. That is, $f_* = g_*$. We refer to g as the *linearization* of f. In the case of the 3-torus, $\mathbb{T}^3$, if f is partially hyperbolic, then so is its linearization [6].

Franks and Manning prove that if f is a hyperbolic diffeomorphism of the torus then f is topologically conjugate to its linearization. Our main result is a similar assertion in the partially hyperbolic case.

Theorem 1.1. *Every partially hyperbolic diffeomorphism of the 3-torus is leaf conjugate to its linearization.*

By a *(center) leaf conjugacy*, we mean a homeomorphism h of the torus to itself that carries the center leaves of f to the center leaves of its linearization g and satisfies

$$h(f(\mathcal{L})) = g(h(\mathcal{L}))$$

for every center leaf $\mathcal{L}$ of f. As h is invertible, the definition is symmetric with respect to f and g. With further hypothesis, we get a result that extends to higher dimensional tori.

Theorem 1.2. *A partially hyperbolic diffeomorphism of the d-torus, $d \geq 3$, is leaf conjugate to its linearization provided that the center bundle is one dimensional and the strong stable and strong unstable foliations, lifted to the universal cover of the torus, are quasi-isometric.*

The universal cover of the d-torus is d-space $\mathbb{R}^d$. The diffeomorphism and its invariant foliations easily lift to the universal cover. Quasi-isometry of a foliation W of $\mathbb{R}^d$ means that the leaves do not fold back on themselves much. There are positive constants a, b such that for all x, y in a common leaf of W we have

$$d_W(x, y) \leq a\|x - y\| + b$$

where d_W refers to distance along the leaf and $\|\cdot\|$ is the ordinary distance in $\mathbb{R}^d$ [10].

Remark. It is unreasonable to expect that a partially hyperbolic diffeomorphism is topologically conjugate to its linearization. For example, suppose $h : M \to M$ is a hyperbolic map with a fixed point x_0, and consider the product $f = h \times id : M \times S^1 \to M \times S^1$. f is partially hyperbolic, with the center direction tangent to the fibers $\{x\} \times S^1$. We may perturb the system along these fibers to introduce all manner of strange dynamics on the invariant submanifold $\{x_0\} \times S^1$ destroying the possibility of a topological conjugacy with f.

Instead, we attempt to conjugate only the hyperbolic part of one system with another. By mapping center leaves to center leaves, a leaf conjugacy ignores any "unhyperbolic" dynamics that can occur along the leaves.

Remark. Theorem 1.1 should be viewed as a classification result. Partially hyperbolic diffeomorphisms of the 3-torus are classified up to their center foliations. A similar remark holds for Theorem 1.2.

Perturbation results were first proved for hyperbolic systems by Anosov and for partially hyperbolic systems by Brin and Pesin and Hirsch, Pugh, and Shub. Under suitable hypotheses, a small perturbation of a partially hyperbolic system is partially hyperbolic and there is a leaf conjugacy from the system to its perturbation. A key ingredient is "plaque expansiveness." See [2, 7, 15] and Appendix A.

In this thesis we are not dealing with small perturbations. The partially hyperbolic diffeomorphism can be very far from its linearization. It is merely the case that the diffeomorphism and its linearization have the same effect on the fundamental group.

In showing that every partially hyperbolic system on the 3-torus is dynamically coherent, Brin, Burago, and Ivanov establish that the stable and unstable foliations are quasi-isometric on the universal cover [5]. Since each of the stable, center, and unstable bundles of such a system must be one-dimensional, Theorem 1.1 is a consequence of Theorem 1.2 and we proceed to prove the latter.

Notation. In the following, we denote the partially hyperbolic diffeomorphism of the torus as f_0 and its lift to the universal cover as f. The reason is that most of the work will be done in the universal cover, and although standard notation would put a bar or tilde over each instance of the lifted map's name, this would soon become cumbersome. Similarly, we denote the linearization of f_0 as g_0 and its lift to the universal cover as g.

The proof splits into five parts:

- Establish "nice" properties for the invariant manifolds. Using quasi-isometry, the foliations of f can readily be compared to the flat foliations of its linearization g. The leaves of f lie close to their linear counterparts, and from this, existence and uniqueness properties hold for intersections between the stable, center, and unstable leaves.

- Adapting the conjugacy proof of Franks [12], construct a conjugacy H between C_g, the space of center leaves of g, and C_f, the space of center leaves of f.

- Construct a section σ^*, a continuous submanifold of $\mathbb{R}^d$ which intersects each center leaf of f exactly once, with the additional properties that σ^* is uniformly continuous and bounded in the E^c_g direction.

- Using H and σ^*, construct a leaf conjugacy on $\mathbb{R}^d$, a homeomorphism $h_0 : \mathbb{R}^d \to \mathbb{R}^d$ such that $h_0(g(\mathcal{L})) = f(h_0(\mathcal{L}))$ for every center leaf $\mathcal{L}$ of g.

- By averaging h_0 over all possible translations by the lattice $\mathbb{Z}^d$, find a leaf conjugacy h on $\mathbb{R}^d$ which descends to a leaf conjugacy on the torus $\mathbb{T}^d$.

Chapter 2

Nice properties of the invariant manifolds

For this chapter, assume $f_0 : \mathbb{T}^d \to \mathbb{T}^d$ is partially hyperbolic and satisfies the hypotheses of Theorem 1.2. Choose a lifting of f_0 to the universal cover $f : \mathbb{R}^d \to \mathbb{R}^d$. Then f is also partially hyperbolic, with the splitting of f_0 lifting to a splitting $T\mathbb{R}^d = E^u_f \oplus E^c_f \oplus E^s_f$. With respect to the standard metric on $\mathbb{R}^d$, there are constants $0 < \lambda < \hat{\gamma} < 1 < \gamma < \mu$ and $C_{\text{ph}} > 1$ such that for $x \in \mathbb{R}^d$

$$\begin{aligned}
\frac{1}{C_{\text{ph}}}\mu^n\|v\| <& \|Tf^n v\| && \text{for } v \in E^u_f(x) \setminus \{0\}, \\
\frac{1}{C_{\text{ph}}}\hat{\gamma}^n\|v\| <& \|Tf^n v\| < C_{\text{ph}}\gamma^n\|v\| && \text{for } v \in E^c_f(x) \setminus \{0\}, \\
& \|Tf^n v\| < C_{\text{ph}}\lambda^n\|v\| && \text{for } v \in E^s_f(x) \setminus \{0\}.
\end{aligned}$$

If $P : \mathbb{R}^d \to \mathbb{T}^d$ is the covering map over $\mathbb{T}^d$, regard the fundamental group $\pi_1(\mathbb{T}^d)$ as the set of deck transformations of the cover, i.e., if τ is in $\pi_1(\mathbb{T}^d)$, then τ is a homeomorphism $\mathbb{R}^d \to \mathbb{R}^d$ such that $P = P \circ \tau$. In the case of the torus, every such τ will be a translation of the form $x \mapsto x + v$ where $v \in \mathbb{Z}^d$. The induced group homomorphism $f_{0*} : \pi_1(\mathbb{T}^d) \to \pi_1(\mathbb{T}^d)$ is defined by $f_{0*}(\tau) = f \circ \tau \circ f^{-1}$.

The action f_{0*} defines a homomorphism $\mathbb{Z}^d \to \mathbb{Z}^d$ which can be extended to a linear map $g : \mathbb{R}^d \to \mathbb{R}^d$. Note that g is the unique linear map $\mathbb{R}^d \to \mathbb{R}^d$ that descends to a map $g_0 : \mathbb{T}^d \to \mathbb{T}^d$ such that the induced actions of f_0 and g_0 on $\pi_1(\mathbb{T}^d)$ are the same: $f_{0*} = g_{0*}$. We call g and g_0 the *linearizations* of f and f_0 respectively. Our goal is to show that there is a conjugacy mapping the space of center leaves of f to the space of center leaves of g, so we must first define a partially hyperbolic splitting for g.

Note that if $\gamma < \tilde{\gamma} < \tilde{\mu} < \mu$ then

$$\begin{aligned} \frac{1}{C_{\text{ph}}}\tilde{\mu}^n\|v\| <& \frac{1}{C_{\text{ph}}}\mu^n\|v\| < \|Tf^n v\| && \text{for } v \in E^u_f(x)\setminus\{0\}, \text{ and} \\ \|Tf^n v\| <& C_{\text{ph}}\gamma^n\|v\| < C_{\text{ph}}\tilde{\gamma}^n\|v\| && \text{for } v \in E^c_f(x)\setminus\{0\}. \end{aligned}$$

so that the equations of partial hyperbolicity hold just as well with $\tilde{\gamma}$ and $\tilde{\mu}$ as with the original constants γ and μ, i.e., without loss of generality the interval $[\gamma, \mu]$ can be replaced by any subinterval $[\tilde{\gamma}, \tilde{\mu}] \subset [\gamma, \mu]$. Since g is a finite-dimensional linear map, it has a finite number of (possibly complex) eigenvalues. Replacing $[\gamma, \mu]$ by a small subinterval if necessary, we can assume that none of the eigenvalues lie in the annulus $\{z \in \mathbb{C} : \gamma \le \|z\| \le \mu\}$, and similarly that none of them lie in the annulus $\{z \in \mathbb{C} : \lambda \le \|z\| \le \hat{\gamma}\}$.

Considered as a linear map on $\mathbb{C}^d$, g has eigenvalues $\lambda_1, \ldots, \lambda_\ell$ with generalized eigenspaces $E^{\mathbb{C}}_{\lambda_1}, \ldots, E^{\mathbb{C}}_{\lambda_\ell}$. Let

$$E^{\mathbb{C},u}_g = \oplus_{\mu<|\lambda_i|} E^{\mathbb{C}}_{\lambda_i}, \quad E^{\mathbb{C},c}_g = \oplus_{\hat{\gamma}<|\lambda_i|<\gamma} E^{\mathbb{C}}_{\lambda_i}, \quad \text{and} \quad E^{\mathbb{C},s}_g = \oplus_{|\lambda_i|<\lambda} E^{\mathbb{C}}_{\lambda_i}.$$

Since conjugate eigenvalues have been grouped together, the subspaces $E^{\mathbb{C},u}_g$, $E^{\mathbb{C},u}_g$, and $E^{\mathbb{C},u}_g$ are just complexifications of real subspaces E^u_g, E^c_g, and E^s_g.

With respect to the splitting $\mathbb{R}^d = E^u_g \oplus E^c_g \oplus E^s_g$, g is partially hyperbolic with the possible caveat that at least one of E^u_g, E^c_g, or E^s_g may be zero. In fact, we will later prove that

$$\dim E^u_f = \dim E^u_g, \quad \dim E^c_f = \dim E^c_g, \quad \text{and} \quad \dim E^s_f = \dim E^s_g,$$

so that g is truly partially hyperbolic and its splitting depends only on the splitting of f and not on the particular choices of λ, $\hat{\gamma}$, γ, and μ.

As with f, define $E_g^{cu} = E_g^c \oplus E_g^u$ and $E_g^{cs} = E_g^c \oplus E_g^s$. Also, define linear projections $\pi_g^u, \pi_g^c, \pi_g^s, \pi_g^{cs}, \pi_g^{cu}, \pi_g^{us}$ with respect to the splitting. For example $\pi_g^u(v) = v^u$ if $v = v^u + v^{cs} \in E_g^u \oplus E_g^{cs} = \mathbb{R}^d$.

In general, the subspaces E_g^u,E_g^c, and E_g^s are not orthogonal with respect to the standard metric on $\mathbb{R}^d$. One could adapt the metric so that they were orthogonal, just as for f one could adapt, point-by-point, the metric on the tangent bundle of $\mathbb{R}^d$ so that at each point the splitting $E_f^u(x) \oplus E_f^c(x) \oplus E_f^s(x)$ is orthogonal, and further to assume that $C_{\mathrm{ph}} = 1$. The point of this paper, however, is to compare f to g, a task made difficult if we cannot compare distances related to one of the diffeomorphisms with distances related to the other. Therefore, the only metric ever used on points and vectors in $\mathbb{R}^d$ will be the standard one. As a side-effect, one must keep in mind that there may be vectors $v \in \mathbb{R}^d$ such that $\|\pi_g^u(v)\| > \|v\|$ and similarly for the other projections.

We now show that at large scales, f and g act in roughly the same way, which will allow us to relate the invariant manifolds of f to those of g.

Proposition 2.1. *For each $k \in \mathbb{Z}$,*

$$\|f^k - g^k\|_0 = \sup_{x \in \mathbb{R}^d} \|f^k(x) - g^k(x)\| < \infty.$$

Proof. This is a purely topological result that follows from the fact that $f_{0*} = g_{0*}$.

Let $K \subset \mathbb{R}^d$ be a compact fundamental domain of the covering $\mathbb{R}^d$ over $\mathbb{T}^d$. For any point $x \in \mathbb{R}^d$, there is a deck transformation $\tau \in \pi_1(\mathbb{T}^d)$ and a point $y \in K$ such that $x = \tau(y)$. Then,

$$\begin{aligned}
\|f^k(x) - g^k(x)\| &= \|f^k(\tau(y)) - g^k(\tau(y))\| \\
&= \|f_{0*}^k(\tau) f^k(y) - g_{0*}^k(\tau) g^k(y)\| \\
&= \|f_{0*}^k(\tau) \left(f^k(y) - g^k(y)\right)\| \\
&= \|f^k(y) - g^k(y)\|
\end{aligned}$$

where the last equality holds as the deck transformation $f_{0*}^k(\tau)$ is an isometry. As a result,

$$\sup_{x \in \mathbb{R}^d} \|f^k(x) - g^k(x)\| = \sup_{y \in K} \|f^k(y) - f^k(y)\| < \infty.$$

□

Of course, we are not saying there is a uniform bound on $\|f^k - g^k\|_0$ independent of $k \in \mathbb{Z}$. In almost all cases, there will in fact be $x \in \mathbb{R}^d$ such that $\|f^k(x) - g^k(x)\| \to \infty$ as $k \to \infty$.

Note that since,

$$\|f^k(x) - f^k(y)\| < \|g^k(x) - g^k(y)\| + 2\|f^k - g^k\|_0$$

and

$$\|g^k(x) - g^k(y)\| < \|f^k(x) - f^k(y)\| + 2\|f^k - g^k\|_0$$

we can prove the following:

Corollary 2.2.

$$\|f^k(x) - f^k(y)\| \sim \|g^k(x) - g^k(y)\| \quad as \quad \|x - y\| \to \infty.$$

More precisely, for each $k \in \mathbb{Z}$ and $C > 1$ there is an $M > 0$ such that for $x, y \in \mathbb{R}^d$,

$$\|x - y\| > M \quad \Rightarrow \quad \frac{1}{C} < \frac{\|f^k(x) - f^k(y)\|}{\|g^k(x) - g^k(y)\|} < C.$$

More generally, for each $k \in \mathbb{Z}$, $C > 1$, and linear map $\pi : \mathbb{R}^d \to \mathbb{R}^d$ there is an $M > 0$ such that for $x, y \in \mathbb{R}^d$,

$$\|\pi(x - y)\| > M \quad \Rightarrow \quad \frac{1}{C} < \frac{\|\pi(f^k(x) - f^k(y))\|}{\|\pi(g^k(x) - g^k(y))\|} < C.$$

The subbundles E_f^u and E_f^s integrate to foliations W_f^u and W_f^s on $\mathbb{R}^d$ which, by hypothesis, are quasi-isometric. For the unstable foliation, this means there are constants $a, b > 0$ such that

$$d_u(x, y) \le a \cdot \|x - y\| + b$$

for all $x \in \mathbb{R}^d$, $y \in W_f^u(x)$. Here, d_u is the distance measured along the unstable leaf.

The foliation W_f^u is tangent to the uniformly continuous distribution E_f^u, so the ratio $d_u(x,y)/\|x-y\|$ converges uniformly to one as $d_u(x,y) \to 0$. One can show, therefore, that by replacing a by a larger constant Q, the constant b can be eliminated altogether. Let $Q > 0$ be such that

$$d_u(x,y) < Q \cdot \|x-y\|$$

for all $x \in \mathbb{R}^d$, $y \in W_f^u(x)$ and

$$d_s(x,y) < Q \cdot \|x-y\|$$

for all $x \in \mathbb{R}^d$, $y \in W_f^s(x)$.

It follows from the work of Brin that f is dynamically coherent [4]; there are unique foliations W_f^{cu}, W_f^{cs}, and W_f^c tangent to E_f^{cu}, E_f^{cs}, and E_f^c respectively. Since g is a linear map, it also possesses (flat) foliations W_g^u, W_g^s, W_g^{cu}, W_g^{cs}, and W_g^c.

