TECHNIQUES FOR ESTABLISHING DOMINATED SPLITTINGS

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ABSTRACT. We give theorems which establish the existence of a dominated splitting and further properties, such as partial hyperbolicity.

1. SPLITTINGS AND INEQUALITIES

Many concepts in dynamical systems are defined by an invariant splitting with one or more inequalities related to the splitting. In many cases, the inequalities need only be verified on the non-wandering set of the system. The results in this section are similar in nature to those established in [Cao03] for hyperbolic systems and earlier work referenced therein. Here, however, we consider generalizations of hyperbolicity, including dominated splittings, weak/strong/pointwise/absolute partial hyperbolicity, center bunching, and other properties.

Throughout this section assume f is a homeomorphism of a compact metric space M. Let $NW(f)$ denote its non-wandering set.

Proposition 1.1. *If U is a neighborhood of $NW(f)$, there is a uniform bound N such that any orbit $\{f^n(x) : n \in \mathbb{Z}\}$ has at most N points lying outside of U.*

Proof. Suppose no such N exists. As $M \setminus U$ is totally bounded, for any $k \in \mathbb{N}$, there is a point $x_k \in M \setminus U$ and an iterate $n_k \geq 1$ such that $d(x_k, f^{n_k}(x_k)) < \frac{1}{k}$. The sequence $\{x_k\}$ accumulates on a non-wandering point outside of U, which gives a contradiction. □

A *cochain* for f (in the context of this section) is a collection of continuous functions $\alpha_n : X \to \mathbb{R}$ for $n \in \mathbb{Z}$. The cochain is *additive* if

$$\alpha_{n+m}(x) = \alpha_n(f^m(x)) + \alpha_m(x)$$

for all $n, m \in \mathbb{Z}$ and $x \in X$. It is *superadditive* if

$$\alpha_{n+m}(x) \geq \alpha_n(f^m(x)) + \alpha_m(x)$$

for all $n, m \in \mathbb{Z}$ and $x \in X$. It is *eventually positive* if there is n_0 such that α_n is positive for all $n > n_0$. Note that any positive linear combination of superadditive cochains is again superadditive.

Proposition 1.2. *If α is a superadditive cochain, the following are equivalent:*

1. *α is eventually positive;*
2. *there is $n \geq 1$ such that $\alpha_n(x) > 0$ for all $x \in M$;*
3. *there is $n \geq 1$ such that $\alpha_n(x) > 0$ for all $x \in NW(f)$;*

Proof. Clearly (1) implies (2) and (2) implies (3).

Proof of (2) implies (1): Suppose (2) holds for some n. As α_n and α_1 are continuous, there are $\delta > 0$ and $C > 0$ such that $\alpha_n(x) > \delta$ and $\alpha_1(x) > -C$ for all $x \in M$. Write $m \in \mathbb{Z}$ as $m = qn + r$ with $q \in \mathbb{Z}$ and $0 \le r < n$. Then $\alpha_m(x) \ge q\delta - Cn$. If m is sufficiently large and positive, then so is $q\delta - Cn$.

Proof of (3) implies (2): First, note that if α is a superadditive cochain for f and $k \ge 1$, then $\beta_n = \alpha_{nk}$ defines a superadditive cochain for f^k. Therefore, we may assume $\alpha_1(x) > 0$ for all $x \in NW(f)$. Next, if γ is the unique additive cochain with $\gamma_1 = \alpha_1$, then $\alpha_n \ge \gamma_n$ for all $n \ge 1$. Therefore, we may assume α is additive. Let $\epsilon > 0$ be small enough that $U := \{x \in M : \alpha_1(x) > \epsilon\}$ is a neighborhood of $NW(f)$. Let N be the bound in proposition 1.1, and let C be such that $\alpha_1(x) > -C$ for all $x \in M$. Then $\alpha_m(x) > \epsilon(m - N) - CN$ for all m and x. Thus, for large m, α_m is positive. □

For a linear operator, A, between normed vector spaces, the norm $\|A\|$ and conorm $m(A)$ are defined by

$$\|A\| = \sup\{\|Av\| : \|v\| = 1\} \quad \text{and} \quad m(A) = \inf\{\|Av\| : \|v\| = 1\}.$$

If f is a diffeomorphism and $E \subset TM$ is a continuous invariant subbundle, then each of $\log m(Df^n|_{E(x)})$ and $-\log\|Df^n_{E(x)}\|$ defines a superadditive cochain. We formulate a number of dynamical concepts in terms of linear combinations of such cochains. Here, all bundles considered are non-zero.

