

PROPERTIES OF COMPACT CENTER-STABLE SUBMANIFOLDS

ANDY HAMMERLINDL

ABSTRACT. We show that a partially hyperbolic system can have at most a finite number of compact center-stable submanifolds. We also give sufficient conditions for these submanifolds to exist and consider the question of whether they can intersect each other.

1. INTRODUCTION

Much of the early theory of partially hyperbolic dynamics was developed by Hirsch, Pugh, and Shub in their foundational text, *Invariant Manifolds* [HPS77]. The book first considers the case of an invariant compact submanifold of the phase space where the dynamics normal to the submanifold is hyperbolic. Later chapters deal with systems where a partially hyperbolic splitting holds on the entire phase space. These two cases may overlap. For instance, if the system has a global splitting of the form $TM = E^u \oplus E^c \oplus E^s$, it may also have a compact submanifold tangent to E^c and such a submanifold is therefore normally hyperbolic.

Recent discoveries show that a slightly different possibility exists. Rodriguez Hertz, Rodriguez Hertz, and Ures constructed an example of a partially hyperbolic system on the 3-torus with a compact submanifold, a 2-torus, tangent to the center and stable directions, $E^c \oplus E^s$ [RHRHU16]. This center-stable submanifold is transverse to the expanding unstable direction and is therefore a normally repelling submanifold. Based on this, the author constructed further examples of compact center-stable submanifolds, both in dimension 3 and higher [Ham16].

This paper establishes general properties for these types of dynamical objects. In particular, we show that any partially hyperbolic system may have at most finitely many compact center-stable submanifolds and we give sufficient conditions under which these objects exist. Finally, we consider the question of whether two of these submanifolds can have non-empty intersection.

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2. STATEMENT OF RESULTS

A diffeomorphism f of a closed connected manifold M is (strongly) *partially hyperbolic* if there is a splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u$$

such that each subbundle is non-zero and invariant under the derivative Df and

$$\|Df v^s\| < \|Df v^c\| < \|Df v^u\| \quad \text{and} \quad \|Df v^s\| < 1 < \|Df v^u\|$$

hold for all $x \in M$ and unit vectors $v^s \in E^s(x)$, $v^c \in E^c(x)$, and $v^u \in E^u(x)$. There exist unique foliations W^s and W^u tangent to E^s and E^u . An immersed submanifold $S \subset M$ is a *center-stable submanifold* if it is tangent to $E^{cs} = E^c \oplus E^s$.

Theorem 2.1. *A partially hyperbolic diffeomorphism has at most a finite number of compact center-stable submanifolds.*

From this, the following result could be proved.

Theorem 2.2. *Every compact center-stable submanifold is periodic.*

However, in section 3 we actually establish theorem 2.2 first and then use the result to show theorem 2.1.

While being tangent to E^{cs} clearly requires the submanifold to be at least C^1 , it is equivalent to a condition which may be stated for C^0 submanifolds.

Theorem 2.3. *Suppose $f : M \rightarrow M$ is partially hyperbolic and $\Lambda \subset M$ is a periodic compact C^0 submanifold. Then Λ is a C^1 submanifold tangent to E^{cs} if and only if $W^u(x) \cap \Lambda = \{x\}$ for all $x \in \Lambda$.*

This also gives a way to find periodic submanifolds from non-periodic ones.

Theorem 2.4. *Suppose $f : M \rightarrow M$ is partially hyperbolic, $k \geq 1$, and $S \subset M$ is a compact C^0 submanifold such that $W^u(x) \cap S = \{x\}$ and $W^u(x) \cap f^k(S) \neq \emptyset$ for all $x \in S$. Then there exists a compact center-stable submanifold.*

Theorems 2.3 and 2.4 are proved in section 4.

The next theorem assumes a one-dimensional unstable direction. It basically states that if a region $M_0 \subset M$ has two boundary components and the ends of unstable curves inside this region tend towards the boundary in a uniform way, then there must be a compact center-stable submanifold inside the region.

Theorem 2.5. *Let f be a partially hyperbolic diffeomorphism of a manifold M , M_0 a compact connected submanifold of M with boundary, $g : M_0 \rightarrow [0, 1]$ a continuous function, and $\ell > 0$ such that*

- (1) $\dim E^u = 1$,
- (2) $\dim M = \dim M_0$,
- (3) $f(M_0) = M_0$,
- (4) if $x \in \partial M_0$, then $W^u(x) \subset \partial M_0$,

- (5) $g(\partial M_0) = \{0, 1\}$,
 (6) if $x \in M_0$, $0 < g(x) < 1$, $y \in W^u(x)$, and $d_u(x, y) > \ell$, then $g(y) \in \{0, 1\}$, and
 (7) if $\alpha : \mathbb{R} \rightarrow M_0$ is a parameterized unstable leaf, then

$$\lim_{t \rightarrow +\infty} g\alpha(t) = \lim_{t \rightarrow +\infty} gf\alpha(t) \quad \text{and} \quad \lim_{t \rightarrow -\infty} g\alpha(t) = \lim_{t \rightarrow -\infty} gf\alpha(t).$$

Then, there is a compact center-stable submanifold in the interior of M_0 .