The foliations W_f^u, W_f^s, and W_f^c are tangent to the distributions E_f^u, E_f^s, and E_f^c at an infinitesimal scale. That is, if $x \in \mathbb{R}^d$ and $\{y_n\}$ is a sequence of points on the stable leaf $W_f^s(x)$ where the distance between x and y_n tends to zero in the limit, then, as unit vectors in $\mathbb{R}^d$, the sequence

$$\frac{x-y_n}{\|x-y_n\|}$$

approaches the subspace $E_f^s(x)$. What if instead we look at a sequence where the distance between x and y_n approaches infinity in the limit? As f_0 and g_0 have the same action on the fundamental group, at large scales, their lifts f and g are nearly indistinguishable. Therefore, at these scales, the invariant manifolds of f should closely resemble those of g. Indeed, if the sequence $\{y_n\}$ lies on the leaf $W_f^s(x)$ and $\|x-y_n\|$ tends to infinity, then the sequence

$$\frac{x-y_n}{\|x-y_n\|}$$

approaches the subspace E_g^s. There is a "tangency" at large scales between W_f^s and E_g^s, and similarly for the unstable and center directions.

Figure 2.1: The leaves of f drawn at three scales. At the microscopic level, the leaves are tangent to the partially hyperbolic splitting of f. At intermediate scales, the leaves may be pathological in nature. At the macroscopic level, however, the leaves closely resemble those of the linearization g.

Proposition 2.3. *If* $\|x-y\| \to \infty$ *where* $y \in W^s_f(x)$ *then* $\frac{x-y}{\|x-y\|} \to E^s_g$ *uniformly.*

More precisely, for $\epsilon > 0$ *there exists* $M > 0$ *such that if* $x \in \mathbb{R}^d$, $y \in W^s_f(x)$, *and* $\|x-y\| > M$ *then*

$$\|\pi^{cu}_g(x-y)\| < \epsilon \|\pi^s_g(x-y)\|.$$

Proof. Note that the spectrum of $g|_{E^s_g}$ lies below λ and the spectrum of $g|_{E^{cu}_g}$ lies above $\hat{\gamma}$. Therefore, there is $k_0 \in \mathbb{Z}$ such that if $v \in \mathbb{R}^d$, $k > k_0$ and

$$\|g^k(v)\| < \hat{\gamma}^k \|v\|$$

then

$$\|\pi^{cu}_g(v)\| < \epsilon \|\pi^s_g(v)\|.$$

Choose $k_1 > k_0$ large enough that $2C_{\text{ph}}Q\lambda^{k_1} < \hat{\gamma}^{k_1}$. Then, by Corollary 2.2, there is $M > 0$ (depending on k_1) such that

$$\|x-y\| > M \quad \Rightarrow \quad \|g^{k_1}(x) - g^{k_1}(y)\| < 2\|f^{k_1}(x) - f^{k_1}(y)\|.$$

Now if $y \in W^s_f(x)$ and $\|x - y\| > M$ then

$$d_s(f^{k_1}(x), f^{k_1}(y)) < C_{\text{ph}}\lambda^{k_1} d_s(x,y) \quad \Rightarrow$$
$$\|f^{k_1}(x) - f^{k_1}(y)\| < C_{\text{ph}} Q \lambda^{k_1} \|x - y\| \quad \Rightarrow$$
$$\|g^{k_1}(x) - g^{k_1}(y)\| < 2C_{\text{ph}} Q \lambda^{k_1} \|x - y\| \quad \Rightarrow$$
$$\|g^{k_1}(x - y)\| < \hat{\gamma}^{k_1} \|x - y\|$$

and so $\|\pi^{cu}_g(x - y)\| < \epsilon \|\pi^s_g(x - y)\|$. □

Remark. As with most of the results proved in this chapter, the above proposition has an analogous statement where the roles of the stable and unstable directions are reversed, proved by exchanging the roles of f and f^{-1}. In many cases, we will make use of such analogues without explicitly stating or proving them.

Corollary 2.4. *If $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathbb{R}^d$ and $y_n \in W^s_f(x_n)$ for all n, then the following are equivalent:*

- $d_s(x_n, y_n) \to \infty$,
- $\|x_n - y_n\| \to \infty$,
- $\|\pi^s_g(x_n - y_n)\| \to \infty$.

For a subset X of $\mathbb{R}^d$ and $R > 0$, let $B_R(X)$ denote the neighbourhood

$$B_R(X) = \{y \in \mathbb{R}^d : \|x - y\| < R \text{ for some } x \in X\}.$$

Proposition 2.5. *There is a constant R_c such that for all $x \in \mathbb{R}^d$,*

- $W^{cs}_f(x) \subset B_{R_c}(W^{cs}_g(x))$,
- $W^{cu}_f(x) \subset B_{R_c}(W^{cu}_g(x))$, *and*
- $W^c_f(x) \subset B_{R_c}(W^c_g(x))$.

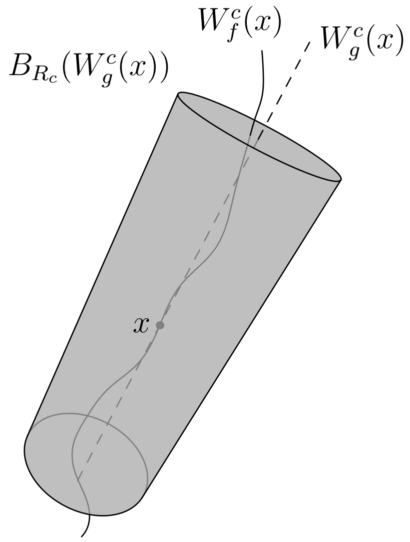

Figure 2.2: A center leaf contained inside a cylinder.

Proof. We will show this for $W^{cs}_f(x)$. The case for $W^{cu}_f(x)$ is similar and then since $W^c_f(x) \subset W^{cu}_f(x) \cap W^{cs}_f(x)$ and $W^c_g(x) = W^{cs}_g(x) \cap W^{cu}_g(x)$, the case for center leaves follows.

It is enough to show that $\|\pi^u_g(x-y)\|$ is uniformly bounded when x and y lie on the same cs-leaf. Let $C > 1$ be such that $\|g^k(v)\| > \frac{\mu^k}{C}\|v\|$ for all $k > 0$ and $v \in E^u_g \subset \mathbb{R}^d$. Fix a number $\beta \in (\gamma, \mu)$ and an integer $k > 0$ such that $\frac{\mu^k}{C\beta^k} > 1$. By Corollary 2.2, there is $M > 0$ such that

$$\begin{aligned} \|\pi^u_g(x-y)\| > M \quad \Rightarrow \quad \|\pi^u_g(f^k(x) - f^k(y))\| &> \beta^k \frac{C}{\mu^k}\|\pi^u_g(g^k(x) - g^k(y))\| \\ &> \beta^k \|\pi^u_g(x-y)\| \\ &> \beta^k M. \end{aligned}$$

Since $\beta^k M > M$ we can continue by induction to show

$$\|\pi^u_g(f^{nk}(x) - f^{nk}(y))\| > \beta^{nk} M \quad \text{for } n > 0.$$

Then, for some constant $a > 0$,

$$\|f^{nk}(x) - f^{nk}(y)\| > a\beta^{kn} M.$$

In particular, if $\|\pi_g^u(x-y)\| > M$ then $\|f^{nk}(x) - f^{nk}(y)\|$ grows at a rate faster than γ^{nk} as $k \to \infty$, so x and y cannot lie on the same center-stable leaf. □

Remark. Unfortunately, this proof does not carry over to the foliations W_f^u and W_f^s since we need the condition $\beta > 1$. This is a rare occasion where we actually know more about the weak foliation W_f^c than the strong ones.

Question. Under the given hypotheses for f, is there necessarily a constant R_u such that $W_f^u(x) \subset B_{R_u}(W_g^u(x))$?

Corollary 2.6. *If $\|x - y\| \to \infty$ where $y \in W_f^c(x)$ then $\frac{x-y}{\|x-y\|} \to E_g^c$ uniformly, in the same sense as in Proposition 2.3.*

Proposition 2.7. *In the universal cover, $\mathbb{R}^d$, a cs-leaf of f can intersect a u-leaf of f at most once. A cu-leaf of f can intersect an s-leaf of f at most once.*

We later show that these leaves, in fact, intersect *exactly* once.

Proof. This is a consequence of quasi-isometry. If x and y lie on the same cs-leaf then

$$\begin{aligned} d_{cs}(f^n(x), f^n(y)) < C_{\text{ph}}\gamma^n d_{cs}(x, y) \quad &\Rightarrow \\ \|f^n(x) - f^n(y)\| < C_{\text{ph}}\gamma^n d_{cs}(x, y) \end{aligned}$$

whereas if they lie on the same u-leaf then

$$\begin{aligned} d_u(f^n(x), f^n(y)) > \frac{1}{C_{\text{ph}}}\mu^n d_u(x, y) \quad &\Rightarrow \\ \|f^n(x) - f^n(y)\| > \frac{1}{QC_{\text{ph}}}\mu^n d_u(x, y) \end{aligned}$$

and $\frac{1}{QC_{\text{ph}}}\mu^n d_u(x, y) > C_{\text{ph}}\gamma^n d_{cs}(x, y)$ for large n. □

Since g is linear it is straightforward to define a foliation W_g^{us} tangent to $E_g^s \oplus E_g^u$. For generic f, however, $E_f^s \oplus E_f^u$ is not integrable [14], so it does not make sense to talk of us-leaves of f. Instead, define the *us-pseudoleaf* of f at x as

$$W_f^{us}(x) = \bigcup_{y \in W_f^u(x)} W_f^s(y).$$

Figure 2.3: The us-pseudoleaf $W_f^{us}(x)$ consists of all points $z \in \mathbb{R}^d$ where $z \in W_f^s(y)$ for some $y \in W_f^u(x)$.

If x_1 and x_2 lie on the same unstable leaf then $W_f^{us}(x_1) = W_f^{us}(x_2)$. If, however, x_1 and x_2 lie on different unstable leaves, then $W_f^{us}(x_1)$ and $W_f^{us}(x_2)$ may be disjoint, may coincide, or may intersect each other in some horribly pathological manner. We use the term pseudoleaf to emphasize the fact that these sets do not naturally yield a foliation.

The choice of defining the pseudoleaf by ranging first along the unstable direction and then along the stable direction is arbitrary, but it is a convention we will maintain through the rest of the thesis.

Proposition 2.8. *$W_f^{us}(x)$ is a properly embedded topological hyperplane.*

Proof. For $y, y' \in W_f^u(x)$, $y \neq y'$, the stable leaves $W_f^s(y)$ and $W_f^s(y')$ are disjoint. $W_f^u(x)$ is homeomorphic to $\mathbb{R}^u$ where $u = \dim E_f^u$. $W_f^s(y)$ depends continuously on y and is homeomorphic to $\mathbb{R}^s$ where $s = \dim E_f^s$. Therefore $W_f^{us}(x)$ is homeomorphic to a bundle of $\mathbb{R}^s$-fibers over $\mathbb{R}^u$ so is homeomorphic to $\mathbb{R}^{u+s}$.

To show the embedding is proper, suppose instead that there is a sequence $\{z_n\}$ on

$W_f^{us}(x)$ that goes to infinity on the pseudoleaf but is bounded in $\mathbb{R}^d$. In other words, there are sequences $y_n \in W_f^u(x)$, $z_n \in W_f^s(y_n)$ where either

$$d_u(y_n, x) \to \infty \quad \text{or} \quad d_s(y_n, z_n) \to \infty$$

while $\|z_n - x\|$ stays bounded.

Then, by Corollary 2.4, either

$$\|y_n - x\| \to \infty \quad \text{or} \quad \|y_n - z_n\| \to \infty.$$

Note, however, that as $(x - y_n) + (y_n - z_n) = x - z_n$ is bounded, it must be that both

$$\|y_n - x\| \to \infty \quad \text{and} \quad \|y_n - z_n\| \to \infty.$$

By replacing $\{y_n\}$ and $\{z_n\}$ by subsequences, we may assume without loss of generality that each of

$$\frac{y_n - x}{\|y_n - x\|} \quad \text{and} \quad \frac{y_n - z_n}{\|y_n - z_n\|}$$

converges to a limit. It is not hard to show that these two sequences in fact converge to the same limit, say $v \in \mathbb{R}^d$, $\|v\| = 1$. Then, by Proposition 2.3, $v \in E_g^u$ as a limit of the first sequence and $v \in E_g^s$ as a limit of the second, a contradiction. □

Corollary 2.9. *If $\|x - z\| \to \infty$ and $z \in W_f^{us}(x)$ then $\frac{x-z}{\|x-z\|} \to E_g^u \oplus E_g^s$ uniformly, in the same sense as in Proposition 2.3.*

To prove this, apply Proposition 2.3 twice, once for the stable direction, and once for the unstable.

Proposition 2.10. *A center leaf of f intersects a us-pseudoleaf of f in at most one point.*

Again, we later show they intersect exactly once.

Proof. Suppose z, z' both lie on $W_f^{us}(x)$ and $z' \in W_f^c(z)$. Then, there are $y, y' \in W_f^u(x)$ such that $z \in W_f^s(y)$ and $z' \in W_f^s(y')$. By following a path $y \overset{s}{\rightsquigarrow} z \overset{c}{\rightsquigarrow} z' \overset{s}{\rightsquigarrow} y'$, we see that y and y' are on the same cs-leaf and the same u-leaf, contradicting Proposition 2.7. □

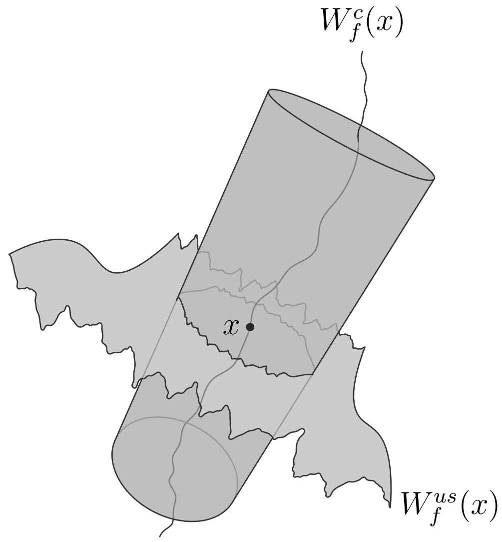

Figure 2.4: The us-pseudoleaf of x cutting through the cylinder which contains the center leaf of x.

We now make use of the assumption that $\dim E_f^c = 1$.

Proposition 2.11. *For $x \in \mathbb{R}^d$, $W_f^c(x)$ is a properly embedded line.*

Proof. Fix $x \in \mathbb{R}^d$. $W_f^{us}(x)$ is not everywhere differentiable in general, but the tangent space of $W_f^{us}(x)$ at the point x is $E_f^u(x) \oplus E_f^s(x)$. This fact is somewhat intuitive, but a rigorous proof of this is annoyingly technical and so has been left to Appendix B.

As $W_f^{us}(x)$ is a properly embedded hyperplane, it cuts $\mathbb{R}^d$ into two half-spaces. Since $E_f^c(x)$ is transverse to $E_f^u(x) \oplus E_f^s(x)$, $W_f^c(x)$ cuts through $W_f^{us}(x)$ at x, moving from one half-space to the other. If $W_f^c(x)$ were a circle, it would have to intersect $W_f^{us}(x)$ a second time to return to the half-space in which it started, contradicting Proposition 2.10. Thus, it must be a line.

As a leaf of a foliation, if $W_f^c(x)$ is not properly embedded, it must accumulate on a point $y \in \mathbb{R}^d$. Then, $W_f^c(x)$ would intersect $W_f^{us}(y)$ an infinite number of times. To see this rigorously, let U and V be the two components of $\mathbb{R}^d \backslash W_f^{us}(y)$, and let $\gamma : [-\epsilon, \epsilon] \to \mathbb{R}^d$ be a small segment of the curve $W_f^c(y)$ centered about y so that $\gamma(-\epsilon) \in U$ and $\gamma(\epsilon) \in V$. Then, as $W_f^c(x)$ accumulates on $W_f^c(y)$, there are distinct segments γ_n of the curve $W_f^c(x)$

which converge uniformly to γ. Then for large n, $\gamma_n(-\epsilon) \in U$ and $\gamma_n(\epsilon) \in V$ showing that γ_n must intersect $W_f^{us}(y)$ at some point. $\square$

Theorem 2.12.

$$\dim E_f^u = \dim E_g^u, \quad \dim E_f^c = \dim E_g^c, \quad \textit{and} \quad \dim E_f^s = \dim E_g^s.$$

Proof. We will show that

$$\dim E_f^u \leq \dim E_g^u, \quad \dim E_f^c \leq \dim E_g^c, \quad \text{and} \quad \dim E_f^s \leq \dim E_g^s.$$

Then, since the dimensions for the splittings of f and g must each sum up to $d = \dim \mathbb{R}^d$, equality follows.

Take any center leaf $W_f^c(x)$. Since it is a properly embedded line, there are $y_n \in W_f^c(x)$ such that $\|x - y_n\| \to \infty$. By Corollary 2.6, $\frac{x-y_n}{\|x-y_n\|} \to E_g^c$. In particular, E_g^c is non-zero, so $\dim E_g^c \geq 1 = \dim E_f^c$.

If $d = 3$, we may similarly show $\dim E_g^u \geq 1 = \dim E_f^u$ and $\dim E_g^s \geq 1 = \dim E_f^s$ to complete the proof. If $d > 3$, however, the proof is more involved.