(1) An invariant subbundle E is *expanding* if
$$\log m(Df^n|_{E(x)})$$
is eventually positive.

(2) An invariant subbundle E is *contracting* if
$$-\log\|Df^n_{E(x)}\|$$
is eventually positive.

(3) An invariant splitting $E^u \oplus E^s$ is *dominated* if
$$\log m(Df^n|_{E^u(x)}) - \log\|Df^n|_{E^s(x)}\|$$
is eventually positive. Write $E^u \oplus_> E^s$ to indicate the direction of the domination.

(4) An invariant splitting $E^u \oplus E^s$ is *absolutely dominated* if there is a constant $c \in \mathbb{R}$ such that both
$$\log m(Df^n|_{E^u(x)}) - cn \quad \text{and} \quad cn - \log\|Df^n|_{E^s(x)}\|$$
are eventually positive.

(5) A dominated splitting $E^u \oplus_> E^s$ is *hyperbolic* if E^s is contracting and E^u is expanding.

(6) A dominated splitting is *weakly partially hyperbolic* if it is either of the form $E^c \oplus_> E^s$ with E^s contracting or $E^u \oplus_> E^c$ with E^u expanding.

(7) An invariant splitting $E^u \oplus E^c \oplus E^s$ is *strongly partially hyperbolic* if both $(E^u \oplus E^c) \oplus_> E^s$ and $E^u \oplus_> (E^c \oplus E^s)$ are dominated splittings, E^s is contracting, and E^u is expanding.

(8) For $r \geq 1$, a strongly partially hyperbolic splitting is *r-partially hyperbolic* if both

$$\log m(Df^n|_{E^u(x)}) - r \log \|Df^n|_{E^c(x)}\|$$

and

$$r \log m(Df^n|_{E^c(x)}) - \log \|Df^n|_{E^s(x)}\|$$

are eventually positive.
Sometimes, one also requires that f is a C^r diffeomorphism [HPS77].

(9) A strongly partially hyperbolic splitting is *center bunched* if both

$$\log m(Df^n|_{E^u(x)}) - \log \|Df^n|_{E^c(x)}\| + \log m(Df^n|_{E^c(x)})$$

and

$$-\log \|Df^n|_{E^c(x)}\| + \log m(Df^n|_{E^c(x)}) - \log \|Df^n|_{E^s(x)}\|$$

are eventually positive.

Corollary 1.3. *Let f be a diffeomorphism on a compact manifold. For an invariant splitting, any of the properties listed above holds on all of M if and only if the property holds on the non-wandering set.*

Since the log of the Jacobian of $Df^n|_{E(x)}$ defines an additive cochain, one could also establish similar results for volume partial hyperbolicity as studied in [BDP03]. Further, the techniques in [Cao03] show that all of these properties hold uniformly if and only if they hold in a non-uniform sense on all invariant measures.

2. SPLITTINGS FROM SEQUENCES

We now present what are hopefully "user-friendly" techniques to prove the existence of a dominated splitting. The techniques here have some similarities with results developed by Mañé to study quasi-Anosov systems [Mañ77, Lemma 1.9], by Hirsch, Pugh, Shub in regards to normally hyperbolicity [HPS77, Theorem 2.17], and by Franks and Williams in constructing non-transitive Anosov flows [FW80, Theorem 1.2]. Here, however, we consider the general case of dominated splittings, and use Conley's Fundamental Theorem of Dynamical Systems to extend a splitting on the chain recurrent set to the entire phase space.

This section uses E^u and E^s to denote the bundles of a dominated splitting, even though the splitting may not necessarily be uniformly hyperbolic. It is far easier, at least for the author, to remember that E^u dominates E^s than to remember which of, say, E^1 and E^2 dominates the other.

Notation. For a non-zero vector $v \in TM$ and $n \in \mathbb{Z}$, let v^n denote the unit vector

$$v^n = \frac{Df^n v}{\|Df^n v\|}.$$

This notation depends on the dynamics $f : M \to M$ being specified in advance.

