DYNAMICALLY INCOHERENT SURFACE ENDOMORPHISMS

LAYNE HALL AND ANDY HAMMERLINDL

Abstract. We explicitly construct a dynamically incoherent partially hyperbolic endomorphisms of $\mathbb{T}^2$ in the homotopy class of any linear expanding map with integer eigenvalues. These examples exhibit branching of centre curves along countably many circles, and thus exhibit a form of coherence that has not been observed for invertible systems.

Keywords: Partial hyperbolicity, Non-invertible dynamics, Dynamical coherence.

1 Introduction

Understanding the integrability of the centre direction is critical for classifying partially hyperbolic dynamics. Non-invertible surface maps demonstrate a broader array of dynamics than their invertible counterparts, for instance, while partially hyperbolic surface diffeomorphisms are dynamically coherent, examples in [HH19] and [HSW19] show that there exist incoherent non-invertible maps. In this paper, we introduce a periodic centre annulus as a mechanism for incoherence of partially hyperbolic surface endomorphisms, and use this to construct incoherent surface endomorpshims which are homotopic to linear expanding maps. The center curves of these examples behave unlike those of the currently known maps on $\mathbb{T}^2$ and diffeomorphisms in higher dimensions.

We begin the statement of our results by recalling the definition of partial hyperbolicity. For non-invertible maps, this is most naturally given in terms of cone families. A *cone family* $\mathcal{C} \subset TM$ consists of a closed convex cone $\mathcal{C}(p) \subset T_pM$ at each point $p \in M$. A cone family is Df*-invariant* if $D_pf\left(\mathcal{C}(p)\right)$ is contained in the interior of $\mathcal{C}(f(p))$ for all $p \in M$. A map $f : M \to M$ is a *(weakly) partially hyperbolic endomorphism* if it is a local diffeomorphism and it admits a cone family $\mathcal{C}^u$ which is Df-invariant and there is $k > 0$ such that $1 < \|Df^k v^u\|$ for all $v^u \in \mathcal{C}^u$. In general, an unstable cone family in the non-invertible setting does not imply the existence

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of an invariant splitting. However, it does imply the existence of a centre direction, that is, a continuous Df-invariant line field $E^c \subset TM$ [CP15, Section 2]. For M an orientable closed surface, the existence of E^c implies that $M = \mathbb{T}^2$.

The homotopy class of a partially hyperbolic surface endomorphism f plays a fundamental role in the existing classification results. Each endomorphism is homotopic to a unique linear map $A : \mathbb{T}^2 \to \mathbb{T}^2$ which we call the *linearisation* of f. It is useful to categorise the linearisation into three types based on the eigenvalues λ_1 and λ_2 of the matrix inducing A which we refer to as follows:

- if $|\lambda_1| < 1 < |\lambda_2|$, we say A is *hyperbolic* if,
- if $1 < |\lambda_1| \leq |\lambda_2|$, we say A is *expanding*, and
- if $1 = |\lambda_1| \leq |\lambda_2|$, we say A is *non-hyperbolic*.

We say a partially hyperbolic endomorphism of $\mathbb{T}^2$ is *dynamically coherent* if there exists an f-invariant foliation tangent to E^c. Otherwise, we say it is *dynamically incoherent*. A closely related property which is sufficient for coherence is unique integrability: the centre direction E^c is said to be uniquely integrable if there is a unique C^1 curve tangent to E^c through every point.

Every endomorphism which has hyperbolic linearisation is both dynamically coherent and leaf conjugate to its linearisation [HH19]. Both [HSW19] and [HH19] show this does not hold in general by constructing incoherent endomorphisms, both of which have non-hyperbolic linearisation. We are naturally left with the question: how does the centre direction behave in the case of an expanding linearisation? Our first result addresses this.

Theorem A. *There exists a partially hyperbolic surface endomorphism f : $\mathbb{T}^2 \to \mathbb{T}^2$ which is homotopic to an expanding map and whose centre direction is not uniquely integrable. Moreover, the centre direction of f is uniquely integrable on an open and dense subset of $\mathbb{T}^2$, but is not uniquely integrable at each point in a countably infinite family of circles.*

We will prove Theorem A by constructing a geometric mechanism called an *invariant centre annulus*, which is an immersed open annulus $X \subset \mathbb{T}^2$ such that $f(X) = X$ and whose boundary, which necessarily consists of either one or two disjoint circles, is tangent to the centre direction. Note that the case when the boundary is one circle is precisely when the closure of the annulus is $\mathbb{T}^2$. If X is an invariant annulus for some positive iterate f^n of f, then we call X a *periodic centre annulus*. The annulus in the example is constructed by taking a linear expanding map, opening up an invariant circle to obtain an invariant annulus, and then applying a shear on the interior of the invariant annulus. Partial hyperbolicity of the example is established by the construction of an unstable cone-family, and the invariant annulus on which the shear was applied becomes an invariant centre annulus. The centre direction is uniquely integrable on this open annulus, but not along its boundary, and so we observe the behaviour of E^c by taking preimages of the annulus. The complete construction of this example is carried out in Section 2.

We present a very specific example in Theorem A for concreteness, but in Section 3 we generalise this procedure to construct a family of examples to prove the following result.