This result will be used in an upcoming paper as a critical step in giving a classification of all partially hyperbolic systems in dimension three which have center-stable tori. Section 5 gives the proof of theorem 2.5.

The proofs of the above results never use the sub-splitting $E^{cs} = E^c \oplus E^s$ of the center-stable bundle. Therefore, all of the above results also hold for *weakly* partially hyperbolic systems, where the diffeomorphism has an invariant splitting of the form $TM = E^{cs} \oplus E^u$. For further discussion of weak versus strong partial hyperbolicity, see sections 1 and 6 of [HP16] and the references therein.

Note that theorem 2.1 above only shows finiteness; it does not say anything about disjointness.

Question 2.6. *Can two distinct compact center-stable submanifolds have non-empty intersection?*

In the case of strongly partially hyperbolic systems in dimension 3, we have a number of special tools at our disposal including branching foliations and Anosov tori [BBI09, RHRHU11], and we can answer this question in the negative.

Theorem 2.7. *In a 3-dimensional strongly partially hyperbolic system, the compact center-stable submanifolds are pairwise disjoint.*

To suggest why such intersections might be possible in higher dimensions, we give an example of an invariant partially hyperbolic subset of a 3-manifold which consists of two surfaces glued together, each of which is tangent to the center-stable direction of the splitting. Moreover, the partially hyperbolic splitting on the subset extends to a dominated splitting defined on the entire manifold.

Theorem 2.8. *There is a diffeomorphism $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ with a sink $z \in \mathbb{T}^3$ and a global dominated splitting into three subbundles $TM = E^u \oplus E^c \oplus E^s$ such that if $B(z)$ denotes the basin of attraction of z , then the boundary of $B(z)$ is the union of two distinct intersecting tori tangent to $E^c \oplus E^s$, and the splitting is partially hyperbolic on all of $\mathbb{T}^3 \setminus B(z)$.*

This shows in particular that there is no local obstruction to having an intersection. We first construct the example which demonstrates theorem 2.8 in section 6 and then prove theorem 2.7 in section 7.

The above results are stated for center-stable submanifolds. By replacing f with its inverse, one may state analogous results for compact center-*unstable* submanifolds. It is easier, in some cases, to prove a result in this alternate setting and so we switch back and forth between the two viewpoints in the proofs below.

In related work, theorems 2.1 and 2.2 generalize results given in [RHRHU15] and their proofs are based on the techniques given there. Theorem 2.3 is closely related to the main result of [BC16], which considers an arbitrary compact invariant set K where $W^u(x) \cap K = \{x\}$ for all $x \in K$. Theorem 2.3 could be proved as a consequence of this result. However, the fact that Λ has the structure of a C^0 submanifold means that we can give a direct, intuitive, and comparatively simple proof of theorem 2.3 in the space of a few pages. For this reason, we give a full self-contained proof in this paper.

3. FINITENESS

In this section, assume $f : M \rightarrow M$ is partially hyperbolic. To prove theorems 2.1 and 2.2, we may freely replace f by an iterate and therefore also assume that $\|Df(v)\| > 2\|v\|$ for all non-zero $v \in E^u$.

Let d_H denote Hausdorff distance. Equipped with d_H , the space of compact subsets of M is a compact metric space. If x and y are points on the same unstable leaf, let $d_u(x, y)$ denote the distance between them as measured along the leaf. If x and y are on distinct unstable leaves, then $d_u(x, y) = +\infty$. Similar to the definition of Hausdorff distance, for subsets $X, Y \subset M$ define

$$\text{dist}_u(x, Y) = \inf_{y \in Y} d_u(x, y)$$

and

$$\text{dist}_u(X, Y) = \max \left\{ \sup_{x \in X} \text{dist}_u(x, Y), \sup_{y \in Y} \text{dist}_u(y, X) \right\}.$$

In what follows, we write *cs*-submanifold as shorthand for a center-stable submanifold. Using the transversality of E^{cs} and E^u , one may prove the following

Lemma 3.1. *There is $r > 0$ such that if S and T are compact *cs*-submanifolds and $d_H(S, T) < r$, then $\text{dist}_u(S, T) < \frac{1}{2}$. \square*

In this section, call a compact subset $X \subset M$ “well positioned” if $d_u(x, y) > 5$ for all distinct $x, y \in X$.