Theorem 2.1. *Suppose f is a diffeomorphism of a closed manifold M and Z is an invariant subset which contains all chain-recurrent points and has a dominated splitting*

$$T_Z M = E^u \oplus E^s$$

with $d = \dim E^u$. Suppose that for every $x \in M \setminus Z$, there is a point y in the orbit of x and a subspace V_y of dimension d such that for any non-zero $v \in V_y$, each of the sequences v^n and v^{-n} accumulates on E^u as $n \to \infty$. Then, the dominated splitting on Z extends to a dominated splitting on all of M.

A key step in proving the theorem is the following

Proposition 2.2. *Let $f : M \to M$ be a diffeomorphism, Λ a compact invariant subset, and let $U \subset \Lambda$ be open in the topology of Λ such that*

1. *$f(U)$ is compactly contained in U,*
2. *each of*
$$\bigcap_{n>0} f^n(U) \quad \textit{and} \quad \bigcap_{n>0} \Lambda \setminus f^{-n}(U)$$
has a dominated splitting with $d = \dim E^u$, and
3. *for each $x \in \overline{U} \setminus f(U)$ there is a d-dimensional subspace V_x such that for all $0 \neq v \in V_x$, both v^n and v^{-n} accumulate on E^u as $n \to \infty$.*

Then, there is a dominated splitting on all of Λ.

From the proof, it will be evident that if $x \in \overline{U} \setminus f(U)$, then $E^u(x) = V_x$ in the resulting dominated splitting on M. Therefore, it is not immediately clear how applying theorem 2.1 or proposition 2.2 would compare favorably to constructing a dominated splitting directly. Still, there are a number of advantages. First, only E^u needs to be known, not E^s, and only on a single fundamental domain where, depending on f, it may be easy to define. Next, to verify the hypotheses, one need only consider individual convergent subsequences rather than an entire cone field or splitting. Finally, as long as one already knows that the original splitting on Z is dominated, there are no further inequalities to verify.

While cone fields do not appear in the statement of proposition 2.2, they are needed for its proof. We follow the conventions given in [CP15, Section 2]. If $\Lambda \subset M$ and $\mathscr{C} \subset T_\Lambda M$ is a cone field, then for each $x \in \Lambda$, the cone $\mathscr{C}(x)$ at x is of the form

$$\mathscr{C}(x) = \{v \in T_x M : Q_x(v) \geq 0\}.$$

where Q_x is a non-positive, non-zero quadratic form which depends continuously on $x \in \Lambda$. The *interior* of $\mathscr{C}(x)$ is

$$\operatorname{int}\mathscr{C}(x) := \{0\} \cup \{v \in T_x M : Q_x(v) > 0\}$$

and the *dual cone* is

$$\mathscr{C}^*(x) := \{v \in T_x M : -Q_x(v) \geq 0\}.$$

Lemma 2.3. *Let $\Lambda \subset M$ be an invariant set with a dominated splitting $T_\Lambda M = E^u \oplus E^s$. Then there is a neighborhood U of Λ and a cone field $\mathscr{C}$ defined on U such that*

(1) *if a sequence $\{v_k\}$ of unit vectors in TM converges to $v \in E^u$, then $v_k \in \mathscr{C}$ for all large positive k;*
(2) *if $x, f(x) \in U$, then $Df(\mathscr{C}(x)) \subset \operatorname{int}\mathscr{C}(f(x))$;*
(3) *if $x \in M$ and $N \in \mathbb{Z}$ are such that $f^{-n}(x) \in U$ for all $n > N$, then*
$$\bigcap_{n>N} Df^n\big(\mathscr{C}(f^{-n}(x))\big)$$
is a subspace of $T_x M$ with the same dimension as E^u;
(4) *if $x \in M$ and $N \in \mathbb{Z}$ are such that $f^n(x) \in U$ for all $n > N$, then*
$$\bigcap_{n>N} Df^{-n}\big(\mathscr{C}^*(f^n(x))\big)$$
is a subspace of $T_x M$ with the same dimension as E^s.
(5) *the subspaces given by (3) and (4) define an extension of the dominated splitting to all of $\bigcap_{n\in\mathbb{Z}} f^n(U)$.*

The proof of lemma 2.3 uses the same techniques as in [CP15, Section 2] and is left to the reader.