Suppose $\dim E_g^u < u$ where $u = \dim E_f^u$. Fix a point $p \in \mathbb{R}^d$ and embed a $(u-1)$-dimensional sphere into the unstable leaf at p by a map $i : S^{u-1} \to W_f^u(p)$. As i is an embedding, antipodal points $x, -x \in S^{u-1}$ are separated by a uniform distance $\delta > 0$:

$$d_u(i(x), i(-x)) > \delta \quad \text{for all } x \in S^{u-1}.$$

Consider the composition $\pi_g^u \circ f^n \circ i : S^{u-1} \to E_g^u$ for $n > 0$. Since $\dim E_g^u \leq u - 1$ by assumption, E_g^u is homeomorphic to a subset of $\mathbb{R}^{u-1}$. Apply the Borsuk-Ulam theorem to find $x_n \in S^{u-1}$ such that

$$\pi_g^u \circ f^n \circ i(x_n) = \pi_g^u \circ f^n \circ i(-x_n).$$

That is,

$$f^n \circ i(x_n) - f^n \circ i(-x_n) \in E_g^{cs}.$$

Let $y_n = f^n \circ i(x_n)$ and $z_n = f^n \circ i(-x_n)$. Then since $i(x_n), i(-x_n) \in W_f^u(p)$,

$$d_u(i(x_n), i(-x_n)) > \delta \quad \Rightarrow \quad d_u(y_n, z_n) > \frac{\delta}{C_{\text{ph}}}\mu^n.$$

Therefore, $\|y_n - z_n\| \to \infty$ and for each $n > 0$, y_n and z_n lie on the same unstable leaf. By Proposition 2.3, $\frac{y_n - z_n}{\|y_n - z_n\|} \to E_g^u$. This contradicts the fact that y_n and z_n were chosen so that $y_n - z_n \in E_g^{cs}$ for all $n > 0$. □

Recall that there is a constant R_c such that for all $x \in \mathbb{R}^d$,

$$W_f^c(x) \subset B_{R_c}(W_g^c(x)).$$

Lemma 2.13. *There is a constant $M_c > 0$ such that for all $x \in \mathbb{R}^d$,*

$$B_{R_c}(W_g^c(x)) \cap W_f^{us}(x) \subset B_{M_c}(x).$$

Proof. Suppose not. Then, there are $x_n \in \mathbb{R}^d$ and $z_n \in B_{R_c}(W_g^c(x_n)) \cap W_f^{us}(x_n)$ such that $\|x_n - z_n\| \to \infty$. Since $z_n \in B_{R_c}(W_g^c(x_n))$, it follows that

$$\frac{x_n - z_n}{\|x_n - z_n\|} \to E_g^c,$$

but since $z_n \in W_f^{us}(x_n)$, by Corollary 2.9

$$\frac{x_n - z_n}{\|x_n - z_n\|} \to E_g^u \oplus E_g^s$$

which cannot also be true. □

The set $W_f^c(x) \setminus \{x\}$ consists of two unbounded connected components. As the center leaf cuts transversely through the us-pseudoleaf, these two components lie in distinct components of the set $\mathbb{R}^d \setminus W_f^{us}(x)$ and so they must also lie in two distinct unbounded components of $B_{R_c}(W_g^c(x)) \setminus W_f^{us}(x)$. This shows that $W_f^{us}(x)$ cuts completely through the cylinder $B_{R_c}(W_g^c(x))$.

By Lemma 2.13,

$$B_{R_c}(W_g^c(x)) \setminus B_{M_c}(x) \subset B_{R_c}(W_g^c(x)) \setminus W_f^{us}(x).$$

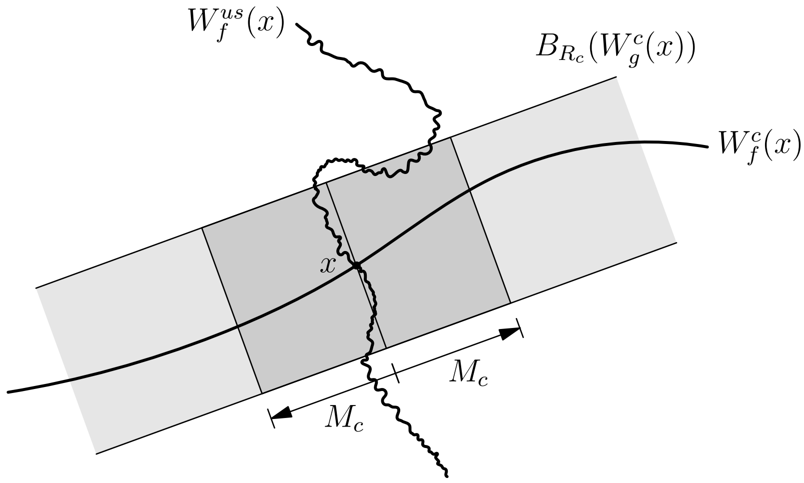

Figure 2.5: The us-pseudoleaf (now shown as one-dimensional for simplicity), dividing the cylinder into components.

The smaller of these sets has at most two unbounded components, so the larger set must have at most two unbounded components as well. This shows that $B_{R_c}(W_g^c(x)) \setminus W_f^{us}(x)$ has exactly two unbounded components, and these are the components containing the two halves of $W_f^c(x) \setminus \{x\}$. Additional bounded components may result from the "jagged" us-pseudoleaf leaving and re-entering the cylinder, as illustrated in Figure 2.5, but these other components are of no consequence.

In essence, Lemma 2.13 says that the us-pseudoleaf of f at x cuts the cylinder $B_{R_c}(W_g^c(x))$ into two pieces, and does so within a bounded distance from x. By possibly increasing the value of the constant M_c, we may also assume it satisfies the property

$$\pi_g^c \left(B_{R_c}(W_g^c(x)) \cap W_f^{us}(x) \right) \subset B_{M_c}(\pi_g^c(x))$$

for $x \in \mathbb{R}^d$. That is, the intersection of the us-pseudoleaf of f and the cylinder containing the c-leaf of f is of a bounded size when measured either in absolute terms or along the E_g^c direction.

Each center leaf of f is homeomorphic to the real line. If x and y lie on the same center leaf, let $[x, y]^c$ denote the line segment along the leaf from x to y.

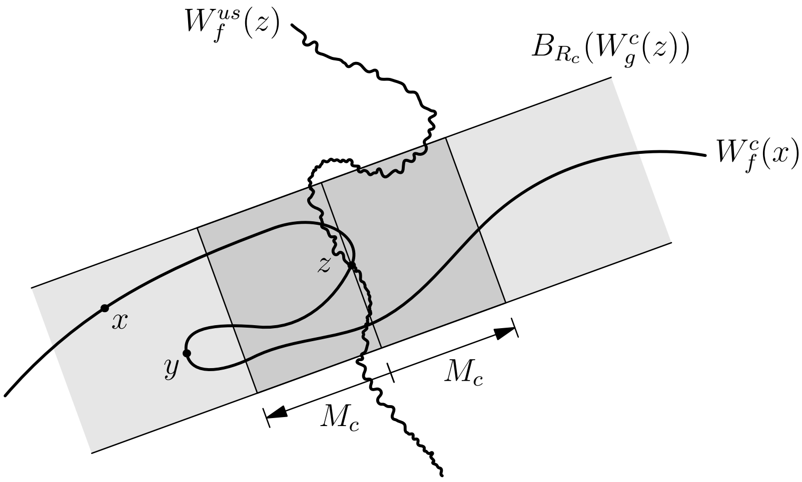

Figure 2.6: A depiction of the impossible situation considered in the proof of Proposition 2.14.

Proposition 2.14. *If $y \in W_f^c(x)$ and $z \in [x,y]^c$ then*

$$\pi_g^c(z) \in B_{M_c}\left([\pi_g^c(x), \pi_g^c(y)]\right)$$

where $[\pi_g^c(x), \pi_g^c(y)]$ is the line segment in $E_g^c \cong \mathbb{R}$ between $\pi_g^c(x)$ and $\pi_g^c(y)$.

In other words, $W_f^c(x)$ extends from one extreme of the cylinder $B_{R_c}(W_g^c(x))$ to the other, and can only backtrack by a distance at most M_c, measured along the E_g^c direction.

Proof. Suppose $z \in [x,y]^c$ fails to satisfy the above inclusion. Then $W_f^{us}(z)$ cuts the cylinder into at least two components, and x and y must be inside the same component due to their distance from z. This contradicts the fact that $W_f^c(x)$ cuts transversally through $W_f^{us}(z)$ at the unique point z and so must move permanently from one component to another. □

Proposition 2.15. *If $x, y \in \mathbb{R}^d$, then the following pairs of sets intersect in a unique point:*

1. *$W_f^{cs}(x)$ with $W_f^u(y)$,*

2. *$W_f^{cu}(x)$ with $W_f^s(y)$,*

3. *$W^c_f(x)$ with $W^u_f(y)$ if $x \in W^{cu}_f(y)$,*

4. *$W^c_f(x)$ with $W^s_f(y)$ if $x \in W^{cs}_f(y)$,*

5. *$W^c_f(x)$ with the pseudoleaf $W^{us}_f(y)$.*

Proof. First consider part 1. Uniqueness has already been established so we need only show existence. First note that the claim is true locally. By uniformity of the partially hyperbolic splitting, there is $\epsilon > 0$ such that for $x, y \in \mathbb{R}^d$, if $\|x - y\| \leq \epsilon$ there exists $z \in W^{cs}_f(x) \cap W^u_f(y)$.

Let

$$B^0(x) = \{y \in \mathbb{R}^d : \text{dist}(y, W^{cs}_f(x)) \leq \epsilon\}.$$

Then the above property restated means that for all $y \in B^0(x)$, there exists $z \in W^{cs}_f(x) \cap W^u_f(y)$. For $n > 0$, let

$$B^n(x) = f^n(B^0(f^{-n}(x)).$$

Since the foliations are invariant under f, if $y \in B^n(x)$, then $f^{-n}(y) \in B^0(f^{-n}(x))$ so that there is

$$z \in W^{cs}_f(f^{-n}(x)) \cap W^u_f(f^{-n}(x)) \quad \Rightarrow \quad f^n(z) \in W^{cs}_f(x) \cap W^u_f(y).$$

It is therefore enough to show that any $y \in \mathbb{R}^d$ lies in $B^n(x)$ for some $n > 0$.

Instead of proving this directly, we will show that for any $M > 0$, there is n such that

$$\text{dist}(\partial B^n(x), W^{cs}_f(x)) > M.$$

Then if $\text{dist}(y, W^{cs}_f(x)) < M$, there is a path from $W^{cs}_f(x)$ to y of length less than M which does not intersect the boundary of $B^n(x)$ and so its endpoint y must be in $B^n(x)$. Since for any $y \in \mathbb{R}^d$, the distance from y to $W^{cs}_f(x))$ is finite, and so less than some $M > 0$, this will complete the proof.

Suppose $n > 0$ and $y \in \partial B^n(x) = f^n\big(\partial B^0(f^{-n}(x))\big)$. As $B^n(x)$ is closed, $y \in B^n(x)$, so there is a unique intersection z of $W_f^{cs}(x)$ and $W_f^u(y)$. Then,

$$\begin{aligned} f^{-n}(y) \in \partial B^0(f^{-n}(x)) \quad &\Rightarrow \quad \|f^{-n}(y) - f^{-n}(z)\| \geq \epsilon \\ &\Rightarrow \quad d_u(f^{-n}(y), f^{-n}(z)) \geq \epsilon \\ &\Rightarrow \quad d_u(y, z) \geq \frac{\epsilon}{C_{\mathrm{ph}}}\mu^n \\ &\Rightarrow \quad \|y - z\| \geq \frac{\epsilon}{QC_{\mathrm{ph}}}\mu^n. \end{aligned}$$

where Q is the constant of quasi-isometry. Now by Proposition 2.3,

$$\|\pi_g^u(y - z)\| \geq C\mu^n$$

for some constant $C > 0$ and sufficiently large n. Since $W_f^{cs}(x)$ is contained in the cylinder $B_{R_c}(W_g^{cs}(x))$, the function π_g^u must be bounded on the leaf. It follows that

$$\mathrm{dist}(y, W_f^{cs}(x)) \geq C\mu^n - R$$

for another constant $R > 0$. (If E_g^{cs} and E_g^u were orthogonal, we could just take $R = R_c$.) Consequently,

$$\mathrm{dist}\left(\partial B^n(x), W_f^{cs}(x)\right) \geq C\mu^n - R.$$

For any $M > 0$, there is n large enough that $M < C\mu^n - R$, completing the proof.

Parts 2. through 4. of the proposition are proved by the same method.

For part 5, recall that uniqueness was proved in Proposition 2.10. For existence, take $x, y \in \mathbb{R}^d$. Note that from part 1. there is $y' \in W_f^{cs}(x) \cap W_f^u(y)$ and then from part 4. there is $z \in W_f^c(x) \cap W_f^s(y')$ so that $z \in W_f^c(x) \cap W_f^{us}(y)$. □

With the knowledge that each us-pseudoleaf uniquely intersects each center leaf, we can show that, like the strong foliations, W_f^c is quasi-isometric.

Proposition 2.16. *W_f^c is quasi-isometric.*

Proof. Fix $v \in E_f^c$ such that $\|v\| > 3M_c$. For $x \in \mathbb{R}^d$, let $\phi(x)$ be the unique intersection of $W_f^c(x)$ with $W_f^{us}(x+v)$. By Proposition 2.5,

$$\phi(x) \in B_{R_c}(W_g^c(x)) = B_{R_c}(W_g^c(x+v)),$$

and by Lemma 2.13,

$$\begin{aligned} \phi(x) \in B_{R_c}(W_g^c(x+v)) \cap W_f^{us}(x+v) \quad &\Rightarrow \\ \big\|\pi_g^c\big(\phi(x) - (x+v)\big)\big\| < M_c \quad &\Rightarrow \\ \|\pi_g^c(\phi(x) - x)\| > \|v\| - M_c > 2M_c. \end{aligned}$$

A (c,s,u)-*path* from $z_0 \in \mathbb{R}^d$ to $z_3 \in \mathbb{R}^d$ can be represented by tuple (z_0, z_1, z_2, z_3) such that $z_1 \in W_f^c(z_0)$, $z_2 \in W_f^s(z_1)$, and $z_3 \in W_f^u(z_2)$. Note that pseudoleaves were defined so that $z_1 \in W_f^{us}(z_3)$.

As the foliations are transverse and their intersections are unique, there is a unique (c,s,u)-path from x to $x+v$ which depends continuously on x. As $\phi(x)$ is determined by this path, the function $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is continuous. Let

$$\rho : \mathbb{R}^d \to \mathbb{R}, \ x \mapsto d_c\big(\phi(x+v), x\big)$$

be the distance measured along the center leaf from x to $\phi(x+v)$. ρ is continuous and invariant under the action of $\pi_1(\mathbb{T}^d)$. Hence, it is bounded above, say by $T > 0$.

Proposition 2.14 says that, measured in the E_g^c direction, a center leaf cannot back-track by more than a distance M_c. Say $\phi(x)$ lies between x and $y \in W_f^c(x)$ along the center leaf, $\phi(x) \in [x,y]^c$ in our notation. If

$$\|\pi_g^c(y - x)\| \le M_c,$$

then, by Proposition 2.14,

$$\|\pi_g^c(\phi(x) - x)\| < 2M_c$$

a contradiction. Therefore, $\|\pi_g^c(y-x)\| > M_c$ for all $y \in W_f^c(x)$ such that $\phi(x) \in [x,y]^c$. By the definition of T, $\|\pi_g^c(y-x)\| > M_c$ for all $y \in W_f^c(x)$ where $d_c(x-y) > T$.

By extension, if $y \in W_f^c(x)$ and $d_c(y,x) > nT$ then $\|\pi_g^c(y-x)\| > nM_c$, so for large values of $d_c(y,x)$,

$$\|\pi_g^c(y-x)\| > \frac{M_c}{2T} d_c(y,x)$$

which is enough to establish quasi-isometry. □

Corollary 2.17. *If $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathbb{R}^d$ and $y_n \in W_f^c(x_n)$ for all n, then the following are equivalent:*

- $d_c(x_n, y_n) \to \infty$,
- $\|x_n - y_n\| \to \infty$,
- $\|\pi_g^c(x_n - y_n)\| \to \infty$.

Let CS_f denote the space of center stable leaves. It is the quotient space derived from $\mathbb{R}^d$ by the equivalence relation $x \sim y$ if $y \in W_f^{cs}(x)$. Define the spaces CU_f of center-unstable leaves and C_f of center leaves in like manner.

Define a metric dist_u on cs-leaves by

$$\text{dist}_u(\mathcal{L}_1, \mathcal{L}_2) = \sup_{x \in \mathcal{L}_1} d_u(x, W_f^u(x) \cap \mathcal{L}_2)$$

where $\mathcal{L}_1, \mathcal{L}_2 \in CS_f$ are regarded as subsets of $\mathbb{R}^d$. Proposition 2.5 implies that $\text{dist}_u(\mathcal{L}_1, \mathcal{L}_2)$ is finite for any two leaves and one can check that dist_u satisfies the axioms of a complete metric on CS_f.