Theorem B. *Let $A : \mathbb{T}^2 \to \mathbb{T}^2$ be a linear map with integer eigenvalues and at least one eigenvalue greater than 1. Then there exists a partially hyperbolic surface endomorphism which is homotopic to A and is dynamically incoherent.*

For the non-hyperbolic case, where $|\lambda_2| > |\lambda_1| = 1$, examples establishing Theorem B have already been constructed in [HSW19] and [HH19]. Thus, we prove the result by constructing incoherent examples homotopic to any expanding linear map with integer eigenvalues.

To contrast the difference between the invertible and non-invertible settings, we briefly survey the known examples of dynamical incoherence in the case of diffeomorphisms. For partially hyperbolic diffeomorphisms with a center bundle of dimension 2 or higher, it is possible to construct examples with smooth center bundles that are not integrable. In this smooth setting, the integrability, or lack thereof, is given by the involutivity condition of Frobenius. See [BW05] and [Ham11] for further details.

For the case of one-dimensional center bundle, the question was open much longer. Since C^1 vector fields are always integrable, a dynamically incoherent example here would, by necessity, have a center direction which is not C^1, or even Lipschitz. Rodriguez-Hertz, Rodriguez-Hertz, and Ures constructed an example on the 3-torus, using an invariant 2-torus tangent to the center-unstable direction [RRU16]. In this example, the non-integrability of the center direction occurs only at this 2-torus and the center direction is smooth and therefore uniquely integrable everywhere else on the 3-torus. In fact, any partially hyperbolic system on the 3-torus, can have only finitely many embedded 2-tori tangent to E^{cs} or E^{cu} and the center direction is integrable outside these regions [HP17]. In any dimension, a partially hyperbolic diffeomorphism can have only finitely many compact submanifolds tangent either to E^{cs} or E^{cu} [Ham18].

More recently, new examples of partially hyperbolic diffeomorphisms have been discovered on the unit tangent bundles of surfaces of negative curvature [Bon+17]. For certain homotopy classes, these systems are dynamically incoherent. Further, these example have unique branching foliations tangent to the E^{cs} and E^{cu} directions and the branching (i.e., merging of distinct leaves) occurs at a dense set of points. The dynamical incoherence of such examples may therefore be regarded as a global phenomenon.

In the examples we construct to prove Theorem B, the branching of the center direction occurs at an infinite collection of circles tangent to the center and the closure of this collection gives a lamination consisting of uncountably many circles. Moreover, outside this lamination, the center direction is uniquely integrable and consists of lines. The branching is therefore not global, nor is it confined to a submanifold. This type of dynamical incoherence is possible due the non-invertible nature of partially hyperbolic endomorphism.

The behaviour of the examples in Theorem B is also distinct from the previously known incoherent endomorphisms on $\mathbb{T}^2$. Namely, the examples in [HSW19] and [HH19] both have centre curves branch on the boundary of a submanifold, and so are analogous to diffeomorphisms in dimension 3.

Theorem B also has an immediate consequence to a previously unanswered question about which homotopy classes admit endomorphisms. Given a partially hyperbolic endomorphism of $\mathbb{T}^2$, its linearisation A is given by an invertible integer-entried 2×2 matrix, and existing techniques show that A must have real eigenvalues. If the eigenvalues have distinct magnitude, then A itself induces a partially hyperbolic endomorphism. This is not true if the eigenvalues have equal magnitude, and it was unknown if there can exist a partially hyperbolic endomorphism which is homotopic to such an A. In particular, the question of existence had been posed for when A is twice the identity, and this is now answered by an immediate corollary of Theorem B.

Corollary C. *There exists a partially hyperbolic surface endomorphism which is homotopic to* $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$.

Finally, there are now two qualitatively different forms of incoherent partially hyperbolic surface endomorphisms: those constructed in this paper, and those in [HSW19] and [HH19]. In preparation is a classification of partially hyperbolic surface endomorphisms in [HH20], where it is shown that any incoherent example is akin to one of these examples.

2 Concrete example

In this section we construct an explicit example to prove Theorem A. The resulting endomorphism will be homotopic to $\left(\begin{smallmatrix} 4 & 0 \\ 2 & 3 \end{smallmatrix}\right)$. In the next section, we show how to generalize the construction to other homotopy classes.

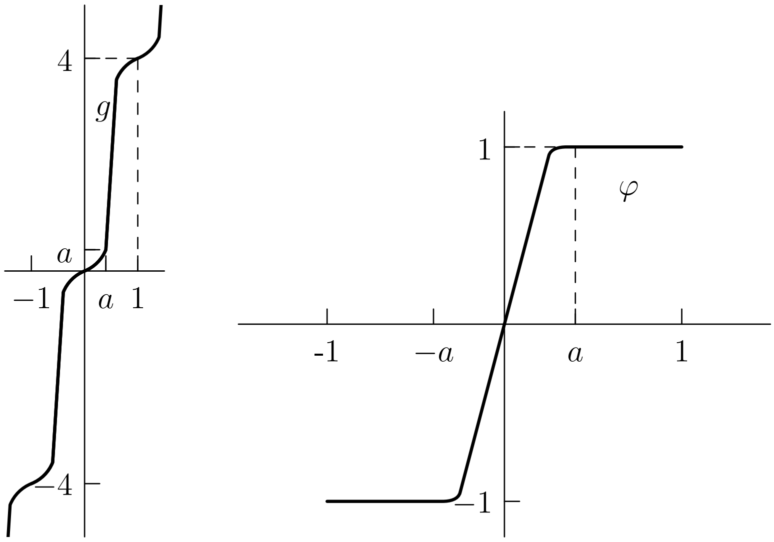

FIGURE 2.1. Graphs of the functions g and φ used to define the example f.