Lemma 2.4. *In the setting of proposition 2.2, if there are cone fields $\mathscr{B}$ defined on $\Lambda \setminus f(U)$ and $\mathscr{C}$ defined on $\overline{U}$ such that $d = \dim\mathscr{B} = \dim\mathscr{C}$ and*
$$Df(\mathscr{B}(x)) \subset \operatorname{int}\mathscr{B}(f(x)) \qquad \text{if } x \in M \setminus U,$$
$$Df(\mathscr{C}(x)) \subset \operatorname{int}\mathscr{C}(f(x)) \qquad \text{if } x \in \overline{f(U)},$$
$$\mathscr{B}(x) \subset \mathscr{C}(x) \qquad \text{if } x \in \overline{U} \setminus f(U),$$
then there is a dominated splitting of dimension d defined on all of Λ.

Proof. Let $\alpha : \Lambda \to [0,1]$ be a continuous function such that $\alpha(M \setminus U) = \{0\}$ and $\alpha(f(U)) = \{1\}$. If P_x is the continuous family of quadratic forms defining $\mathscr{B}$ and Q_x is the family defining $\mathscr{C}$, then
$$(1-\alpha(x))P_x + \alpha(x)Q_x$$
defines a cone field $\mathscr{A}$ on Λ such that $Df(\mathscr{A}(x)) \subset \operatorname{int}\mathscr{A}(f(x))$ for all $x \in \Lambda$. This inclusion implies the existence of a dominated splitting. □

Proof of proposition 2.2. Let Λ_C and Λ_B denote the two intersections respectively in item (2) of the proposition. By lemma 2.3, there is a cone field $\mathscr{C}_0$ defined on a neighborhood U_C of Λ_C. For $n \in \mathbb{Z}$, define a cone field $\mathscr{C}_n$ on $f^n(U_C)$ by $\mathscr{C}_n(x) = Df^n(\mathscr{C}_0(f^{-n}(x)))$. Similarly, define a cone field $\mathscr{B}_0$ on a neighborhood U_B of Λ_B and for each $n \in \mathbb{Z}$ define the cone field $\mathscr{B}_n(x) = Df^n(\mathscr{B}_0(f^{-n}(x)))$.

We claim here that $\bigcap_m \mathscr{B}_m(x) = V_x$ for all $x \in \overline{U} \setminus f(U)$ where the intersection is over all $m \in \mathbb{Z}$ for which $\mathscr{B}_m(x)$ is defined and V_x is the subspace given in the statement of the proposition. Indeed, if $v \in V_x$ is non-zero, then there is a sequence $n_j \to \infty$ such that v^{-n_j} converges to a vector in E^u. Hence, $v^{-n_j} \in \mathscr{B}_0$ for all large j. Equivalently, $v \in \mathscr{B}_{n_j}$ for all large j. Since the sequence $\mathscr{B}_n$ is nested,

$$\bigcap_j \mathscr{B}_{n_j}(x) = \bigcap_n \mathscr{B}_n(x).$$

This shows that $V_x \subset \bigcap_n \mathscr{B}_n(x)$. Since both sets are d-dimensional subspaces of $T_x M$, they must be equal. This proves the claim

If, for some $m, n \in \mathbb{Z}$, the cone fields $\mathscr{B}_m$ and $\mathscr{C}_n$ satisfied the conditions of lemma 2.4, then the desired dominated splitting would exist. Hence, we may assume that for every m, $n \in \mathbb{Z}$, the open set

$$\{x : \mathscr{B}_m(x) \subset \operatorname{int}\mathscr{C}_n(x)\}$$

does not cover all of $\overline{U} \setminus f(U)$. By compactness, there is $y \in \overline{U} \setminus f(U)$ such that

$$\mathscr{B}_m(y) \setminus \operatorname{int}\mathscr{C}_n(y)$$

is non-empty for all m, $n \in \mathbb{Z}$. By compactness of the unit sphere in $T_y M$, the intersection

$$\bigcap_{m,n} \mathscr{B}_m(y) \setminus \operatorname{int}\mathscr{C}_n(y)$$

is non-empty. Let u be a unit vector in this intersection. Since $u \in \bigcap_m \mathscr{B}_m(y)$, the above claim shows that $u \in V_y$. Therefore, there is $n_j \to \infty$ such that u^{n_j} converges to a vector in E^u. Then $u^{n_j} \in \mathscr{C}_0$ for all large j, and therefore $u \in \mathscr{C}_{-n_j} \subset \operatorname{int}\mathscr{C}_{-n_j-1}$ for all large j as well. This gives a contradiction. □

Proof of theorem 2.1. By the so-called "Fundamental Theorem of Dynamical Systems" due to Conley [Nor95], there is a continuous function $\ell : M \to \mathbb{R}$ such that $\ell(f(x)) \le \ell(x)$ with equality if and only if x is in the set $R(f)$ of chain-recurrent points. Further, $\ell(R(f))$ is a compact, nowhere dense subset of $\mathbb{R}$.