Any function which preserves the foliation descends to a map on the quotient space. In particular, it makes sense to talk of the leaf $f(\mathcal{L})$ or $\tau(\mathcal{L})$, $\tau \in \pi_1(\mathbb{T}^d)$ if $\mathcal{L} \in CS_f$ (or CU_f or C_f). From the definition, dist_u has the useful property that for $\mathcal{L}_1, \mathcal{L}_2 \in CS_f$ and $n \in \mathbb{Z}$,

$$\frac{1}{C_{\text{ph}}} \mu^n \text{dist}_u(\mathcal{L}_1, \mathcal{L}_2) < \text{dist}_u(f^n(\mathcal{L}_1), f^n(\mathcal{L}_2)).$$

In order to use this metric, however, we must first check that it induces the quotient topology on CS_f.

Proposition 2.18. dist_u *induces the quotient topology on* CS_f.

Proof. Fix an unstable leaf $W^u_f(x_0)$. Then, for $\mathcal{L}_1, \mathcal{L}_2 \in CS_f$, let y_i be the unique intersection of $\mathcal{L}_i$ and $W^u_f(x_0)$. The metric $D(\mathcal{L}_1, \mathcal{L}_2) = d_u(y_1, y_2)$ induces the quotient topology on CS_f.

By definition, $D(\mathcal{L}_1, \mathcal{L}_2) \le \text{dist}_u(\mathcal{L}_1, \mathcal{L}_2)$, so to show the two metrics are equivalent, it suffices to show that for $\epsilon > 0$ there is $\delta > 0$ such that

$$D(\mathcal{L}_1, \mathcal{L}_2) < \delta \quad \Rightarrow \quad \text{dist}_u(\mathcal{L}_1, \mathcal{L}_2) < \epsilon.$$

Using Propositions 2.3 and 2.5, one can show there are constants $A > 0$ and $b > 0$ such that for any $\mathcal{L}_1, \mathcal{L}_2 \in CS_f$,

$$\text{dist}_u(\mathcal{L}_1, \mathcal{L}_2) < A \cdot D(\mathcal{L}_1, \mathcal{L}_2) + b.$$

For $\epsilon > 0$, choose n large enough that $\frac{\epsilon}{C_{\text{ph}}}\mu^n > b$ and set

$$\delta = \frac{1}{\|Tf^n\| \cdot A}\left(\frac{\epsilon}{C_{\text{ph}}}\mu^n - b\right) > 0$$

where $\|Tf^n\| = \sup\{\frac{\|T_x f^n v\|}{\|v\|} : x \in \mathbb{R}^d, v \in T_x\mathbb{R}^d\}$. Then,

$$\begin{aligned} D(\mathcal{L}_1, \mathcal{L}_2) < \delta \quad &\Rightarrow \\ D(f^n(\mathcal{L}_1), f^n(\mathcal{L}_2)) < \frac{1}{A}\left(\frac{\epsilon}{C_{\text{ph}}}\mu^n - b\right) \quad &\Rightarrow \\ \text{dist}_u(f^n(\mathcal{L}_1), f^n(\mathcal{L}_2)) < \frac{\epsilon}{C_{\text{ph}}}\mu^n \quad &\Rightarrow \\ \text{dist}_u(\mathcal{L}_1, \mathcal{L}_2) < \epsilon. \end{aligned}$$

□

As a consequence of Proposition 2.15 any cs-leaf of f meets any cu-leaf in a unique center leaf of f. The space C_f is therefore canonically homeomorphic to the product space $CS_f \times CU_f$. In the linear case, it can be observed directly that $C_g \cong CS_g \times CU_g$ where C_g, CS_g, and CU_g are the corresponding spaces of leaves of g.

Chapter 3

A conjugacy of leaves

Before constructing a leaf conjugacy, we first construct a true conjugacy mapping between the spaces C_g and C_f of center leaves. Once the center direction has been quotiented out, the actions of f and g are hyperbolic, and the techniques of Franks [12] to find a conjugacy apply with only minor modifications.

Lemma 3.1. *Let f_0 and g_0 be partially hyperbolic diffeomorphisms on $\mathbb{T}^d$ with liftings f and g on the universal cover $\mathbb{R}^d$. Suppose that*

- *the foliations W^u_f, W^s_f, W^u_g, and W^s_g are quasi-isometric,*
- $\dim E^c_f = 1 = \dim E^c_g$,
- *$f_{0*} = g_{0*}$ as endomorphisms of $\pi_1(\mathbb{T}^d)$, and*
- *there is $R > 0$ such that for $x, y \in \mathbb{R}^d$*

$$y \in W^{cs}_g(x) \quad \Rightarrow \quad \operatorname{dist}_u(W^{cs}_f(x), W^{cs}_f(y)) < R$$

and

$$y \in W^{cu}_g(x) \quad \Rightarrow \quad \operatorname{dist}_s(W^{cu}_f(x), W^{cu}_f(y)) < R.$$

Then there is a unique continuous map $H : C_g \to C_f$ such that

- *$H(g(\mathcal{L})) = f(H(\mathcal{L}))$ for $\mathcal{L} \in C_g$, and*
- *$H(\tau(\mathcal{L})) = \tau(H(\mathcal{L}))$ for $\tau \in \pi_1(\mathbb{T}^d)$ and $\mathcal{L} \in C_g$.*

In other words, the diagrams

$$\begin{array}{ccc} C_g & \xrightarrow{g} & C_g \\ \downarrow H & & \downarrow H \\ C_f & \xrightarrow{f} & C_f \end{array} \quad \textit{and} \quad \begin{array}{ccc} C_g & \xrightarrow{\tau} & C_g \\ \downarrow H & & \downarrow H \\ C_f & \xrightarrow{\tau} & C_f \end{array}$$

commute.

In the previous chapter, we assumed throughout a fixed diffeomorphism f_0 with linearization g_0. For this lemma, however, all of the assumptions for f_0 and g_0 are written out explicitly, as the lemma will be applied in different contexts to construct a full (invertible) conjugacy between C_f and C_g. When reading through the proof, it is easiest to think of f and g as they are in the last chapter.

Proof. Since $C_f \cong CS_f \times CU_f$, we will construct maps $H^{cs} : CS_g \to CS_f$ and $H^{cu} : CU_g \to CU_f$ with analogous properties, and then define $H = H^{cs} \times H^{cu}$.

Let

$$\mathcal{H}^{cs} = \{h \in C^0(\mathbb{R}^d, CS_f) : h \circ \tau = \tau \circ h \text{ for } \tau \in \pi_1(\mathbb{T}^d)\}.$$

$\mathcal{H}^{cs}$ is a closed subspace of the continuous mappings from $\mathbb{R}^d$ to CS_f. It is non-empty as it contains the quotient map $x \mapsto W^{cs}_f(x)$. Define a complete metric on $\mathcal{H}^{cs}$ by

$$D(h_1, h_2) = \sup_{x \in \mathbb{R}^d} \operatorname{dist}_u(h_1(x), h_2(x)).$$

Since $h_i \circ \tau = \tau \circ h_i$ for $\tau \in \pi_1(\mathbb{T}^d)$, if K is a compact fundamental domain of the covering, then

$$D(h_1, h_2) = \sup_{x \in K} \operatorname{dist}_u(h_1(x), h_2(x))$$

which is finite. The axioms of a metric space are straightforward to check, and completeness follows from the completeness of dist_u.

Consider the map $F : \mathcal{H}^{cs} \to \mathcal{H}^{cs}$ given by $F(h) = f^{-1} \circ h \circ g$. This is well-defined, as $f_{0*} = g_{0*}$ implies that $F(h) \circ \tau = \tau \circ F(h)$ for $\tau \in \pi_1(\mathbb{T}^d)$. Then for $h_1, h_2 \in \mathcal{H}^{cs}$ and $n > 0$

$$\begin{aligned} D(F^n(h_1), F^n(h_2)) &= \sup_{x \in \mathbb{R}^d} \operatorname{dist}_u(f^{-n} \circ h_1 \circ g^n(x), f^{-n} \circ h_2 \circ g^n(x)) \\ &= \sup_{x \in \mathbb{R}^d} \operatorname{dist}_u(f^{-n} \circ h_1(x), f^{-n} \circ h_2(x)) \\ &\le C_{\mathrm{ph}} \mu^{-n} \sup_{x \in \mathbb{R}^d} \operatorname{dist}_u(h_1(x), h_2(x)) \\ &\le C_{\mathrm{ph}} \mu^{-n} D(h_1, h_2) \end{aligned}$$

with the constants $C_{\mathrm{ph}} > 0$ and $\mu > 1$ coming from the partially hyperbolic splitting for f. Therefore, F is a contraction with respect to the metric D and has a unique fixed point $h^{cs} \in \mathcal{H}^{cs}$. This means that $h^{cs} : \mathbb{R}^d \to CS_f$ is the unique continuous map with the properties that $h^{cs} \circ g = f \circ h^{cs}$ and $h^{cs} \circ \tau = \tau \circ h^{cs}$ for $\tau \in \pi_1(\mathbb{T}^d)$.

We now show that h^{cs} descends to a map $H^{cs} : CS_g \to CS_f$. Suppose $x, y \in \mathbb{R}^d$ and $y \in W^{cs}_g(x)$. Then, $g^n(y) \in W^{cs}_g(g^n(x))$ for all n, so

$$\operatorname{dist}_u \left(W^{cs}_f(g^n(x)), W^{cs}_f(g^n(y)) \right) < R$$

by the hypotheses of the lemma. Let $q : \mathbb{R}^d \to CS_f$ denote the quotient map $x \mapsto W^{cs}_f(x)$. The above inequality may be restated as

$$\begin{aligned} &\operatorname{dist}_u(q \circ g^n(x), q \circ g^n(y)) < R \quad \Rightarrow \\ &\operatorname{dist}_u(F^n(q)(x), F^n(q)(y)) = \operatorname{dist}_u(f^{-n} \circ q \circ g^n(x), f^{-n} \circ q \circ g^n(y)) \\ &\qquad < C_{\mathrm{ph}} \mu^{-n} R, \end{aligned}$$

Since F is a contraction, $F^n(q)$ tends to the fixed point h^{cs} as $n \to \infty$. Therefore

$$\begin{aligned} \operatorname{dist}_u(h^{cs}(x), h^{cs}(y)) \le \lim_{n \to \infty} C_{\mathrm{ph}} \mu^{-n} R = 0 \quad \Rightarrow \\ h^{cs}(x) = h^{cs}(y) \end{aligned}$$

showing that $h^{cs} : \mathbb{R}^d \to CS_f$ descends to $H^{cs} : CS_g \to CS_f$.

By the same reasoning, there is a unique map $h^{cu} : \mathbb{R}^d \to CU_f$ satisfying $h^{cu} \circ g = f \circ h^{cu}$ and $h^{cu} \circ \tau = \tau \circ h^{cu}$ for $\tau \in \pi_1(\mathbb{T}^d)$. This map descends to a map $H^{cu} : CU_g \to CU_f$. Since C_f and C_g are canonically identified with $CS_f \times CU_f$ and $CS_g \times CU_g$ respectively, define $H : C_g \to C_f$ by $H = H^{cs} \times H^{cu}$. The desired properties of H follow from the corresponding properties of H^{cs} and H^{cu}.

To establish the uniqueness of H, suppose $H_1 : C_g \to C_f$ also satisfies the conclusions of the lemma. Define $h_1^{cs} : \mathbb{R}^d \to CS_f$ by $h_1^{cs}(x) = W_f^{cs}(H_1(W_g^c(x)))$, that is, $h_1^{cs}(x)$ is the cs-leaf of f which contains the center leaf $H_1(W_g^c(x))$. One can verify that h_1^{cs} is in H^{cs} and

$$H_1 \circ g = f \circ H_1 \quad \Rightarrow \quad h_1^{cs} \circ g = f \circ h_1^{cs}$$

so by uniqueness, $h_1^{cs} = h^{cs}$. This means that for $\mathcal{L} \in C_g$, the c-leaves $H(\mathcal{L})$ and $H_1(\mathcal{L})$ are subleaves of the same cs-leaf of f. Using the uniqueness of h^{cu}, one shows similarly, that $H(\mathcal{L})$ and $H_1(\mathcal{L})$ are subleaves of the same cu-leaf of f. This implies that $H(\mathcal{L}) = H_1(\mathcal{L})$ which establishes that H is unique. □

The prototypical candidates for f_0 and g_0 in the lemma are, of course, the partially hyperbolic diffeomorphism f_0 and its linearization g_0 from the previous chapter. If the roles of f_0 and g_0 are interchanged, the lemma also applies to produce a unique map $C_f \to C_g$ and it applies to f_0 with itself and g_0 with itself. These applications combine to show that the map in the lemma is, in fact, a homeomorphism.

Theorem 3.2. *Let $f_0 : \mathbb{T}^d \to \mathbb{T}^d$ be partially hyperbolic with lifting $f : \mathbb{R}^d \to \mathbb{R}^d$ such that W_f^u and W_f^s are quasi-isometric and $\dim E_f^c = 1$. If $g : \mathbb{R}^d \to \mathbb{R}^d$ is the linearization of f, then there is a unique homeomorphism $H : C_g \to C_f$ such that*

- *$H(g(\mathcal{L})) = f(H(\mathcal{L}))$ for $\mathcal{L} \in C_g$, and*
- *$H(\tau(\mathcal{L})) = \tau(H(\mathcal{L}))$ for $\tau \in \pi_1(\mathbb{T}^d)$ and $\mathcal{L} \in C_g$.*

Proof. By the lemma, there is $H : C_g \to C_f$ such that $H \circ g = f \circ H$ and $H \circ \tau = \tau \circ H$ for $\tau \in \pi_1(\mathbb{T}^d)$. Also by the lemma, there is $K : C_f \to C_g$ such that $K \circ f = g \circ K$ and $K \circ \tau = \tau \circ K$. Then $K \circ H$ is a map from $C_g \to C_g$ and

$$(K \circ H) \circ g = g \circ (K \circ H) \quad \text{and} \quad (K \circ H) \circ \tau = \tau \circ (K \circ H).$$

Of course, the identity map $id : C_g \to C_g$ also satisfies

$$id \circ g = g \circ id \quad \text{and} \quad id \circ \tau = \tau \circ id.$$

By the uniqueness claim of the lemma, $K \circ H = id$. Similarly, $H \circ K = id$ on C_f, so $K = H^{-1}$ and H is a homeomorphism. □

One way of understanding this theorem is to note that the actions of f and g on the corresponding metric spaces C_f and C_g are *Anosov homeomorphisms* [1]. Using this topological form of hyperbolicity, one could construct the conjugacy H by means of a topological Shadowing Lemma. In fact, $H(\mathcal{L})$ is the unique center leaf of f such that $g^n(\mathcal{L})$ and $f^n(H(\mathcal{L}))$ stay within a bounded distance of other for all $n \in \mathbb{Z}$.

Since the conjugation H respects deck transformations, it quotients down to a bijection between center leaves of f_0 and those of g_0 on $\mathbb{T}^d$. These spaces of leaves, however, rarely have pleasant topologies. For instance, if the eigenvalues of the linear map g_0 are all irrational, every center leaf of g_0 will be dense in $\mathbb{T}^d$ and the space of leaves will have the chaotic topology.

A more useful construction is a *leaf conjugacy* as defined in [15], a homeomorphism $h : \mathbb{T}^d \to \mathbb{T}^d$ such that if $\mathcal{L}$ is a c-leaf of g then $h(\mathcal{L})$ is a c-leaf of f and

$$h \circ g(\mathcal{L}) = f \circ h(\mathcal{L}).$$

The task of the remaining chapters is to construct such a leaf conjugacy from H.

Chapter 4

Leaf sections

Let f_0, f, g_0, and g be as in Chapter 2. The conjugacy $H : C_g \to C_f$ constructed in the last chapter tells us how to map leaves of g to those of f, so to construct a leaf conjugacy on $\mathbb{R}^d$, we need only specify how points on one leaf are mapped to points on another. To define this mapping, we will construct solid slabs in $\mathbb{R}^d$, one for f and one for g, such that each center leaf intersects the appropriate slab in a compact segment. Then, h_0 can be defined from one slab to the other by mapping each center line segment of g to the corresponding center line segment of f in the simplest way possible.

For the linear map g, the solid slab is trivial to construct; just take the space between two flat us-leaves. For f, each of the two boundaries of the slab needs to intersect each center leaf exactly once. A us-pseudoleaf satisfies this condition, but its pathological nature makes its use intractable. Instead, we define a section of the center foliation as a map whose image intersects each leaf exactly once. We then construct two sections so that the slab between them has the properties we desire.

Since C_f is a quotient space of $\mathbb{R}^d$, define a *section* of C_f as a map $\sigma : C_f \to \mathbb{R}^d$ such that $\sigma(\mathcal{L}) \in \mathcal{L} \subset \mathbb{R}^d$ for every center leaf $\mathcal{L} \in C_f$. For any $x \in \mathbb{R}^d$, since $W_f^{us}(x)$ intersects each center leaf exactly once, the map $C_f \to \mathbb{R}^d$, $\mathcal{L} \mapsto W_f^{us}(x) \cap \mathcal{L}$ is an example of a section. However, it is not a particularly useful section to use in constructing a conjugacy.