For small $0 < a < 1$, let $g : \mathbb{S}^1 \to \mathbb{S}^1$ be a monotone map which:

- is homotopic to the quadrupling map $x \mapsto 4x$,
- has fixed points $x = a, -a, 0$,
- satisfies $g'(0) < 1$,
- is linear on the complement of $(-a, a)$, and
- $g'(x) \geq 4$ for $x \in \mathbb{S} \setminus [-a, a]$.

Define $f_0 : \mathbb{T}^2 \to \mathbb{T}^2$ by $f_0(x, y) = (g(x), 3y)$. Then f_0 is a map which is homotopic to the linear map $(x, y) \mapsto (4x, 3y)$, and fixes the annulus $[-a, a] \times \mathbb{S}^1$. Let $\varphi : [-1/2, 1/2] \to [-1, 1]$ be a monotone function which is odd, that is $\varphi(-x) = -\varphi(x)$, such that φ' is zero off of $(-a, a)$ while satisfying $\varphi(-a) = -1$, $\varphi(0) = 0$, and $\varphi(a) = 1$. Then φ defines a map to $\mathbb{S}^1 \to \mathbb{S}^1$ which is a shear that is seen only inside the interval $(-a, a)$. We further take $\varphi'(0) > 1$, noting that we can take $\varphi'(0)$ to be arbitrarily large by taking the support of φ to be smaller. It will also be convenient to take φ and g to be linear on a small neighbourhood about each of the points $-a$, 0 and a. Graphs of suitable functions g and φ are shown in Fig. 2.1. Our example $f : \mathbb{T}^2 \to \mathbb{T}^2$ is an explicit deformation of f_0 defined by $f(x, y) = (g(x), 3y + \varphi(x))$. From our construction, we see the linearisation $A : \mathbb{T}^2 \to \mathbb{T}^2$ of f is given by

$$B = \begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix},$$

and so f is indeed homotopic to a linear expanding map.

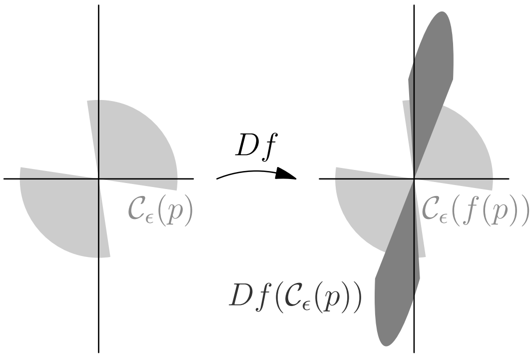

FIGURE 2.2. The cone family $\mathcal{C}_\varepsilon$ with the behaviour as proved in Lemma 2.1

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Next we establish that f is indeed partially hyperbolic by building an unstable cone family. The derivative of f at a point $(x, y) \in \mathbb{T}^2$ is given by

$$D_{(x,y)}f = \begin{pmatrix} g'(x) & 0 \\ \varphi'(x) & 3 \end{pmatrix}.$$

Note that $\Lambda = \{0\} \times \mathbb{S}^1 \subset \mathbb{T}^2$ is an f-invariant circle, and that on this circle, the derivative is given by

$$D_\Lambda f = \begin{pmatrix} g'(0) & 0 \\ \varphi'(0) & 3 \end{pmatrix}.$$

Since $g'(0) < 1$, then Λ is an invariant normally hyperbolic repeller, and we will use this to define a cone-family on a neighbourhood of Λ. Let U_Λ be a small open neighbourhood of Λ on which φ' and g' are constant. For $p \in U_\Lambda$ and $\varepsilon > 0$, define a constant cone family $\mathcal{C}_\varepsilon(p)$ as the cone containing the first quadrant with boundary given by the slopes of $(1, -\varepsilon)$ and $(-\varepsilon, 1)$ as depicted in Fig. 2.2.

Lemma 2.1. *There is $\varepsilon > 0$ such that $\mathcal{C}_\varepsilon$ is expanded by Df, and $Df(\mathcal{C}_\varepsilon(p)) \subset \operatorname{int}(\mathcal{C}_\varepsilon(f(p)))$ for all $p \in U_\Lambda$.*

Proof. If $p \in U_\Lambda$, then Df is given by the constant matrix $\left(\begin{smallmatrix} g'(0) & 0 \\ \varphi'(0) & 3 \end{smallmatrix}\right)$ and one can use this to show that $Df(\mathcal{C}_\varepsilon(p)) \subset \operatorname{int}(\mathcal{C}_\varepsilon(f(p)))$. We leave the proof of this to the reader.

To show that there is ε such that Df expands $\mathcal{C}_\varepsilon$, by continuity, it suffices to show that Df expands all vectors in the first quadrant. So let $(u_x, u_y) \in T_p\mathbb{T}^2$ lie the first quadrant, that is, $u_x, u_y \geq 0$. We compute

$$D_p f(u_x, u_y) = (g'(0)u_x, 3u_y + \varphi'(0)u_x).$$

Since $\varphi'(0) > 1$, then $\varphi'(0)^2 + g'(0)^2 > 1$, which implies $\|D_p f(u_x, u_y)\| > \|(u_x, u_y)\|$. □

For the remainder of this section, we fix ε, and thus $\mathcal{C}_\varepsilon$, to be as in the preceding lemma.