Lemma 3.2. *Let T be a compact *cs*-submanifold. Then, there is an integer n_0 such that $f^n(T)$ is well positioned for all $n > n_0$.*

Proof. As E^{cs} is transverse to E^u , there is $\epsilon > 0$ such that $d_u(x, y) > \epsilon$ for all distinct $x \in T$ and $y \in W^s(x) \cap T$. Then take n_0 such that $2^{n_0}\epsilon > 5$ and use the above assumption on E^u . \square

Lemma 3.3. *If S is a well-positioned compact cs -submanifold and $\text{dist}_u(S, f^k(S)) < \frac{1}{2}$ for some $k \geq 1$, then there is a unique well-positioned periodic C^0 submanifold Λ such that $\text{dist}_u(S, \Lambda) < 1$.*

Proof. For $x \in f^k(S)$, define $h(x)$ as the unique point in $f^k(S)$ such that

$$d_u(f^{-k}(x), h(x)) < \frac{1}{2}.$$

Existence follows from $\text{dist}_u(S, f^k(S)) < \frac{1}{2}$ and uniqueness from the fact that $f^k(S)$ is well positioned. By the same reasoning, there an inverse map h^{-1} and so $h : f^k(S) \rightarrow f^k(S)$ is a homeomorphism. For $x \in M$, define $W_1^u(x) = \{y \in M : \text{dist}_u(x, y) < 1\}$. One can show that $f^{-k}(W_1^u(x)) \subset W_1^u(h(x))$ for all $x \in f^{-k}(S)$. In other words, f^{-k} restricted to a neighbourhood of $f^k(S)$ is a fiber contraction of a C^0 fiber bundle. By the fiber contraction theorem [HPS77, Theorem 3.1], there is an f^{-k} invariant C^0 submanifold Λ in this neighbourhood. Applying f^{-k} , one sees that $\text{dist}_u(S, \Lambda) < 2^{-k} < 1$.

Suppose Λ' is a well-positioned periodic submanifold with $\text{dist}_u(S, \Lambda') < 1$. Then $\text{dist}_u(\Lambda, \Lambda') < 2$ and $\text{dist}_u(f^{-n}(\Lambda), f^{-n}(\Lambda'))$ tends to zero as $n \rightarrow \infty$. This shows that $\Lambda = \Lambda'$. \square

Lemma 3.4. *Let S be a compact cs -submanifold. For any $\epsilon > 0$, there is a well-positioned periodic C^0 submanifold Λ_ϵ such that $\text{dist}_u(S, \Lambda_\epsilon) < \epsilon$.*

Proof. Let n_0 be such that $2^{-n} < \epsilon$ and $f^n(S)$ is well positioned for all $n > n_0$. As Hausdorff distance defines a compact metric space, there are $m > n > n_0$ such that $d_H(f^n(S), f^m(S)) < r$. By lemma 3.1 and lemma 3.3, there is Λ such that $\text{dist}_u(f^n(S), \Lambda) < 1$. Then $\text{dist}_u(S, f^{-n}(\Lambda)) < 2^{-n}$, so take $\Lambda_\epsilon = f^{-n}(\Lambda)$. \square

Proof of theorem 2.2. Let S be a compact cs -submanifold. By lemma 3.4, there is a sequence of periodic submanifolds $\{\Lambda_n\}$ such that $\text{dist}_u(S, \Lambda_n)$ tends to zero. The uniqueness in lemma 3.3 implies that $\{\Lambda_n\}$ is eventually constant. Therefore, $S = \Lambda_n$ for all large n . \square

Proof of theorem 2.1. All compact cs -submanifolds are periodic. By lemma 3.2, they are all well positioned. If $S \neq T$ are two of these submanifolds, then lemmas 3.1 and 3.3 imply that $d_H(S, T) > r$. A compactness argument using Hausdorff distance implies that there are only finitely many. \square

4. REGULARITY OF SUBMANIFOLDS

This section proves theorems 2.3 and 2.4. Using the results of the previous section, the latter follows easily from the former.

Proof of theorem 2.4. Note that $\text{dist}_u(S, f^k(S)) < \infty$ and therefore

$$\text{dist}_u(f^{-n}(S), f^{k-n}(S)) < \frac{1}{2}$$

for sufficiently large n . The condition $W^u(x) \cap S = \{x\}$ implies that $f^{-n}(S)$ is well positioned. Lemma 3.3 then shows that there is a periodic C^0 submanifold Λ which satisfies the hypotheses of theorem 2.3. \square

The above proof further shows that the submanifolds in theorem 2.4 satisfy $\text{dist}_u(S, \Lambda) < \infty$.