For one, we have not established that the us-pseudoleaf $W_f^{us}(x)$ stays a bounded distance from the flat us-leaf $W_g^{us}(x)$. For another, the section may fail to be uniformly continuous for any reasonable choice of metric on C_f.

To avoid these issues, we will construct a continuous section $\sigma^* : C_f \to \mathbb{R}^d$ such that the image of σ^* lies a bounded distance from the us-leaf of g and so that σ^* is uniformly continuous for any metric on C_f that is invariant under the action of $\pi_1(\mathbb{T}^d)$.

If σ^* were defined on a compact domain, it would follow immediately from continuity that σ^* is uniformly continuous. Of course, C_f is homeomorphic to $\mathbb{R}^{u+s}$ and so is not compact. Instead, we establish a "finiteness" property for σ^* called Axiom F that, for our purposes, is just as good as having a compact domain.

If X_1, X_2, Y_1, and Y_2 are metric spaces, then $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are *isometrically equivalent* if there are isometries $\alpha : X_1 \to X_2$ and $\beta : Y_1 \to Y_2$ such that $f_2 \circ \alpha = \beta \circ f_1$. That is, with regard to their structure as functions on metric spaces, f_1 and f_2 cannot be distinguished.

Let X and Y be metric spaces. A continuous map $f : X \to Y$ satisfies *Axiom F* if

- there is a finite collection $\mathcal{G} = \{g_1, \dots, g_n\}$ of maps on metric spaces $g_j : X_j \to Y_j$,
- there is a collection $\{K_i : i \in I\}$ of compact subsets of X such that their interiors form an open cover of X, and
- for each K_i, the restriction $f|_{K_i}$ is isometrically equivalent to an element of $\mathcal{G}$, that is, for $i \in I$ there are isometries α_i and β_i such that $f|_{K_i} \circ \alpha_i = \beta_i \circ g_j$ for some $j \in \{1, \dots, n\}$.

The index set I could, in principle, be of any cardinality, but in the examples of Axiom F maps arising in this paper, the index set will be countably infinite.

As a concrete example, the function $\phi : \mathbb{R} \to \mathbb{R}$, $x \mapsto 3x + \sin(2\pi x)$ satisfies Axiom F. Using the cover $\{K_i\}_{i \in \mathbb{Z}}$ where $K_i = [i, i+2]$ each $\phi|_{K_i}$ is isometrically equivalent to $\phi|_{K_0}$ by the isometries $\alpha_i(x) = x + i$ and $\beta_i(x) = x + 3i$.

In some sense, functions satisfying Axiom F can be thought of as being constructed by “tiling” together a handful of locally defined functions. This notion of tiling functions, however, is difficult to define rigorously, difficult to establish for a given function, and more than is needed.

A continuous function on a compact domain is uniformly continuous and this fact generalizes to the Axiom F case.

Proposition 4.1. *If $f : X \to Y$ satisfies Axiom F, it is uniformly continuous.*

The converse is not true, as witnessed, for example, by the logarithm function restricted to $(1, \infty)$. Here, log is uniformly continuous as its derivative is bounded, but for $a \in (1, \infty)$, the limit

$$\lim_{x \to a} \frac{|\log(x) - \log(a)|}{|x - a|} = \frac{1}{a}$$

is an invariant up to isometric equivalence. This shows that the restriction of the logarithm function to any subset cannot be isometrically equivalent to its restriction to any other subset. $\log|_{(1,\infty)}$ therefore fails to satisfy Axiom F.

The proof of Proposition 4.1 is left to the reader.

Our aim is to construct a continuous section $\sigma^* : C_f \to \mathbb{R}^d$ such that σ^* stays within a bounded distance of a linear *us*-leaf of g and so that σ^* is uniformly continuous for any $\pi_1(\mathbb{T}^d)$-invariant metric on C_f. To achieve uniform continuity, we will construct σ^* in a way that ensures it satisfies Axiom F.

Theorem 4.2. *There is a continuous section $\sigma^* : C_f \to \mathbb{R}^d$ such that*

- $\pi_g^c \circ \sigma^*$ *is bounded; equivalently, there is $x_0 \in \mathbb{R}^d$ and $M > 0$ such that* image $\sigma^* \subset B_M(W_g^{us}(x_0))$*; and*

- *for any metric on C_f, invariant under the action of $\pi_1(\mathbb{T}^d)$, σ^* satisfies Axiom F.*

From this point on, assume that a $\pi_1(\mathbb{T}^d)$-invariant metric on C_f has been chosen. One such metric is the Hausdorff distance, the infimum over all $r \geq 0$ for which

$$\mathcal{L}_2 \subset B_r(\mathcal{L}_1) \quad \text{and} \quad \mathcal{L}_1 \subset B_r(\mathcal{L}_2)$$

where $\mathcal{L}_1, \mathcal{L}_2 \in C_f$ are considered as subsets of $\mathbb{R}^d$. Since these subsets are not compact, one has to check using Proposition 2.5 that this distance is always finite.

To construct σ^*, take bounded size plaques of a countable collection of us-pseudoleaves so that each center leaf of f passes through at least one of these plaques. Then, stitch these plaques together into a section by averaging them along each center leaf. To achieve this averaging, note that each center leaf $W^c_f(x)$ can be identified with $\mathbb{R}$ by a curve $\gamma : \mathbb{R} \to W^c_f(x)$ parameterized by arc-length. If $x_i \in W^c_f(x)$ and $a_i \in \mathbb{R}$, $\sum a_i = 1$, define summation along the leaf by

$$\sum\nolimits^c a_i x_i = \gamma\left(\sum a_i \gamma^{-1}(x_i)\right).$$

This is well-defined regardless of the choice of γ, and shows that $W^c_f(x)$ is an affine space.

The space of sections is also an affine space. If $\sigma_i : C_f \to \mathbb{R}^d$ are sections and $\sum a_i = 1$, define $\sum\nolimits^c a_i \sigma_i$ leafwise by

$$\left(\sum\nolimits^c a_i \sigma_i\right)(\mathcal{L}) = \sum\nolimits^c a_i \sigma_i(\mathcal{L}) \quad \text{for } \mathcal{L} \in C_f.$$

Here, the a_i may be constants or functions $C_f \to \mathbb{R}$.

If $U_i \subset C_f$ and $\alpha_i : U_i \to \mathbb{R}$ give a partition of unity for C_f, then continuous local sections $\sigma_i : U_i \to \mathbb{R}^d$ may be averaged together to give a continuous global section $\sum\nolimits^c \alpha_i \sigma_i : C_f \to \mathbb{R}^d$. The section σ^* will be the result of such an averaging.

Let $z_0 = (0, 0, \ldots, 0)$ denote the origin in $\mathbb{R}^d$.[1] Let $\sigma : C_f \to \mathbb{R}^d$ be the section $\mathcal{L} \mapsto W^{us}_f(z_0) \cap \mathcal{L}$ which has the us-pseudoleaf of z_0 as its image. For $z \in \mathbb{Z}^d$, define $\sigma_z : C_f \to \mathbb{R}^d$ as $\sigma_z(\mathcal{L}) = \sigma(\mathcal{L} - z) + z$. As z is a lattice point, if $\mathcal{L}$ is a center leaf of f then the translation $\mathcal{L} - z$ is also a center leaf. Hence, the function σ_z is well-defined. In fact, σ_z is the section with $W^{us}_f(z)$ as its image.

Our desired section σ^* will be constructed as σ_Λ, an averaging of sections σ_z for a subset Λ of $\mathbb{Z}^d$ where the average is weighted by appropriate bump functions α_z. Similar

[1] We use z_0 in place of simply 0 in order to distinguish it as a point of the manifold, and also because the choice of the origin on $\mathbb{R}^d$, the universal cover of $\mathbb{T}^d$, is essentially arbitrary.

to the translates σ_z, each α_z will be a translate of a carefully constructed bump function $\alpha : C_f \to \mathbb{R}$.

Lemma 4.3. *There is $R_{\mathbb{Z}} > 0$ such that every point $x \in \mathbb{R}^d$ lies in $B_{R_{\mathbb{Z}}}(W_g^c(z))$ for some $z \in \mathbb{Z}^d \cap B_{R_{\mathbb{Z}}}(W_g^{us}(z_0))$.*

Proof. For the linear map g, there is a symmetry

$$x \in B_{R_{\mathbb{Z}}}(W_g^c(z)) \quad \Leftrightarrow \quad z \in B_{R_{\mathbb{Z}}}(W_g^c(x)).$$

It is enough to show that the intersection

$$B_{R_{\mathbb{Z}}}(W_g^c(x)) \cap B_{R_{\mathbb{Z}}}(W_g^{us}(z_0))$$

contains a lattice point. The intersection of these two cylinders contains a sphere centered at $\pi_g^{us}(x) \in W_g^c(x) \cap W_g^{us}(z_0)$ having a radius proportional to $R_{\mathbb{Z}}$. By choosing a large value of $R_{\mathbb{Z}}$, this sphere can be assumed to have a radius large enough to ensure it contains a lattice point $z \in \mathbb{Z}^d$. □

Lemma 4.4. *For $R > 0$, there is a continuous, non-negative function $\alpha : C_f \to \mathbb{R}$ such that*

- *$\alpha(\mathcal{L}) > 0$ for every $\mathcal{L} \in C_f$ which intersects $B_R(W_g^c(z_0))$, and*
- *α has compact support.*

Proof. Since C_f is homeomorphic to $\mathbb{R}^{u+s}$, to create the bump function α it is enough to show that

$$A = \{\mathcal{L} \in C_f : \mathcal{L} \cap B_R(W_g^c(z_0)) \neq \varnothing\}$$

is relatively compact.

If $x \in B_R(W_g^c(z_0))$ then $\|\pi_g^{us}(x)\|$ is bounded, say by M. If

$$\mathcal{L} \cap B_R(W_g^c(z_0)) \neq \varnothing$$

and $x \in \mathcal{L}$, then by Proposition 2.5, $\|\pi_g^{us}(x)\| \le M + R_c$. Let K denote the compact set

$$\left\{x \in \mathbb{R}^d : \|\pi_g^{us}(x)\| \le M + R_c \text{ and } \pi_g^c(x) = 0\right\}.$$

Regarded as a quotient map, $W_f^c : \mathbb{R}^d \to C_f$ is continuous, so $W_f^c(K)$ is a compact subset of C_f. For any leaf $\mathcal{L}$ in C_f there is at least one $x \in \mathcal{L}$ such that $\pi_g^c(x) = 0$, showing that $A \subset W_g^c(K)$. □

Let α be the bump function given by Lemma 4.4 using the radius $R_{\mathbb{Z}}$ given by Lemma 4.3. For $z \in \mathbb{Z}^d$ define $\alpha_z : C_f \to \mathbb{R}$ by

$$\alpha_z(\mathcal{L}) = \alpha(\mathcal{L} - z).$$

Let Γ be a subset of $\mathbb{Z}^d$ such that for $\mathcal{L} \in C_f$, $\alpha_z(\mathcal{L}) > 0$ for only finitely many $z \in \Gamma$. Define $\alpha_\Gamma : C_f \to \mathbb{R}$ as the sum of the bump functions:

$$\alpha_\Gamma(\mathcal{L}) = \sum_{z \in \Gamma} \alpha_z(\mathcal{L}).$$

Then on the subset of C_f where α_Γ is positive, the functions $\mathcal{L} \mapsto \frac{\alpha_z(\mathcal{L})}{\alpha_\Gamma(\mathcal{L})}$ give a partition of unity. Define a (local) section σ_Γ on the domain

$$\mathrm{Dom}(\sigma_\Gamma) = \{\mathcal{L} \in C_f : \alpha_\Gamma(\mathcal{L}) > 0\}$$

by

$$\sigma_\Gamma = \sum_{z \in \Gamma}^{c} \frac{\alpha_z}{\alpha_\Gamma} \sigma_z.$$

Lemma 4.5. *If $\Gamma, \Upsilon \subset \mathbb{Z}^d$ and $\Upsilon = \Gamma + w$ for some $w \in \mathbb{Z}^d$ then*

$$\mathrm{Dom}(\sigma_\Upsilon) = \mathrm{Dom}(\sigma_\Gamma) + w$$

and

$$\sigma_\Upsilon(\mathcal{L} + w) = \sigma_\Gamma(\mathcal{L}) + w$$

for $\mathcal{L} \in \mathrm{Dom}(\sigma_\Gamma)$.

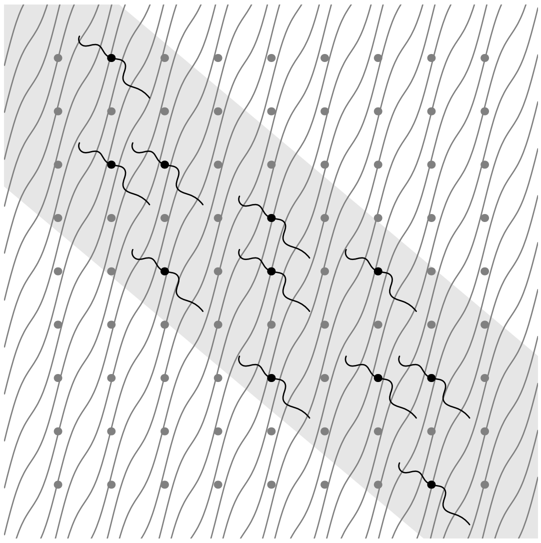

Figure 4.1: Here, the grey lines represent the foliation W^c_f. The lattice points of $\mathbb{Z}^d$ are drawn as grey dots, save for the points in a subset Γ which are drawn in black. The squiggle through a point $z \in \Gamma$ is the image of σ_z restricted to the support of α_z.

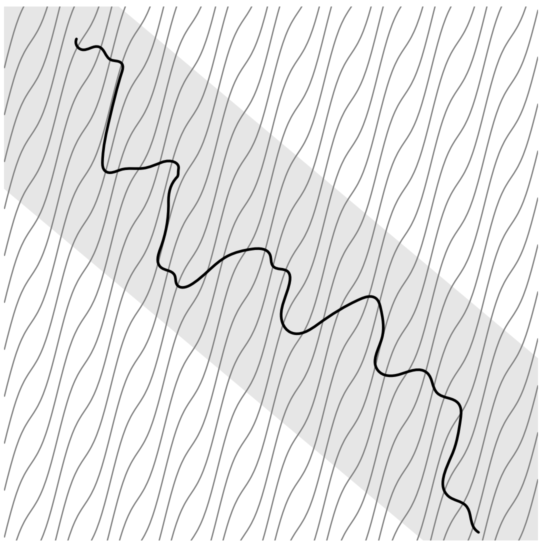

Figure 4.2: The image of the section σ_Γ, defined by averaging along center leaves.

This follows from the definitions of σ_z and α_z as translates of σ and α.

We next show that, in some sense, $\sigma_\Gamma(\mathcal{L})$ in a neighbourhood of $\mathcal{L}$ is uniquely determined by a subset of Γ lying near $\mathcal{L}$.

Lemma 4.6. *For $K \subset C_f$ compact, there is $R > 0$ such that if $\Gamma \subset \mathbb{Z}^d$ and $K \subset \mathrm{Dom}(\sigma_\Gamma)$ then*

$$\sigma_\Gamma|_K = \sigma_{\hat{\Gamma}}|_K$$

where $\hat{\Gamma} = \Gamma \cap B_R(W_g^c(z_0))$.

Proof. Since K is compact, its image under the continuous section $\sigma : C_f \to \mathbb{R}^d$ is also compact. In particular, $\pi_g^{us}(\sigma(K))$ is bounded. If $x \in \mathbb{R}^d$ lies on a leaf $\mathcal{L}$, then $\pi_g^{us}(x - \sigma(\mathcal{L}))$ is bounded by the constant given by Proposition 2.5. All together, this establishes that there is R_1 such that $\|\pi_g^{us}(x)\| < R_1$ for all $x \in \mathcal{L}$ where $\mathcal{L} \in K$.

Since the support of α is also a compact subset of C_f, there is R_2 such that $\|\pi_g^{us}(x)\| < R_2$ for all $x \in \mathcal{L}$ where $\alpha(\mathcal{L}) > 0$.

Suppose $z \in \Gamma$ is such that $\alpha_z(\mathcal{L}) > 0$ for some $\mathcal{L} \in K$. Then

$$\begin{aligned} \alpha_z(\mathcal{L}) > 0 \quad &\Rightarrow \\ \alpha(\mathcal{L} - z) > 0 \quad &\Rightarrow \\ \|\pi_g^{us}(x - z)\| < R_2 & \end{aligned}$$

where $x \in \mathcal{L}$. Since $\mathcal{L} \in K$, $\|\pi_g^{us}(x)\| < R_1$ and by the triangle inequality,

$$\|\pi_g^{us}(z)\| < R_1 + R_2.$$

Consequently, for $\mathcal{L} \in K$,

$$\sigma_\Gamma(\mathcal{L}) = \sum_{z\in\Gamma}^{c} \frac{\alpha_z(\mathcal{L})}{\alpha_\Gamma(\mathcal{L})}\sigma_z(\mathcal{L}) = \sum_{z\in\hat{\Gamma}}^{c} \frac{\alpha_z(\mathcal{L})}{\alpha_\Gamma(\mathcal{L})}\sigma_z(\mathcal{L})$$

where $\hat{\Gamma} = \{z \in \Gamma : \|\pi_g^{us}(z)\| < R_1 + R_2\}$. To conclude the proof, take R large enough that

$$\|\pi_g^{us}(z)\| < R_1 + R_2 \quad \Rightarrow \quad z \in B_R(W_g^c(z_0)).$$

□

Corollary 4.7. *For $K \subset C_f$ compact, there is $R > 0$ such that if $\Gamma \subset \mathbb{Z}^d$, $w \in \mathbb{Z}^d$, and $K + w \subset \mathrm{Dom}(\sigma_\Gamma)$ then*

$$\sigma_\Gamma(\mathcal{L} + w) = \sigma_{\hat{\Gamma}}(\mathcal{L}) + w$$

for all $\mathcal{L} \in K$ where $\hat{\Gamma} = (\Gamma - w) \cap B_R(W_g^c(z_0))$.