Next, consider the compact set $K = \mathbb{T}^2 \setminus \big((-a, a) \times \mathbb{S}^1\big)$. This set is 'backward invariant' in the sense that $f^{-1}(K) = K$. Then f is linear and expanding on K, which will allow us to define a natural unstable cone family on a neighbourhood of K. Let U_K be a small open neighbourhood of K which is disjoint from U_Λ and is such that $\varphi'(x) = 0$ and $g'(x) > 3$ for all $(x, y) \in U_K$.

Lemma 2.2. *Suppose that $\mathcal{B}_\delta$ is a cone family on U_K which for $0 < \delta < 1$ is given at each point by the cone that contains the horizontal and is bounded by the slopes $(1, \delta)$ and $(1, -\delta)$. Then $\mathcal{B}_\delta$ is expanded by Df, and $Df(\mathcal{B}_\delta(p)) \subset \operatorname{int}(\mathcal{B}_\delta(f(p)))$ for all $p \in U_K$ which satisfy $f(p) \in U_K$.*

Proof. By definition, if $(x, y) \in U_K$, then

$$Df_{(x,y)} = \begin{pmatrix} g'(x) & 0 \\ 0 & 3 \end{pmatrix}.$$

Since $g'(x) > 3$, $Df_{(x,y)}$ is expanding. Moreover, $Df_{(x,y)}$ preserves both the horizontal and vertical, and expands the horizontal more than the vertical. This implies the result. □

A depiction of a cone family $\mathcal{B}_\delta$ as in the preceding lemma is shown in Fig. 2.3.

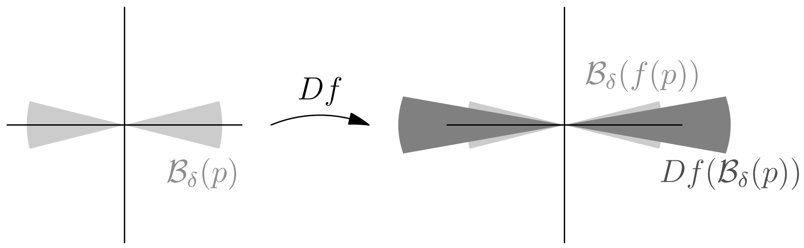

FIGURE 2.3. A cone family $\mathcal{B}_\delta$ with the properties stated in Lemma 2.2.

In Fig. 2.4 we depict the regions U_Λ and U_K over which we have defined $\mathcal{C}_\varepsilon$ and $\mathcal{B}_\delta$. The next step is to 'stitch together' these two cones families to obtain a global one. Let $V \subset (-a, a) \times \mathbb{S}^1$ be an open strip such that that $f(V) \cup U_K = \mathbb{T}^2$. Note that the definition of the circle map g implies that $f(V) \Subset V$, and so there is N such that $f^N(V) \subset U_\Lambda$. We pull back $\mathcal{C}_\varepsilon$ to define a cone family $\mathcal{C}^N$ on V by $\mathcal{C}^N(p) = Df^{-N}(\mathcal{C}_\varepsilon(f^N(p)))$ at $p \in V$.

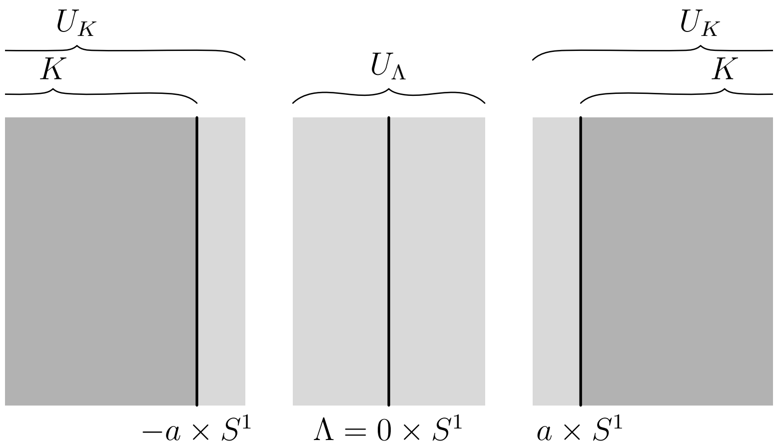

FIGURE 2.4. The regions U_Λ and U_K in $\mathbb{T}^2$ we over which we have defined $\mathcal{C}_\varepsilon$ and B_δ. We glue these cone families together across the gap between these regions to obtain $\mathcal{C}^u$.