Let $\mathscr{C}$ be a cone field defined on a neighborhood U of $R(f)$ as in lemma 2.3. Then, there is $\delta > 0$ such that $\ell(x) - \ell(f(x)) > \delta$ for all $x \notin U$. Define

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_q < b_q$$

such that $b_i - a_i < \delta$ for all i and the union of closed intervals $[a_i, b_i]$ covers $\ell(R(f))$. For $a, b \in \mathbb{R}$ define

$$\Lambda[a,b] := \{x \in M : \ell(f^n(x)) \in [a,b] \text{ for all } n \in \mathbb{Z}\}$$

If $x \in \Lambda[a_i, b_i]$, then $b_i - a_i < \delta$ implies that $f^n(x) \in U$ for all n. Therefore, the dominated splitting may be extended to each $\Lambda[a_i, b_i]$. By the inductive hypothesis, assume the dominated splitting has been extended to all of $\Lambda[a_1, b_k]$. Choose $t_k \in (b_k, a_{k+1})$ and use

$$\Lambda = \Lambda[a_i, b_{k+1}] \quad \text{and} \quad U = \{x \in \Lambda : \ell(x) < t_k\}$$

in proposition 2.2 to extend the dominated splitting to all of Λ. By induction, the dominated splitting extends to all of $\Lambda[a_1, b_q] = M$. □

When applying theorem 2.1, it may be a hassle to show directly that v^n accumulates on E^u. Suppose instead we know that there is a sequence $\{n_j\}$ with $\lim_j n_j = +\infty$ such that v^{n_j} converges to a unit vector $w \in T_Z M$ which does not lie in E^s. As with v, we use the notation

$$w^m = \frac{Df^m(w)}{\|Df^m(w)\|}.$$

The properties of the dominated splitting on Z imply that there is a sequence $\{m_j\}$ tending to $+\infty$ such that $\lim_j w^{m_j}$ exists and lies in E^u. By replacing $\{n_j\}$ with a further subsequence, one may establish that $\lim_j v^{n_j+m_j} = \lim_j w^{m_j}$. This reasoning shows that if v^n accumulates on a vector in $T_Z M \setminus E^s$, it also accumulates on a vector in E^u.

Iterating in the opposite direction, suppose there is a sequence $\{n_j\}$ tending to $+\infty$ such that $\{v^{-n_j}\}$ converges to $w \in T_Z M \setminus E^s$. Then there is a sequence $\{m_j\}$ tending to $+\infty$ such that $\lim_j w^{m_j}$ exists and lies in E^u. By replacing $\{n_j\}$ with a subsequence, one may establish both that $\lim_j(-n_j + m_j) = -\infty$ and $\lim_j v^{-n_j+m_j} = \lim_j w^{m_j}$. Hence, if v^{-n} accumulates on a vector in $T_Z M \setminus E^s$, it also accumulates on a vector in E^u.

With these observations in mind, we now state a slightly generalized version of theorem 2.1. The proof is highly similar and is left to the reader.

Theorem 2.5. *Suppose f is a diffeomorphism of a manifold M, and Y and Z are compact invariant subsets such that*

1. *all chain recurrent points of $f|_Y$ lie in Z,*
2. *Z has a dominated splitting $T_Z M = E^u \oplus E^s$ with $d = \dim E^u$, and*
3. *for every $x \in Y \setminus Z$, there is a point y in the orbit of x and a subspace V_y of dimension d such that for any non-zero $v \in V_y$, each of the sequences v^n and v^{-n} accumulates on a vector in $T_Z M \setminus E^s$ as $n \to +\infty$.*

Then the dominated splitting on Z extends to a dominated splitting on $Y \cup Z$.

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