This is just the combination of the last two lemmas.

Lemma 4.8. *Let $\Lambda \subset \mathbb{Z}^d$ be such that*

- $\mathrm{Dom}(\sigma_\Lambda) = C_f$, *and*
- $\pi_g^c(\Lambda)$ *is bounded.*

Then $\sigma_\Lambda : C_f \to \mathbb{R}^d$ satisfies Axiom F.

Proof. Let $K \subset C_f$ be the support of α. Then $\mathrm{Dom}(\sigma_\Lambda) = C_f$ implies that the interiors of the sets $K_z = K + z$ for $z \in \Lambda$ give an open cover of C_f. Using this K, let $R > 0$ be given by Corollary 4.7. Let M be the bound on $\pi_g^c(\Lambda)$ and define

$$\mathcal{Z} = \{z \in \mathbb{Z}^d \cap B_R(W_g^c(z_0)) : \|\pi_g^c(z)\| < 2M\}.$$

$\mathcal{Z}$ is a finite set, so the collection $\Sigma = \{\sigma_\Gamma : \Gamma \subset \mathcal{Z}\}$ is also finite.

Now, if $z \in \Lambda$, then by Corollary 4.7, $\sigma_\Lambda(\mathcal{L} + z) = \sigma_\Gamma(\mathcal{L}) + z$ for $\mathcal{L} \in K$ where

$$\Gamma = (\Lambda - z) \cap B_R(W_g^c(z_0)).$$

Further, if $w \in \Gamma$, then

$$\begin{aligned} w \in \Lambda - z \quad &\Rightarrow \quad w + z \in \Lambda \\ &\Rightarrow \quad \|\pi_g^c(w)\| \le \|\pi_g^c(w + z)\| + \|\pi_g^c(z)\| < 2M \end{aligned}$$

showing that $\Gamma \subset \mathcal{Z}$ and so $\sigma_\Lambda|_{K_z}$ is isometrically equivalent to $\sigma_\Gamma \in \Sigma$. □

In addition to Axiom F, our desired section σ_Λ must be bounded in the E^c_g component. Fortunately, this is even easier to ensure.

Lemma 4.9. *If* $\Lambda \subset \mathbb{Z}^d$ *and* $\pi^c_g(\Lambda)$ *is bounded, then* $\pi^c_g \circ \sigma_\Lambda$ *is bounded.*

Proof. First note that $\pi^c_g \circ \sigma$ is bounded on the compact support of α. Then, as $\pi^c_g(z)$ is bounded for $z \in \Lambda$ and the σ_z are merely translates of σ, it follows that $\pi^c_g \circ \sigma_z(\mathcal{L})$ is uniformly bounded for $z \in \Lambda$ and $\mathcal{L} \in \operatorname{supp} \alpha_z$. Since each point $\sigma_\Lambda(\mathcal{L})$ is an averaging of such $\sigma_z(\mathcal{L})$, $\pi^c_g \circ \sigma_\Lambda$ is bounded by Proposition 2.14. □

To complete the construction of σ_Λ, define

$$\Lambda = \mathbb{Z}^d \cap B_{R_\mathbb{Z}}(W^{us}_g(z_0))$$

where $R_\mathbb{Z}$ is given by Lemma 4.3. For $\mathcal{L} \in C_f$ take any point $x \in \mathcal{L}$ and by the same lemma, there is $z \in \Lambda$ such that $x \in B_{R_\mathbb{Z}}(W^c_g(z))$. As $\mathcal{L} - z$ contains $x - z \in B_{R_\mathbb{Z}}(W^c_g(z_0))$, by Lemma 4.4, $\alpha_z(\mathcal{L}) = \alpha(\mathcal{L} - z) > 0$. Therefore, $\operatorname{Dom}(\sigma_\Lambda) = C_f$. Since $\pi^c_g(\Lambda)$ is bounded, $\sigma_\Lambda : C_f \to \mathbb{R}^d$ satisfies Axiom F and $\pi^c_g \circ \sigma_\Lambda$ is bounded. This completes the proof of Theorem 4.2.

Chapter 5

A leaf conjugacy on $\mathbb{R}^d$

Let $\sigma_0 : C_f \to \mathbb{R}^d$ be the uniformly continuous section given by Theorem 4.2 (there denoted by σ^*). Let z_0 denote the origin $(0, 0, \ldots, 0) \in \mathbb{R}^d$. Fix $v \in \mathbb{Z}^d$ such that

$$W_g^{us}(z_0) \cap W_g^{us}(z_0 + v) = \varnothing$$

and image(σ_0) and image$(\sigma_0) + v$ lie a bounded distance away from each other. We know such a v exists as $\pi_g^c \circ \sigma_0$ is bounded.

Define a section $\sigma_1 : C_f \to \mathbb{R}^d$ so that

$$\sigma_1(\mathcal{L}) = \sigma_0(\mathcal{L} - v) + v$$

or equivalently

$$\text{image } \sigma_1 = \text{image}(\sigma_0) + v.$$

For $\mathcal{L} \in C_f$, let $[\sigma_0(\mathcal{L}), \sigma_1(\mathcal{L})]^c$ denote the segment of the center leaf $\mathcal{L}$ between $\sigma_0(\mathcal{L})$ and $\sigma_1(\mathcal{L})$. The standard metric of $\mathbb{R}^d$ induces a metric on the leaf $\mathcal{L}$. Let

$$\rho(\mathcal{L}) = \text{length}\big([\sigma_0(\mathcal{L}), \sigma_1(\mathcal{L})]^c\big) = d_c(\sigma_0(\mathcal{L}), \sigma_1(\mathcal{L})).$$

As σ_0 is uniformly continuous, its translate σ_1 is uniformly continuous, as is ρ, the distance between them. As the images of σ_0 and σ_1 are a bounded distance apart, ρ is bounded away from zero. ρ is also bounded above, since if there were $\mathcal{L}_n \in C_f$ such that the center distance $d_c(\sigma_0(\mathcal{L}_n), \sigma_1(\mathcal{L}_n))$ diverged to infinity, then by Corollary 2.17, $\|\pi_g^c(\sigma_0(\mathcal{L}_n) - \sigma_1(\mathcal{L}_n))\| \to \infty$, contradicting the fact that $\pi_g^c \circ \sigma_0$ and $\pi_g^c \circ \sigma_1$ are bounded.

The next step in establishing a leaf conjugacy on $\mathbb{T}^d$ is to first construct one on $\mathbb{R}^d$, that is, a homeomorphism $h_0 : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$h_0 \circ g(\mathcal{L}) = f \circ h_0(\mathcal{L}) \quad \text{for } \mathcal{L} \in C_f.$$

This identity is ensured if h_0 is defined so that $h_0(x) \in H(\mathcal{L})$ for $x \in \mathcal{L}$ where $H : C_g \to C_f$ is the homeomorphism between spaces of leaves constructed in Chapter 3.

Define h_0 on the linear us-leaf $W_g^{us}(z_0)$ by

$$h_0(x) = \sigma_0(H(W_g^c(x)))$$

and on $W_g^{us}(z_0 + v)$ by

$$h_0(x) = \sigma_1(H(W_g^c(x))).$$

Let S denote the solid (closed) slab between the hyperplanes $W_g^{us}(z_0)$ and $W_g^{us}(z_0+v)$ inclusively. Identifying E_g^c with $\mathbb{R}$, this slab can be written as

$$S = \{x \in \mathbb{R}^d : \pi_g^c(z_0) \le \pi_g^c(x) \le \pi_g^c(z_0 + v)\}.$$

Let $v' \in \mathbb{R}^d$ denote the unique intersection of $W_g^c(z_0)$ with $W_g^{us}(z_0 + v)$. Then, extend h_0 to all of S by mapping the segment $[x, x + v']_g^c \subset W_g^c(x)$ to the segment $[h_0(x), h_0(x + v')]_f^c \subset H(W_g^c(x))$ at a constant speed.

To express h_0 in formulas, define $t : S \to [0, 1]$ as a function on the solid slab that takes the value 0 on $W_g^{us}(z_0)$, the value 1 on $W_g^{us}(z_0+v)$ and interpolates linearly between them:

$$t(x) = \frac{\|\pi_g^c(x - z_0)\|}{\|\pi_g^c(v)\|}.$$

Since E_f^c is uniquely integrable and one-dimensional, if we give it an orientation, it defines a flow which progresses at unit speed $\varphi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$. Orient E_f^c so that this flow goes from the image of σ_0 to the image of σ_1. Note that ρ is defined so that

$$\varphi_{\rho(\mathcal{L})}(\sigma_0(\mathcal{L})) = \sigma_1(\mathcal{L})$$

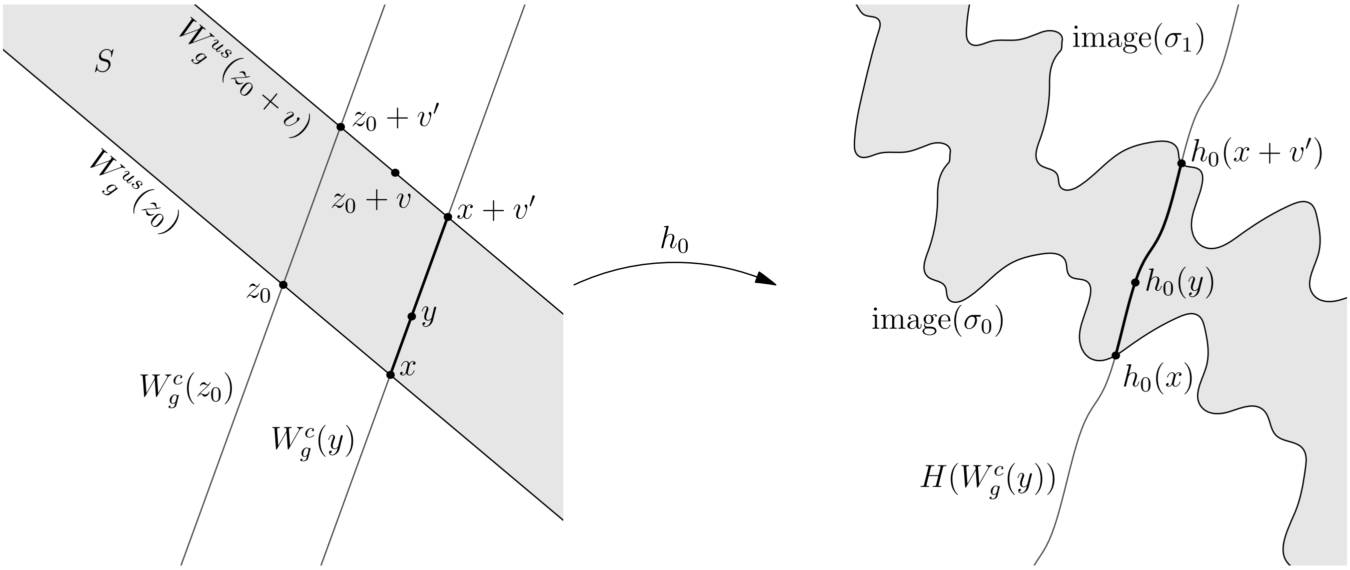

Figure 5.1: The map h_0.

for $\mathcal{L} \in C_f$. Finally, h_0 on the domain S is defined as

$$h_0(x) = \varphi_{\rho(\mathcal{L})\cdot t(x)}(\sigma_0(\mathcal{L}))$$

where $\mathcal{L} = H(W_g^c(x))$.

Lemma 5.1. h_0 *is uniformly continuous on* S *and uniformly bilipschitz along each center leaf.*

Proof. In essence, h_0 is uniformly continuous as it is constructed from uniformly continuous functions.

Equip C_f and C_g with metrics invariant under the action of $\pi_1(\mathbb{T}^d)$. As ρ is bounded both above and away from zero, let $0 < m < M$ be such that image $\rho \subset [m, M]$. The functions $\sigma_0 : C_f \to \mathbb{R}^d$ and $\rho : C_f \to [m, M]$ have been shown to be uniformly continuous. That $H : C_g \to C_f$ is uniformly continuous can be deduced from the relation $H \circ \tau = \tau \circ H$ for $\tau \in \pi_1(\mathbb{T}^d)$. $t : S \to [0, 1]$ and the quotient map $W_g^c : \mathbb{R}^d \to C_g$ are linear and therefore uniformly continuous.

The multiplication map $(a, b) \mapsto a \cdot b$ is not uniformly continuous on $\mathbb{R} \times \mathbb{R}$, but in the case of $\rho(\mathcal{L}) \cdot t(x)$, $\rho(\mathcal{L})$ takes values in $[m, M]$, $t(x)$ takes values in $[0, 1]$ and multiplication restricted to $[m, M] \times [0,1]$ is uniformly continuous.

Since, for the center flow, $\varphi_t \circ \tau = \tau \circ \varphi_t$ where $t \in \mathbb{R}$ and $\tau \in \pi_1(\mathbb{T}^d)$, one can show that φ is uniformly continuous on $\mathbb{R}^d \times [0, M]$ by looking a fundamental domain of the covering.

Finally, as a composition of uniformly continuous function, h_0 is uniformly continuous.

Since h_0 maps center leaves of f, each of length $\|\pi_g^c(v)\|$, to center leaves of g, each of length between m and M, and maps each at a constant speed, h_0 is bilipschitz along center leaves. □

Lemma 5.2. *$h_0 : S \to \mathbb{R}^d$ is a bounded distance from the identity:*

$$\sup_{x \in S} \|h_0(x) - x\| < \infty.$$

Proof. Let $K \subset \mathbb{R}^d$ be a fundamental domain of the covering $\mathbb{R}^d \to \mathbb{T}^d$. By compactness of K, there is an $R > 0$ such that for all $x \in K$

$$H(W_g^c(x)) \subset B_R(W_g^c(x))$$

where the sets are regarded as subsets of $\mathbb{R}^d$. Then, if $\tau \in \pi_1(\mathbb{T}^d)$,

$$\tau(H(W_g^c(x))) \subset \tau(B_R(W_g^c(x))) \quad \Rightarrow$$
$$H(W_g^c(\tau(x))) \subset B_R(W_g^c(\tau(x)))$$

since $H \circ \tau = \tau \circ H$, so this inclusion holds for all points in $\mathbb{R}^d$. In particular,

$$\sigma_0(H(W_g^c(x))) \in B_R(W_g^c(x)),$$

so

$$\left\|\pi_g^{us}\big(\sigma_0(H(W_g^c(x))) - x\big)\right\|$$

is bounded for $x \in S$.

By Theorem 4.2, $\pi_g^c \circ \sigma_0$ is bounded, and by the definition of the solid slab S, $\|\pi_g^c(x)\| \leq \|\pi_g^c(v)\|$ for $x \in S$, so

$$\left\|\pi_g^c\big(\sigma_0(H(W_g^c(x))) - x\big)\right\|$$

is bounded for $x \in S$ implying that

$$\|\sigma_0(H(W_g^c(x))) - x\|$$

is bounded as well, say by M_σ.

Now $h_0(x)$ is defined as $\varphi_{\rho(\mathcal{L})\cdot t(x)}(\sigma_0(\mathcal{L}))$ where $\mathcal{L} = H(W_g^c(x))$. We have established that $\|\sigma_0(\mathcal{L}) - x\| < M_\sigma$. Recall that $\rho(\mathcal{L}) \cdot t(x)$ is bounded by a constant referred to in the previous proof as M. φ is a unit-speed flow, so

$$\begin{aligned}
\|\varphi_{\rho(\mathcal{L})\cdot t(x)}(\sigma_0(\mathcal{L})) - \sigma_0(\mathcal{L})\| < M \quad \Rightarrow& \\
\|h_0(x) - x\| &\leq \|h_0(x) - \sigma_0(\mathcal{L})\| + \|\sigma_0(\mathcal{L}) - x\| \\
&< M + M_\sigma
\end{aligned}$$

for all $x \in S$. □

Extend h_0 to all of $\mathbb{R}^d$ by requiring $h_0(x + v) = h_0(x) + v$. Since for $x \in \mathbb{R}^d$, there is $k \in \mathbb{Z}$ such that $x + kv \in S$, this uniquely defines h_0. If, however, $x \in W_g^{us}(z_0 + kv)$ for some k, then $x - kv \in W_g^{us}(z_0) \subset S$ and $x - (k-1)v \in W_g^{us}(z_0 + v) \subset S$ so we must verify that

$$h_0(x + v) = h_0(x) + v$$

for $x \in W_g^{us}(z_0)$ to ensure that this extension of h_0 is well-defined. If $x \in W_g^{us}(z_0)$, then

$$\begin{aligned}
h_0(x + v) &= \sigma_1(H(W_g^c(x + v))) \\
&= \sigma_1(H(W_g^c(x)) + v) && \text{(by properties of } H\text{)} \\
&= \sigma_0(H(W_g^c(x))) + v && \text{(by the definition of } \sigma_1\text{)} \\
&= h_0(x) + v
\end{aligned}$$

as desired.