Lemma 2.3. *For $p \in V$, the cone $\mathcal{C}^N(p)$ is a closed neighbourhood of the horizontal.*

Proof. Let $q = (x, y) \in U_\Lambda$ and $(u_x, u_y) \in T_q\,\mathbb{T}^2$. Consider a preimage p of q which also lies in V and let f^{-1} denote the local inverse, so that $p = f^{-1}(q)$. Then the tangent vector $(\hat{u}_x, \hat{u}_y) = D_q f^{-1}(u_x, u_y) \in T_p\,\mathbb{T}^2$ is given by

$$(\hat{u}_x, \hat{u}_y) = D_q f^{-1}(u_x, u_y) = \left(\frac{u_x}{g'(x)}, -\frac{\varphi'(x)}{2g'(x)} u_x + 2u_y \right).$$

Since $\varphi', g' \geq 0$, then $u_y < 0 < u_x$ implies $\hat{u}_y < 0 < \hat{u}_x$. Similarly, $u_x < 0 < u_y$ implies $\hat{u}_x < 0 < \hat{u}_y$. Therefore $D_q f^{-1}\,\mathcal{C}$ is a closed neighbourhood of the first and third quadrants on all of $f^{-1}(U_\Lambda) \cap V$. By induction, $\mathcal{C}^N(p)$ is a closed neighbourhood of the horizontal for all $p \in V$. $\square$

Lemma 2.4. *There is a cone family $\mathcal{B}$ on U_K such that following hold:*

- *if $p \in U_K$ and $f(p) \in U_K$, then $Df\,\mathcal{B}(p) \subset \operatorname{int} \mathcal{B}(f(p))$;*
- *if $p \in U_K$ and $f(p) \in V$, then $Df\,\mathcal{B}(p) \subset \operatorname{int} \mathcal{C}^N(f(p))$;*
- *if $p \in V \cap U_K$, then $\mathcal{B}(p) \subset \operatorname{int} \mathcal{C}^N(p)$.*

Proof. The first property will hold so long as we take $\mathcal{B} = \mathcal{B}_\delta$ as in Fig. 2.3 for any small $\delta > 0$. Since we can take δ arbitrarily small, then to prove the second and third claims, it suffices to show that $\mathcal{C}^N$ is a closed neighbourhood of the horizontal at each point. This was established in Lemma 2.3. $\square$

We recall (see [CP15], §2.2) that a continuous cone family $\mathcal{C}$ is equivalent to a continuous quadratic form Q on the tangent space. In this correspondence, the cone $\mathcal{C}(p)$ at a point p is determined by the quadratic Q_p by $\mathcal{C}(p) = \{v \in T_p\mathbb{T}^2 : Q_p(v) \geq 0\}$.

Lemma 2.5. *The map f admits a cone family $\mathcal{C}^u$ such that $Df(\mathcal{C}^u) \subset \operatorname{int}\mathcal{C}^u$ and which coincides with $\mathcal{B}$ on U_K and $\mathcal{C}^N$ on $f(V)$.*

Proof. Let $\mathcal{C}^N$ be prescribed by a quadratic form P on V, and $\mathcal{B}$ by a form Q on U_K. Let $\alpha : \mathbb{T}^2 \to [0,1]$ be a continuous function such that $\alpha(\mathbb{T}^2 \setminus V) = \{0\}$ and $\alpha(f(V)) = \{1\}$. Define a continuous cone family $\mathcal{C}^u$ by the quadratic form

$$(1-\alpha)P + \alpha Q.$$

Then $\mathcal{C}^u$ coincides with $\mathcal{C}^N$ on $f(V)$ and $\mathcal{B}$ on U_K, and it is invariant by Lemma 2.4. □

Proposition 2.6. *The endomorphism f is partially hyperbolic.*

Proof. By Lemma 2.5, $\mathcal{C}^u$ is an invariant cone family. It remains to show that $\mathcal{C}^u$ is expanded by Df^k for some k. Observe that there is $m > 0$ such that if $p \in \mathbb{T}^2$, then the orbit of p has at most m points which do not lie in either U_Λ or U_K. When $p \in U_\Lambda$ or $p \in U_K$, we know by construction that $\mathcal{C}^u \subset \mathcal{C}^N \subset \mathcal{C}_\varepsilon$ or $\mathcal{C}^u = \mathcal{B}_\delta$, respectively. By Lemma 2.1 and Lemma 2.2, Df expands $\mathcal{C}_\varepsilon$ and $\mathcal{B}_\delta$. Thus by choosing k sufficiently large, if $v^u \in \mathcal{C}^u(p)$, we have $\|Df^k v^u\| > \|v^u\|$. □

We now know that f is partially hyperbolic, and so it admits an invariant centre direction E^c [HH19].

Define an annulus $X = (-a,a) \times \mathbb{S}^1$. Lift f to a diffeomorphism $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ which fixes the lift $\tilde{X} = (-a,a) \times \mathbb{R}$ of X. As the lift of a partially hyperbolic surface endomorphism, the map $\tilde{f}$ admits an invariant splitting $E^c \oplus E^u$, with E^c descending to the centre direction of f on $\mathbb{T}^2$ [MP75]. Moreover, $\tilde{f}$ is a finite distance from its linearisation $A : \mathbb{R}^2 \to \mathbb{R}^2$, which we remind the reader was defined in the introduction.

Proposition 2.7. *The map f admits an invariant centre annulus X, and the centre direction E^c is uniquely integrable on X.*

Proof. The boundary components of the invariant annulus $(-a,a) \times \mathbb{S}^1$ are the circles $\{a\} \times \mathbb{S}^1$ and $\{-a\} \times \mathbb{S}^1$. Restricted to these circles, Df is given by

$$\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}.$$

Thus $\{a\} \times \mathbb{S}^1$ and $\{-a\} \times \mathbb{S}^1$ are tangent to E^c, and so $X = (-a,a) \times \mathbb{S}^1$ is an invariant centre annulus.