Corollary 5.3. *$h_0 : \mathbb{R}^d \to \mathbb{R}^d$ is uniformly continuous and uniformly bilipschitz along center leaves.*

Corollary 5.4. $h_0 : \mathbb{R}^d \to \mathbb{R}^d$ *is a bounded distance from the identity:*

$$\|h_0 - id\|_0 := \sup_{x \in \mathbb{R}^d} \|h_0(x) - x\| < \infty.$$

These follow from the corresponding lemmas for h_0 as first defined on $S \subset \mathbb{R}^d$ by use of the relation $h_0(x + v) = h_0(x) + v$.

We have constructed a leaf conjugacy from $g : \mathbb{R}^d \to \mathbb{R}^d$ to $f : \mathbb{R}^d \to \mathbb{R}^d$; $h_0 : \mathbb{R}^d \to \mathbb{R}^d$ is a homeomorphism such that $h_0(g(\mathcal{L})) = f(h_0(\mathcal{L}))$ for each center leaf $\mathcal{L}$ of g. Our work is not done, unfortunately, as our goal is a leaf conjugacy on the closed manifold $\mathbb{T}^d$, and there is no reason to believe that h_0 descends to that space. Instead, we will "average" shifts of h_0 to produce a homeomorphism h that does descend to $\mathbb{T}^d$.

Chapter 6

A leaf conjugacy on $\mathbb{T}^d$

For $z \in \mathbb{Z}^d$, let $\tau_z : \mathbb{R}^d \to \mathbb{R}^d$ denote the translation $x \mapsto x + z$ and define $h_z : \mathbb{R}^d \to \mathbb{R}^d$ as a shift of h_0:

$$h_z = \tau_z \circ h \circ \tau_z^{-1} = \tau_z \circ h \circ \tau_{-z}.$$

As h_0 is uniformly continuous, the collection

$$\mathcal{H}_0 = \{h_z : z \in \mathbb{Z}^d\}$$

is uniformly equicontinuous. Also, since h_0 is a bounded distance from the identity, the functions h_z are a bounded distance away from the identity, and from each other:

$$\|h_z - id\|_0 = \|h_0 - id\|_0 \qquad \text{for } z \in \mathbb{Z}^d, \text{ and}$$

$$\|h_z - h_{z'}\|_0 \leq 2\|h_0 - id\|_0 \qquad \text{for } z, z' \in \mathbb{Z}^d.$$

As with sections, we want to average the functions h_z along center leaves. To do so, all of the functions must map a point x to the same center leaf. Functions $h, h' : \mathbb{R}^d \to \mathbb{R}^d$ are *c-equivalent* (with respect to f) if $h'(x) \in W^c_f(h(x))$ for all $x \in \mathbb{R}^d$. If $h_1, \ldots, h_n$ are c-equivalent and $a_1, \ldots, a_n \in \mathbb{R}$, $\sum a_i = 1$, define the affine sum $\sum^c a_i h_i : \mathbb{R}^d \to \mathbb{R}^d$ pointwise by

$$\left(\sum\nolimits^c a_i h_i\right)(x) = \sum\nolimits^c a_i h_i(x) \quad \text{for } x \in \mathbb{R}^d,$$

where $\sum^c$ on the right-hand side is summation along the center leaf as defined at the start of the subsection.

Note that since each h_z maps $x \in \mathbb{R}^d$ to a point on the leaf $H(W_g^c(x))$, all shifts h_z of h_0 are c-equivalent. Let $\mathcal{H}_1$ be the set of all (finite) affine combinations of elements of $\mathcal{H}_0$:

$$\mathcal{H}_1 = \left\{ \sum^c a_i h_{z_i} : h_{z_i} \in \mathcal{H}_0, a_i \in \mathbb{R}, \sum a_i = 1 \right\}.$$

Lemma 6.1. $\mathcal{H}_1$ *is equicontinuous.*

Proof. Fix $x \in \mathbb{R}^d$ and $\epsilon > 0$. Since all of the functions h_z, $z \in \mathbb{Z}^d$ lie a finite distance from each other, fix a unit-speed curve $\gamma : [0,T] \to \mathbb{R}^d$ along the center leaf $\mathcal{L}_x := H(W_g^c(x))$ such that all points $h_z(x)$, $z \in \mathbb{Z}^d$ lie in the interior of the curve. There is a tubular neighbourhood of the image of γ such that within this neighbourhood the center foliation is trivial.

In fact, take a local section $\sigma : D \to \mathbb{R}^d$ defined so that $D \subset C_f$ is a small topological closed disk containing $\mathcal{L}_x$ in its interior and such that $\sigma(\mathcal{L}_x) = \gamma(0)$. Define $\phi : D \times [0,T] \to \mathbb{R}^d$ by letting $\phi(\mathcal{L}, t)$ be the result of sliding $\sigma(\mathcal{L})$ a distance t along the leaf. Further assume that D is small enough that

$$\|\phi(\mathcal{L}_x, t) - \phi(\mathcal{L}, t)\| < \frac{\epsilon}{2}$$

for all $\mathcal{L} \in D$ and $t \in [0,T]$, which is possible as the foliation W_f^c is continuous. ϕ is an embedding and $K = \phi(D \times [0,T]) \subset \mathbb{R}^d$ is a tubular neighbourhood of $\gamma([0,T]) = \phi(\{\mathcal{L}_x\} \times [0,T])$.

Let $\psi : K \to \mathbb{R}$ be defined to satisfy the equation $\psi(\phi(\mathcal{L}, t)) = t$. For $x \in K$, $\psi(x)$ gives the distance from x to the image of σ as measured along $W_f^c(x)$ and is therefore continuous.

As K is compact, ψ is uniformly continuous and there exists $\delta > 0$ such that for $y, z \in K$,

$$\|y - z\| < \delta \quad \Rightarrow \quad |\psi(y) - \psi(z)| < \frac{\epsilon}{2}.$$

Now take $y \in \mathbb{R}^d$ close enough to x that $h(y) \in K$ and $\|h(x) - h(y)\| < \delta$ for all $h \in \mathcal{H}_0$. (This is possible due to the equicontinuity of $\mathcal{H}_0$.) Let $\mathcal{L}_y$ denote $H(W^c_g(y))$.

Take any affine combination $\sum^c a_i h_i$ where $h_i \in \mathcal{H}_0$. Then (using that $\sum a_i = 1$ and ϕ is a unit-speed flow),

$$\begin{aligned} \|h_i(x) - h_i(y)\| < \delta \quad \text{for all } i \quad &\Rightarrow \\ |\psi(h_i(x)) - \psi(h_i(y))| < \frac{\epsilon}{2} \quad \text{for all } i \quad &\Rightarrow \\ \left|\sum a_i\ \psi(h_i(x)) - \sum a_i\ \psi(h_i(y))\right| < \frac{\epsilon}{2} \quad &\Rightarrow \\ \left\|\phi\left(\mathcal{L}_x, \sum a_i\ \psi(h_i(x))\right) - \phi\left(\mathcal{L}_x, \sum a_i\ \psi(h_i(y))\right)\right\| < \frac{\epsilon}{2}. \end{aligned}$$

Also,

$$\left\|\phi\left(\mathcal{L}_x, \sum a_i\ \psi(h_i(y))\right) - \phi\left(\mathcal{L}_y, \sum a_i\ \psi(h_i(y))\right)\right\| < \frac{\epsilon}{2}$$

from the choice of $D \subset C_f$. Thus

$$\left\|\sum^c a_i h_i(x) - \sum^c a_i h_i(y)\right\| < \epsilon$$

since, by the definition of the affine summation $\sum^c$,

$$\sum^c a_i h_i(x) = \phi\left(\mathcal{L}_x, \sum a_i\ \psi(h_i(x))\right)$$

and

$$\sum^c a_i h_i(y) = \phi\left(\mathcal{L}_y, \sum a_i\ \psi(h_i(y))\right).$$

As the estimates for continuity did not depend on the particular choice of affine combination $\sum^c a_i h_i$, this shows that $\mathcal{H}_1$ is equicontinuous. $\square$

Now for $n > 0$, define

$$C_n = \left\{(k_1, \ldots, k_d) \in \mathbb{Z}^d : \max\{|k_1|, \ldots, |k_d|\} \le n\right\}$$

and

$$h_n = \sum_{z \in C_n}^{c} \frac{1}{(2n+1)^d} h_z.$$

When restricted to any compact subset of $\mathbb{R}^d$, the sequence $\{h_n\} \subset \mathcal{H}_1$ is uniformly equicontinuous and uniformly bounded, so, by Arzelà-Ascoli, has a uniformly convergent subsequence. By a diagonalization argument, we find a sequence $\{h_{n_k}\}$ that converges uniformly on compact subsets of $\mathbb{R}^d$. Let h be the limit of this sequence.

Lemma 6.2. *h is c-equivalent to h_0.*

Proof. Each h_{n_k} is c-equivalent to h_0 by construction. Then for $x \in \mathbb{R}^d$,

$$h_{n_k}(x) \in W_f^c(h_0(x)) \quad \Rightarrow \quad h(x) = \lim_{k\to\infty} h_{n_k}(x) \in W_f^c(h_0(x))$$

since the leaf is a closed subset of $\mathbb{R}^d$. □

Then, from the construction of h_0,

$$h(x) \in H(W_g^c(x)) \quad \text{for } x \in \mathbb{R}^d$$

so $h(g(\mathcal{L})) = f(h(\mathcal{L}))$ for any center leaf of g.

Lemma 6.3. *h is injective.*

Proof. Since H is a homeomorphism of leaves, h maps points on distinct center leaves of g to points on distinct center leaves of f. We need only show that if $y \in W_g^c(x)$ and $x \neq y$ then $h(x) \neq h(y)$.

From Corollary 5.3, h_0 is bilipschitz on center leaves, so there is $r > 0$ such that if $y \in W_g^c(x)$ then

$$d_f^c(h_0(x), h_0(y)) \geq r \cdot d_g^c(x, y).$$

Because elements of $\mathcal{H}_0$ are simply shifts of h_0, the same inequality holds for h_z, $z \in \mathbb{R}^d$. Then for an affine combination $\sum^c a_i h_{z_i}$,

$$\begin{aligned} d_f^c\left(\sum^c a_i h_{z_i}(x), \sum^c a_i h_{z_i}(y)\right) &= \sum a_i \, d_f^c(h_{z_i}(x), h_{z_i}(y)) \\ &\geq r \cdot d_g^c(x, y) \end{aligned}$$

where instead of a triangle inequality, we have a true equality as h_0 and its shifts h_{z_i} preserve the orientation of the leaves.

Since $h_{n_k} \to h$,

$$d_f^c(h_{n_k}(x), h_{n_k}(y)) \geq r \cdot d_g^c(x, y) \quad \Rightarrow$$
$$d_f^c(h(x), h(y)) \geq r \cdot d_g^c(x, y),$$

so $x \neq y \ \Rightarrow \ h(x) \neq h(y)$. □

Lemma 6.4. *$h(x+z) = h(x) + z$ for all $z \in \mathbb{Z}^d$.*

Proof. We show $h(x+w) = h(x) + w$ for $w = (1, 0, 0, \ldots, 0) \in \mathbb{Z}^d$. The other coordinates are proved similarly and the result follows. Recall that τ_z denotes the translation $x \mapsto x + z$. We want to show that $\tau_w \circ h \circ \tau_{-w} = h$.

Fix $x \in \mathbb{R}^d$. Let $\mathcal{L} = h(W_g^c(x))$ and let φ be an isometry mapping $\mathcal{L}$ to $\mathbb{R}$. Then

$$\begin{aligned}\varphi(h_n(x)) &= \varphi\left(\sum_{z \in C_n}^{c} \frac{1}{(2n+1)^d} h_z(x)\right) \\ &= \frac{1}{(2n+1)^d} \sum_{z \in C_n} \varphi(h_z(x))\end{aligned}$$

whereas

$$\begin{aligned}\varphi(\tau_w \circ h_n \circ \tau_{-w}(x)) &= \varphi\left(\sum_{z \in C_n}^{c} \frac{1}{(2n+1)^d} \tau_w \circ h_z \circ \tau_{-w}(x)\right) \\ &= \frac{1}{(2n+1)^d} \sum_{z \in C_n} \varphi(h_{z+w}(x)).\end{aligned}$$

Then

$$\begin{aligned}&\varphi(\tau_w \circ h_n \circ \tau_{-w}(x)) - \varphi(h_n(x)) \\ &\qquad = \frac{1}{(2n+1)^d}\left(\sum_{z \in C_n^+} \varphi(h_z(x)) - \sum_{z \in C_n^-} \varphi(h_z(x))\right)\end{aligned}$$

where

$$C_n^+ = \left\{(n+1, k_2, k_3, \ldots, k_d) \in \mathbb{Z}^d : \max\{|k_2|, \ldots, |k_d|\} \leq n\right\}$$

and

$$C_n^- = \big\{(0, k_2, k_3, \ldots, k_d) \in \mathbb{Z}^d : \max\{|k_2|, \ldots, |k_d|\} \leq n\big\}$$

C_n^+ and C_n^- each have exactly $(2n+1)^{d-1}$ elements. Note, also, that the collection $\{\varphi(h_z(x)) : z \in \mathbb{Z}^d\}$ is bounded, since the functions h_z are all at most a uniform distance away from each other. Say $|\varphi(h_z(x))| < M$ for $z \in \mathbb{Z}^d$. Then

$$\begin{aligned} &\big|\varphi\big(\tau_w \circ h_{n_k} \circ \tau_{-w}(x)\big) - \varphi\big(h_{n_k}(x)\big)\big| \\ &\qquad \leq \frac{1}{(2n_k+1)^d}\left(\sum_{z\in C_{n_k}^+} |\varphi(h_z(x))| + \sum_{z\in C_{n_k}^-} |\varphi(h_z(x))|\right) \\ &\qquad \leq \frac{1}{(2n_k+1)^d}\; 2\;(2n_k+1)^{d-1}\; M = \frac{2M}{2n_k+1} \end{aligned}$$

which tends to zero as $k \to \infty$ showing that

$$\varphi(\tau_w \circ h \circ \tau_{-w}(x)) = \varphi(h(x))$$

and therefore $\tau_w \circ h \circ \tau_{-w} = h$. □

Now that this invariance is established, $h : \mathbb{R}^d \to \mathbb{R}^d$ descends to a map $\tilde{h} : \mathbb{T}^d \to \mathbb{T}^d$ and this map is the leaf conjugacy between $f_0 : \mathbb{T}^d \to \mathbb{T}^d$ and its linearization $g_0 : \mathbb{T}^d \to \mathbb{T}^d$. As h is injective and $h \circ \tau = \tau \circ h$ for $\tau \in \pi_1(\mathbb{T}^d)$, $\tilde{h}$ is injective and homotopic to the identity map to $\mathbb{T}^d$, showing that it is a homeomorphism. Finally, the relation $h(g(\mathcal{L})) = f(h(\mathcal{L}))$ for any center leaf of g, implies that $\tilde{h}(g_0(\mathcal{L})) = f_0(\tilde{h}(\mathcal{L}))$ for any center leaf of g_0.

Chapter 7

Further Questions

In the preceding proof, we assumed that the center foliation was one-dimensional, largely for technical reasons, and made use of this simplification in several places. Is this a necessary assumption, or can the proof be generalized for higher dimensional center foliations? The most difficult concept to extend may be the “averaging” along center leaves used in the construction of the section in Chapter 4. In higher dimensions, center-of-mass constructions exist for weighted averages on manifolds, but these make use of the exponential map and are only defined when averaging points in a small neighbourhood. It is unclear if this concept can be applied as a replacement for the method of averaging along center leaves.

The assumption of quasi-isometry of the stable and unstable foliations is fundamental to the proof, allowing us to compare the leaves of the diffeomorphism with its linearization. While I see no reason to believe it, one could also ask if Theorem 1.2 holds without the assumption of quasi-isometry. There are examples of partially hyperbolic systems whose strong foliations are not quasi-isometric, but we know of no such examples on higher dimensional tori. If, indeed, all partially hyperbolic systems on tori enjoy this property of quasi-isometry, then its supposition in the theorem is redundant.

In his thesis, Franks first studied tori as the simplest spaces on which to establish conjugacies. With the work of Manning, the proof was extended to all nilmanifolds and infranilmanifolds. While $\mathbb{R}^d$ is mentioned throughout this thesis, many of the properties proved hold just as well for the universal cover of any nilmanifold. The main result, therefore, likely has an analogue which describes partially hyperbolic systems on these spaces.