To see that E^c is uniquely integrable on X, note that the restriction of $\tilde{f}$ to $\tilde{X}$ is a diffeomorphism. Under this diffeomorphism, the set $\{0\} \times \mathbb{R}^1$ is an invariant normally hyperbolic manifold with splitting $E^u \oplus E^s$. Here, E^s corresponds locally to E^c. Using classical stable manifold theory, one may then show that E^c is uniquely integrable on a neighbourhood U of Λ [HPS77]. But every point in $p \in X$ is in the preimage of some point in $q \in U$, so E^c is uniquely integrable on all of X. $\square$

We now establish that the centre curves must branch on the boundary of X.

Lemma 2.8. *Let $U \subset \tilde{X}$ be a set of the form $\mathbb{R} \times (-r_0, r_0)$ for some $r_0 > 0$. Then there is r such that then $\tilde{f}^{-n}(U) \subset \mathbb{R} \times (-r, r)$ for all $n > 0$.*

Proof. If $U \subset \mathbb{R} \times (-r_0, r_0)$ for $r_0 > 0$, then $A^{-1}(U) \subset \mathbb{R} \times (-r_0/3, r_0/3)$. Since f is a finite distance from A, then $\tilde{f}^{-1}(U) \subset \mathbb{R} \times (-r_0/3 - C, r_0/3 + C)$. Thus by choosing r so that $r > \frac{r}{3} + C$ and $r > r_0$, the claim follows by induction. $\square$

Now we establish that the centre direction is not-uniquely integrable, proving Theorem A.

Proof of Theorem A. Suppose that E^c is uniquely integrable on all of $\mathbb{R}^2$, so that it integrates to a foliation which descends to $\mathbb{T}^2$. Consider a small centre curve $J^c \subset \mathbb{R}^2$ with one endpoint at the origin and the other endpoint at $p \in (0, a) \times \mathbb{R}$, so that $J^c \subset (0, a) \times \mathbb{R}$. Under $\tilde{f}^{-n}$, the endpoints of $\tilde{f}^{-n}(p)$ at the origin remains fixed, while the x-coordinate of the other endpoint monotonically approaches the line $\{a\} \times \mathbb{R}$. By unique integrability of X on E^c, we have $\tilde{f}^{-n}(J^c) \subset \tilde{f}^{-(n+1)}(J^c)$, with both of these curves being leaf segments of the leaf of the centre foliation through $(0, 0)$. The curve $\{a\} \times \mathbb{R}$ is tangent to the centre, so it is necessarily a leaf of the centre foliation, implying that the endpoint of $\tilde{f}^{-n}(J^u)$ cannot converge to a point on $\{a\} \times \mathbb{R}$. Thus $\tilde{f}^{-n}(J^u)$ must grow unbounded in the vertical direction as $n \to \infty$. But by Lemma 2.8, $\tilde{f}^{-n}(J^u)$ must be uniformly bounded in the vertical direction, giving a contradiction.

Now we prove that the centre curves branch only on a set of countably many annuli. For this, we recall that g is homotopic to the the circle map $x \mapsto 4x$. Then a single preimage of the interval $[-a, a]$ under the circle map g consists of $[-a, a]$ and three other disjoint intervals, and the union of all backward iterates of $[-a, a]$ under g is dense in $\mathbb{S}^1$. Thus the preimage of the invariant annulus X under f consists of X and three other disjoint annuli, and $\bigcup_{n \geq 0} f^{-n}(X)$ is dense in $\mathbb{T}^2$. Since E^c is uniquely integrable on X, then it is uniquely integrable on $\bigcup_{n \geq 0} f^{-n}(X)$. The centre direction is not

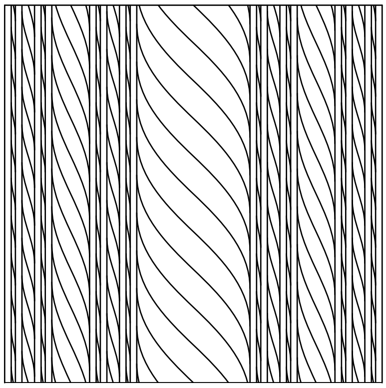

FIGURE 2.5. The centre curves of the endomorphism f on $\mathbb{T}^2$.

uniquely integrable on each boundary circle of X, and is thus not uniquely integrable on each of their preimages. Thus E^c behaves as desired. $\square$

With Theorem A proved, we conclude this by obtaining a clear depiction of the centre curves as shown in Fig. 2.5. If p is a point whose orbit is disjoint from X, then the orbit of p lies in K. Recalling that for $(x, y) \in K$ that

$$Df_{(x,y)} = \begin{pmatrix} g'(x) & 0 \\ 0 & 3 \end{pmatrix} \tag{2.1}$$

and that $g'(x)$ is a constant greater than 4, we see that the centre direction on the orbit of such p is vertical. Elsewhere, we can use the following.

Lemma 2.9. *The centre direction E^c on X has negative slope.*

Proof. Recall that E^c lies in the complement of $\mathcal{C}^u$. For $p \in U_\Lambda$, the unstable cone is $\mathcal{C}_\varepsilon$, so all vectors not in $\mathcal{C}^u$ have negative slope. Hence the proposition holds at p. For $p \in X$, then $p = f^n(q)$ for some $q \in U_\Lambda$, and $E^c(p) = Df^{-n}(E^c(q))$. But $E^c(q)$ has negative slope, and it was shown in the proof of Lemma 2.5 that $D_p f^{-1}$ takes vectors of negative slope to vectors of negative slope for $p \in X$, so the claim also holds at q. $\square$

Since Df is constant given by the equation Eq. (2.1) on the compliment of X, then a connected component of $f^{-1}(X)$ which is not itself X is a rescaled copy of X. This copy is contracted more strongly horizontally than vertically. This procedure continues while iterating X backwards, and so one can show that the centre curves are in Fig. 2.5.

We conclude by noting that the authors believe that this example f is dynamically incoherent. However, showing this would require more effort than required for our purposes, since it will be easier to prove incoherence in the examples we construct in the next section.

3 General construction

In this section, we prove Theorem B by generalising the construction used to establish Theorem A. Our approach is eased by first observing we only need to construct an example homotopic to linear maps of a certain form:

Lemma 3.1. *Let $A : \mathbb{T}^2 \to \mathbb{T}^2$ be a linear expanding map with integer eigenvalues λ and μ. Then A is conjugate as a map on $\mathbb{T}^2$ to a linear map $B : \mathbb{T}^2 \to \mathbb{T}^2$ given by*

$$B = \begin{pmatrix} \mu & 0 \\ t & \lambda \end{pmatrix}$$

for some $t \in \mathbb{Z}$.

Proof. If the eigenvalue λ of A is an integer, it has an associated eigenvector $v = (a, b) \in \mathbb{Z}^2$ for a and b with $\gcd(a, b) = 1$. The proposition will be satisfied if we can find $P \in SL(2, \mathbb{Z})$ such that $Pv = (0, 1)$, as we may take $B = PAP^{-1}$. This amounts to solving two coupled equations over $\mathbb{Z}$, which since $\gcd(a, b) = 1$, always has a solution. □

Now to prove Theorem B, it suffices to construct examples which are homotopic to the linear map B of the form in the preceding lemma.

We remark that the deformation approach to obtain f from map the f_0 in Section 2 changed the homotopy class of the map. For our general example, we will adapt this approach to construct an example in a desired homotopy class. The idea is to define the initial map f_0 with two invariant annuli, and then apply two shears in opposing directions along each annulus. To encourage visualising this idea, we refer the reader to Fig. 3.1, which shows how the centre curves look for our example in the case $t = 0$. We suggest comparing this to the centre curves of the example constructed in the previous section, as shown in Fig. 2.5. The depiction in Fig. 3.1 will be justified later.

Proposition 3.2. *If $A : \mathbb{T}^2 \to \mathbb{T}^2$ is an expanding linear map with integer eigenvalues, there exists a partially hyperbolic surface endomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ which admits a periodic centre annulus and is homotopic to A.*

Proof. Begin by letting B be as in Lemma 3.1, and assume that λ, $\mu > 0$ and $t > 0$. We will later explain how an example can be constructed for general λ and μ, while when $t < 0$, the examples are similar, so the details are left to the reader. Define $a > 0$ so that there is a monotone function $g : \mathbb{S} \to \mathbb{S}$ which:

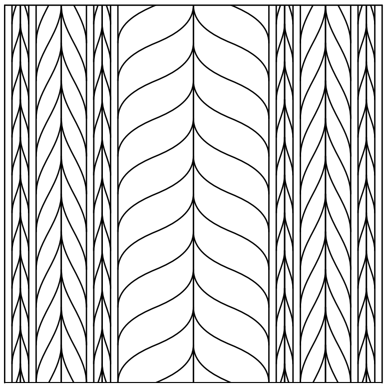

FIGURE 3.1. The centre curves of the general example when $t = 0$.

- is homotopic to the map $x \mapsto \mu x$,
- has fixed points $x = 0$, a, $2a$, $-a$, $-2a$,
- satisfies $g'(a) = g'(-a) < 1$,
- is linear on the complement of $(-2a, 0) \cup (0, 2a)$, on which we have $g' > \lambda$.

Note that we have not assumed any ordering on λ and μ, so that if $\mu < \lambda$, then a must be chosen close enough to either 0 or $1/4$ to allow g to satisfy the final property above. Now define $f_0 : \mathbb{T}^2 \to \mathbb{T}^2$ by $f_0(x, y) = (g(x), \lambda y)$. Then f_0 is homotopic to B and has two invariant annuli $(-2a, 0) \times \mathbb{S}^1$ and $(0, 2a) \times \mathbb{S}^1$ which share a boundary circle.

Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be monotone and such that $\varphi(-2a) = 0$, $\varphi(a) = (t+1)/2$, $\varphi(0) = t + 1$ with the support of φ' contained in $(-2a, 0)$. Define another monotone function $\psi : \mathbb{S}^1 \to \mathbb{S}^1$ to be such that $\psi(-2a) = 0$, $\psi(-a) = -1/2$ and $\psi(0) = -1$ and the support of ψ contained in $(0, 2a)$. Then φ is a shear upwards by a factor of $t + 1$ in one invariant annulus, while ψ is a shear downwards by a factor of 1 in the other. The explicit deformation to give the desired example is given by $f(x, y) = (g(x), \lambda y + \varphi(x) + \psi(x))$. The shears φ and ψ acting simultaneously result in f being homotopic to B.