C. Bonatti and A. Wilkinson examined transitive partially hyperbolic systems on 3-manifolds, giving strong evidence that all such systems have already been discovered [3]. Brin, Burago, and Ivanov showed that there are no partially hyperbolic systems on either $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{S}^1$, as their fundamental groups are too simple to support these systems [6, 5]. Hopefully, this thesis is a link in a chain of reasoning that will lead eventually to the description of all partially hyperbolic systems on three-dimensional manifolds.

Appendix A

Quasi-isometry implies plaque expansiveness

An ϵ-*pseudo orbit* of $f : M \to M$ that respects an invariant foliation W is a bi-infinite sequence $\{x_n\}$ in M such that for all $n \in \mathbb{Z}$, $f(x_{n-1})$ and x_n lie on the same leaf of W and $d_W(f(x_{n-1}), x_n) < \epsilon$. The diffeomorphism f is *plaque expansive* with respect to W if for every $\epsilon_0 > 0$ there exists $\epsilon > 0$ such that the following holds:

> If $\{x_n\}$ and $\{y_n\}$ are ϵ-pseudo orbits of f that respect W and $d(x_n, y_n) < \epsilon$ for all $n \in \mathbb{Z}$ then x_0 and y_0 lie on the same leaf of W and $d_W(x_0, y_0) < \epsilon_0$.

Theorem A.1. *Let f be a partially hyperbolic diffeomorphism of a compact Riemannian manifold M. Suppose the stable W^s and unstable W^u foliations of f are quasi-isometric in the universal cover $\tilde{M}$. Then the distributions E^c, E^{cs} and E^{cu} integrate uniquely to plaque expansive foliations.*

Remark. This theorem is inspired by the proof of dynamical coherence under the same hypotheses due to Brin [4]. One great advantage of establishing plaque expansiveness for a partially hyperbolic diffeomorphism f is that perturbations of f are also plaque expansive and therefore dynamically coherent. In this case, however, one can show that the hypothesis of quasi-isometry is stable under perturbation, so plaque expansiveness is not needed to establish stable dynamical coherence. The result is still useful, though, in establishing that f is leaf conjugate to its neighbors, and engenders hope of answering the open question of whether all dynamically coherent, partially hyperbolic systems are plaque expansive.

This result will also appear as a self-contained note [13].

Proof. That the distributions are uniquely integrable is shown by Brin [4]. We will prove that W^{cs} is plaque expansive. The case for W^{cu} is similar, and then it follows from the definition that the intersection W^c of the foliations W^{cs} and W^{cu} is also plaque expansive.

Given $\epsilon > 0$ small, let $\{x_n\}$ and $\{y_n\}$ be ϵ-pseudo orbits respecting W^{cs} such that for all $n \in \mathbb{Z}$, $d(x_n, y_n) < \epsilon$. There exist paths $\alpha_n, \beta_n : [0,1] \to M$ of length at most ϵ and tangent to E^{cs} such that

$$\alpha_n(0) = f(x_{n-1}), \quad \alpha_n(1) = x_n,$$

$$\beta_n(0) = f(y_{n-1}), \quad \beta_n(1) = y_n.$$

Because x_0 and y_0 are close together, by sliding y_0 along its W^{cs} leaf, we may assume, without loss of generality, that x_0 and y_0 lie on the same local unstable leaf.[1] To establish plaque expansiveness, we can then show that $x_0 = y_0$.

The diffeomorphism f lifts from M to its universal cover $\tilde{M}$ where, by abuse of notation, we still call it f. Lift x_0 and y_0 to $\tilde{x}_0, \tilde{y}_0 \in \tilde{M}$ so that the two points still lie close together. Then inductively for $n > 0$, lift the paths α_n, β_n on M to paths $\tilde{\alpha}_n, \tilde{\beta}_n$ on $\tilde{M}$ such that $\tilde{\alpha}_n(0) = f(\tilde{x}_{n-1})$ and $\tilde{\beta}_n(0) = f(\tilde{y}_{n-1})$ and define $\tilde{x}_n := \tilde{\alpha}_n(1)$ and $\tilde{y}_n := \tilde{\beta}_n(1)$. Because the lengths of α_n and β_n are small and $\tilde{M}$ is locally identified with M, it follows that $d(\tilde{x}_n, \tilde{y}_n) = d(x_n, y_n) < \epsilon$.

[1]Because W^{cs} and W^u are uniformly transverse, there is a constant $0 < c < \frac{1}{2}$ such that if $d(x_0, y_0) < c\epsilon$ then there is a point z_0 on the unstable leaf of x_0 and the center-stable leaf of y_0 and $d_u(x_0, z_0)$, $d_{cs}(y_0, z_0)$, and $d_{cs}(f(y_0), f(z_0))$ are each less than $\epsilon/2$. Therefore, a $c\epsilon$-pseudo orbit is turned into an ϵ-pseudo orbit by replacing y_0 with z_0.

As f is partially hyperbolic (on both M and $\tilde{M}$), there are constants $1 < \gamma < \mu$ and $C \geq 1$ such that

$$\|df^n(x)v^{cs}\| \leq C\gamma^n \|v^{cs}\| \quad \text{for } v^{cs} \in E^{cs}_x \text{ and } n > 0,$$

and

$$C^{-1}\mu^n \|v^u\| \leq \|df^n(x)v^u\| \quad \text{for } v^u \in E^u_x \text{ and } n > 0.$$

Consequently, as the $\tilde{\alpha}_n$ are tangent to E^{cs},

$$\text{length}(f^k \circ \tilde{\alpha}_n) \leq C\gamma^k \text{ length}(\tilde{\alpha}_n)$$

so

$$d\big(f^k(f(\tilde{x}_n)), f^k(\tilde{x}_{n+1})\big) < C\gamma^k \epsilon$$

and

$$\begin{aligned} d(f^n(\tilde{x}_0), \tilde{x}_n) &\leq \sum_{k=0}^{n-1} d(f^{k+1}(\tilde{x}_{n-k-1}), f^k(\tilde{x}_{n-k})) \\ &< \sum_{k=0}^{n-1} C\gamma^k \epsilon = C\frac{\gamma^n - 1}{\gamma - 1}\epsilon. \end{aligned}$$

Similarly, $d(f^n(\tilde{y}_0), \tilde{y}_n) < C\frac{\gamma^n-1}{\gamma-1}\epsilon$, so

$$\begin{aligned} d(f^n(\tilde{x}_0), f^n(\tilde{y}_0)) &\leq d(f^n(\tilde{x}_0), \tilde{x}_n) + d(\tilde{x}_n, \tilde{y}_n) + d(\tilde{y}_n, f^n(\tilde{y}_0)) \\ &< \left(2C\frac{\gamma^n - 1}{\gamma - 1} + 1\right)\epsilon. \end{aligned}$$

On the other hand, $\tilde{x}_0$ and $\tilde{y}_0$ lie on the same unstable leaf, so

$$d_u(f^n(\tilde{x}_0), f^n(\tilde{y}_0)) \geq C^{-1}\mu^n d_u(\tilde{x}_0, \tilde{y}_0)$$

where d_u is distance measured along the unstable leaf. By quasi-isometry

$$\begin{aligned} &d_u(f^n(\tilde{x}_0), f^n(\tilde{y}_0)) \leq a \cdot d(f^n(\tilde{x}_0), f^n(\tilde{y}_0)) + b \quad \Rightarrow \\ &\quad d(f^n(\tilde{x}_0), f^n(\tilde{y}_0)) \geq (d_u(f^n(\tilde{x}_0), f^n(\tilde{y}_0)) - b)/a \geq (C^{-1}\mu^n d_u(\tilde{x}_0, \tilde{y}_0) - b)/a. \end{aligned}$$

Since $\gamma < \mu$, these two estimates are irreconcilable for large $n > 0$ unless $d_u(\tilde{x}_0, \tilde{y}_0) = 0$. This means that $\tilde{x}_0 = \tilde{y}_0$, so $x_0 = y_0$ and plaque expansiveness is proved. □

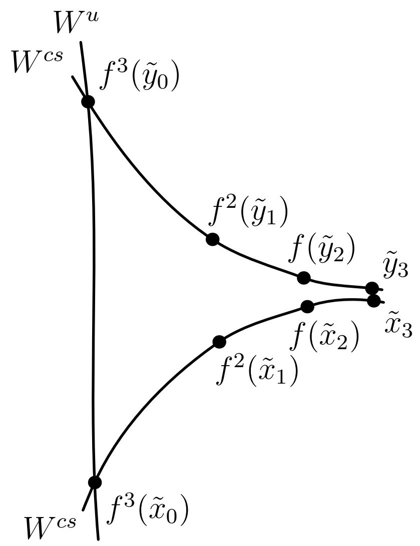

Figure A.1: The invariant manifolds through $f^n(\tilde{x}_0)$ and $f^n(\tilde{y}_0)$ for $n = 3$.

Brin, Burago, and Ivanov have shown that all partially hyperbolic diffeomorphisms on the 3-torus are dynamically coherent [6, 8, 5]. Since this is proved by establishing quasi-isometry as in the hypotheses of the preceeding theorem, it yields the following.

Corollary A.2. *All partially hyperbolic systems on the 3-torus are plaque expansive.*

Appendix B

$W_f^{us}(p)$ is differentiable at p

Lemma B.1. *$W_f^{us}(p)$ is (once) differentiable at $p \in \mathbb{R}^d$ and the tangent space of $W_f^{us}(p)$ at p is $E_f^u(p) \oplus E_f^s(p)$.*

Proof. Fix $p \in \mathbb{R}^d$. Let $u = \dim E_f^u$ and $s = \dim E_f^s$. $W_f^u(p)$ is a C^1-leaf, so there is a neighbourhood $0 \in U \subset \mathbb{R}^u$ and a C^1-embedding $\phi : U \to W_f^u(p)$ such that $\phi(0) = p$. As W_f^s is a continuous foliation with C^1-leaves tangent to a continuous distribution E_f^s, there is a neighbourhood $0 \in V \subset \mathbb{R}^s$ and a continuous function $\psi : U \to C^1(V, \mathbb{R}^d)$ so that for each $x \in U$, $\psi(x)(0) = \phi(x)$ and $\phi(x)$ is a C^1-embedding of V into $W_f^s(\psi(x))$.

By abuse of notation, write $\psi(x, y) = \psi(x)(y)$. Then we can consider ψ as a map $U \times V \to W_f^{us}(p)$ and show it is differentiable at $(0, 0)$.

For $(x, y) \in U \times V$ and $v \in \mathbb{R}^s$, let $D_v\psi(x, y)$ denote the directional derivative

$$D_v\psi(x, y) = \lim_{t\to 0} \frac{1}{t} \left(\psi(x, y + tv) - \psi(x, y)\right) \in \mathbb{R}^d.$$

By construction of ψ, $D_u\psi(x, y)$ is a continuous function of u, x, and y. Let $V_1 \subset \mathbb{R}^s$ be open such that $0 \in V_1 \subset \overline{V_1} \subset V$. Then

$$\lim_{x\to 0} \sup\{D_u\psi(x, y) - D_u\psi(0, y) : y \in V_1, u \in \mathbb{R}^s, \|u\| = 1\} = 0$$

for otherwise, there are sequences $\{x_n\}$ in U, $\{y_n\}$ in V_1 and $\{u_n\}$ in $\mathbb{R}^s$ where

$$x_n \to 0, \quad y_n \to y \in V_1, \quad \text{and} \quad u_n \to u \in \mathbb{R}^s,$$

but $D_{u_n}\psi(x_n, y_n) - D_{u_n}(0, y_n)$ does not converge to zero, a contradiction.

By the Fundamental Theorem of Calculus,

$$\psi(x,y) - \psi(x,0) = \int_0^1 D_u\psi(x,ty)dt \ \cdot \ \|y\|$$

where $u = \frac{y}{\|y\|}$, so if $(x,y) \to (0,0)$ then

$$\begin{aligned}\frac{\|\psi(x,y) - \psi(x,0) - \psi(0,y) + \psi(0,0)\|}{\|y\|} &= \left\| \int_0^1 D_u\psi(x,ty)dt - \int_0^1 D_u\psi(0,ty)dt \right\| \\ &\leq \int_0^1 \|D_u\psi(x,ty) - D_u\psi(0,ty)\|dt \to 0.\end{aligned}$$

Since $\psi(\cdot, 0) = \phi$ is C^1, there is a linear map $A : \mathbb{R}^u \to \mathbb{R}^d$ such that

$$\lim_{x \to 0} \frac{\|\psi(x,0) - \psi(0,0) - Ax\|}{\|x\|} = 0$$

and since $\psi(0, \cdot)$ is C^1, there is a linear map $B : \mathbb{R}^s \to \mathbb{R}^d$ such that

$$\lim_{y \to 0} \frac{\|\psi(0,y) - \psi(0,0) - By\|}{\|y\|} = 0.$$

Then, if $x \neq 0$ and $y \neq 0$,

$$\begin{aligned}\frac{\|\psi(x,y) - \psi(0,0) - Ax - By\|}{\|(x,y)\|} &\leq \frac{\|\psi(x,y) - \psi(x,0) - \psi(0,y) + \psi(0,0)\|}{\|(x,y)\|} \\ &\quad + \frac{\|\psi(x,0) - \psi(0,0) - Ax\|}{\|(x,y)\|} \\ &\quad + \frac{\|\psi(0,y) - \psi(0,0) - By\|}{\|(x,y)\|} \\ &\leq \frac{\|\psi(x,y) - \psi(x,0) - \psi(0,y) + \psi(0,0)\|}{\|y\|} \\ &\quad + \frac{\|\psi(x,0) - \psi(0,0) - Ax\|}{\|x\|} \\ &\quad + \frac{\|\psi(0,y) - \psi(0,0) - By\|}{\|y\|}\end{aligned}$$

and each of these terms tends to zero as $(x,y) \to (0,0)$. The cases where $x = 0$ and $y = 0$ need to be proved separately, but follow by the same logic. Then, $\psi : U \times V \to \mathbb{R}^d$ is differentiable at $(0,0)$ with derivative $(x,y) \mapsto Ax + By$.

Finally, $\psi(U \times \{0\}) \subset W_f^u(p)$ so image $A = E_f^u(p)$ and $\psi(\{0\} \times V) \subset W_f^s(p)$ so image $B = E_f^s(p)$ showing the tangent plane of $W_f^{us}(p)$ at p is $E_f^u(p) \oplus E_f^s(p)$. □

Bibliography

[1] E. Akin. *The General Topology of Dynamical Systems.* American Mathematical Society, 1993.

[2] D. V. Anosov. Geodesic flows on closed Riemannian manifolds with negative curvature. *Proc. Inst. Steklov*, 90:1–235, 1967.

[3] C. Bonatti and A. Wilkinson. Transitive partially hyperbolic diffeomorphisms on 3-manifolds. *Topology*, 44(3):475–508, 2005.

[4] M. Brin. On dynamical coherence. *Ergod. Th. and Dynam. Sys.*, 23:395–401, 2003.

[5] M. Brin, D. Burago, and S. Ivanov. Dynamical coherence of partially hyperbolic diffeomorphisms of the 3-torus. *Journal of Modern Dynamics.* to appear.

[6] M. Brin, D. Burago, and S. Ivanov. On partially hyperbolic diffeomorphisms of 3-manifolds with commutative fundamental group. *Modern dynamical systems and applications*, pages 307–312, 2004.

[7] M. Brin and Ja. Pesin. Partially hyperbolic dynamical systems. *Math. USSR Izvestija*, 8:177–218, 1974.

[8] D. Burago and S. Ivanov. Partially hyperbolic diffeomorphisms of 3-manifolds with abelian fundamental groups. *Journal of Modern Dynamics*, 2(4):541–580, 2008.

[9] K. Burns and A. Wilkinson. Dynamical coherence and center bunching. *Discrete and Continuous Dynamical Systems*, 22(1&2):89–100, 2008.

[10] S. R. Fenley. Quasi-isometric folations. *Topology*, 31(3):667–676, 1992.

[11] J. Franks. Anosov diffeomorphisms on tori. *Transactions of the American Mathematical Society*, 145:117–124, 1969.

[12] J. Franks. Anosov diffeomorphisms. *Global Analysis: Proceedings of the Symposia in Pure Mathematics*, 14:61–93, 1970.

[13] A. Hammerlindl. Quasi-isometry and plaque expansiveness. *Canadian Math. Bull.*, to appear.

[14] F. Rodriguez Hertz, M. A. Rodriguez Hertz, and R. Ures. Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1d-center bundle. *Invent. Math.*, 172(2):353–381, 2008.

[15] M. Hirsch, C. Pugh, and M. Shub. *Invariant Manifolds*, volume 583 of *Lecture Notes in Mathematics*. Springer-Verlag, 1977.

[16] A. Manning. There are no new Anosov diffeomorphisms on tori. *Amer. J. Math.*, 96(3):422–42, 1974.

[17] C. Pugh, M. Shub, and A. Wilkinson. Hölder foliations. *Duke Math. J.*, 86(3):517–546, 1997.

[18] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73:747,817, 1967.