To see that f is partially hyperbolic, we adapt the main ideas of Section 2. Note that the orbits of points in the annuli $X_1 = (-2a, 0) \times \mathbb{S}^1$ and $X_2 = (0, 2a) \times \mathbb{S}^1$ are disjoint, so the approach is to define cone families much like the one annulus for the concrete example on each of them. The invariant circles $\{-a\} \times \mathbb{S}^1$ and $\{a\} \times \mathbb{S}^1$ are normally hyperbolic invariant manifolds, so there is a natural unstable cone-family defined on each of these circles akin to $\mathcal{C}_\varepsilon$ in Lemma 2.1. Meanwhile, on the complement of $X_1 \cup X_2$, the map f is linear, and a cone family which is a small uniform neighbourhood of the horizontal will be an unstable cone family, similar to $\mathcal{B}$ in Lemma 2.2.

Arguing as in Lemma 2.4 and Lemma 2.5, we can stitch these cones together to construct $\mathcal{C}^u$, a cone family which satisfies $Df\,\mathcal{C}^u(p) \subset \operatorname{int}\mathcal{C}^u(f(p)))$. Since f expands $\mathcal{C}^u$ on the linear region close to the invariant hyperbolic circles, then by using arguments of Proposition 2.6, we can show that $\mathcal{C}^u$ is in fact an unstable cone family and that f is partially hyperbolic. Now by applying the arguments of Proposition 2.7 to each invariant annulus X_1 and X_2, we see that they are invariant centre annuli of f.

Now we drop the assumption that $\mu, \lambda > 0$. So let the matrix B be of the form $\left(\begin{smallmatrix} \mu & 0 \\ t & \lambda \end{smallmatrix}\right)$ where one or both of μ and λ may be negative. Since $|\mu| > 1$, the linear map $g_0 : S^1 \to S^1$, defined by $x \mapsto \mu x$ has at least one point with period exactly two. Using a deformation, one can define a map $g : S^1 \mapsto S^1$ and interval $I \subset S^1$ such that such that

- $g(I)$ is disjoint from I,
- $g^2(I)$ is equal to I, and
- g is linear on the complement of I with derivative $|g'| > |\lambda|$.

Up to conjugation with a rigid rotation, we may assume I is centered at zero. That is, there is $a > 0$ such that $I = (2a, -2a)$. Moreover, by replacing $g|_I$ (but leaving g on $g(I)$ linear and unchanged), we may assume that g^2 has fixed points at $x = -2a, -a, 0, a, 2a$, and that $(g^2)'(-a) = (g^2)'(a) < 1$. In other words, g^2 here has the properties that g had in the case when μ and λ were assumed positive in the earlier section of this proof. Let the shearing functions φ and ψ be defined exactly as before and define $f(x, y) = (g(x), \lambda y + \varphi(x) + \psi(x))$. By again adapting the previous techniques, one can show that the resulting endomorphism is partially hyperbolic. It is easy to see that the annulus $I \times S^1$ is an invariant centre annulus for f^2, so that $I \times S^1$ is a periodic centre annulus for f. $\square$

To establish Theorem B, we now show that the examples constructed in the preceding proof are dynamically incoherent. While it is unclear whether or not the original example of Section 2 was incoherent, the presence of two adjacent periodic annuli as depicted in Fig. 3.1 makes observing coherence straightforward.

Proof of Theorem B. Let f be an example established in the proof of the preceding proposition with the assumption that λ, μ, $t \geq 0$. The other cases are again similar and left to the reader. When we restrict f to X_1, the map admits an invariant splitting $E^c \oplus E^u$ and the circle $\{a\} \times \mathbb{S}^1$ is a hyperbolic repeller tangent to the unstable direction. The centre direction is thus uniquely integrable on a neighbourhood of this circle, which in turn implies it is uniquely integrable on all of X_1. Similarly, E^c is uniquely integrable on X_2.

Using the ideas of Lemma 2.9, one can show that E^c has positive slope on $X_1 = (-2a, 0) \times \mathbb{S}^1$, while it has negative slope on $X_2 = (0, 2a) \times \mathbb{S}^1$. Due to the shearing being in the opposite direction on each annulus, the centre curves approach the centre circle $\{0\} \times \mathbb{S}^1$ with slopes of opposite sign, as is shown in the depiction Fig. 3.1. On a neighbourhood of the circle $\{0\} \times \mathbb{S}^1$ there cannot exist a foliation chart. Hence f is dynamically incoherent. □

We conclude by justifying the the rest of the depiction of the centre curves as is shown in Fig. 3.1. Once more, outside the orbits of the two invariant annuli X_1 and X_2, the map f is a linear map preserving the horizontal and vertical directions, expanding stronger in the horizontal. An annulus in the preimage of either X_1 and X_2 is a linear rescaling of the curves on the invariant annulus that is contracted a greater amount in the horizontal than the vertical, and so when $t = 0$ in the linearisation B, the centre curves are as shown. Note that the slopes on each annulus X_1 and X_2 will not be symmetric as in the figure when $t \neq 0$, though they will always be of opposite sign.

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School of Mathematics and Statistics, The University of Melbourne, Victoria 3010 Australia

Email address: lrhall@student.unimelb.edu

School of Mathematical Sciences, Monash University, Victoria 3800 Australia

URL: http://users.monash.edu.au/~ahammerl/

Email address: andy.hammerlindl@monash.edu