One direction of theorem 2.3 readily follows from results in the last section. To prove the other direction, it will be easier to exchange the roles of E^u and E^s . Therefore, we assume $f : M \rightarrow M$ is partially hyperbolic and Λ is a periodic C^0 submanifold such that $W^s(x) \cap \Lambda = \{x\}$ for all $x \in \Lambda$. Our goal is then to show that Λ is a C^1 submanifold tangent to E^{cu} . To prove this, we may freely replace f by an iterate. In particular, assume $f(\Lambda) = \Lambda$. Also assume that associated to the partially hyperbolic splitting $TM = E^s \oplus E^{cu}$ is a continuous function $\lambda : M \rightarrow (0, \frac{1}{2})$ such that $\|Df v^s\| < \lambda(x) < 2\lambda(x) < \|Df v^{cu}\|$ for all $x \in M$ and unit vectors $v^s \in E^s(x)$ and $v^{cu} \in E^{cu}(x)$.

Let $\mathcal{C} \subset TM$ be a cone family associated to the dominated splitting. That is, for every $x \in M$, $\mathcal{C}(x) = \mathcal{C} \cap T_x M$ is a closed convex subset of TM such that $E^s(x) \subset \mathcal{C}(x)$, $E^{cu}(x) \cap \mathcal{C}(x) = 0$, and $\mathcal{C}(x)$ depends continuously on x . Define the dual cone family \mathcal{C}^* as the closure of $TM \setminus \mathcal{C}$. The properties of the splitting imply that

$$\bigcap_{n \geq 0} Df^{-n}(\mathcal{C}) = E^s \quad \text{and} \quad \bigcap_{n \geq 0} Df^n(\mathcal{C}^*) = E^{cu},$$

Replacing \mathcal{C} by some $Df^{-n}(\mathcal{C})$, f by a large iterate f^m , and the function λ by

$$x \mapsto \lambda(f^{m-1}(x)) \cdots \lambda(f(x)) \lambda(x),$$

assume for any $x \in M$ and non-zero vector $v \in T_x M$ that

- (1) if $v \in \mathcal{C}$, then $\|Df v\| < \lambda(x) \|v\|$;
- (2) if $v \in \mathcal{C}^*$, then $Df(v) \in \mathcal{C}^*$; and
- (3) if $v \in Df(\mathcal{C}^*)$, then $\|Df v\| > 2\lambda(x) \|v\|$.

Let $\exp_x : T_x M \rightarrow M$ be the exponential map. Up to rescaling the Riemannian metric on M , assume that if $d(x, y) < 1$, then there is a unique vector $v \in T_x M$ with $\|v\| < 1$ such that $y = \exp_x(v)$. Define a continuous map

$$F : \{v \in TM : \|v\| < 1\} \rightarrow TM$$

by requiring that $\exp_{f(x)}(F(v)) = f(\exp_x(v))$ for all $x \in M$ and $v \in T_x M$ with $\|v\| < 1$.

Lemma 4.1. *There is $0 < \delta < 1$ such that for any $x \in M$ and $v \in T_x M$ with $\|v\| < \delta$*

- (1) *if $v \in \mathcal{C}$, then $\|F(v)\| < \lambda(x) \|v\|$;*
- (2) *if $v \in \mathcal{C}^*$, then $F(v) \in \mathcal{C}^*$; and*
- (3) *if $v \in F(\mathcal{C}^*)$, then $\|F(v)\| > 2\lambda(x) \|v\|$.*

Proof. The properties of the exponential map imply that $\frac{1}{\|v\|} \|F(v) - Df(v)\|$ tends uniformly to zero as $v \rightarrow 0$. The lemma may then be proved from the corresponding properties of Df . \square

Later on, we will also need the following fact.

Lemma 4.2. *For any $n > 0$, there is $r_n > 0$ such that if $w \in \mathcal{C}^*$ and $\|w\| < r_n$, then $F^{n+1}(w) \in Df^n(\mathcal{C}^*)$.*

Proof. There is a lower bound on the angle between any non-zero vectors $u \in Df^n(\mathcal{C})$ and $v \in Df^{n+1}(\mathcal{C}^*)$. Take r_n small enough that if $\|w\| < r_n$, then the angle between $F^{n+1}(w)$ and $Df^{n+1}(w)$ is smaller than this bound. \square

Lemma 4.3. *If $v \in T_x M$ is such that $\|F^n(v)\| < \delta$ and $F^n(v) \in \mathcal{C}$ for all $n \geq 0$, then $\exp_x(v)$ lies on the stable leaf through x .*

This lemma is more or less one of the steps in establishing the existence of the stable foliation. See for instance [HPS77, Section 5]. For completeness, we give a proof which assumes that the stable foliation exists.

Proof. Since E^{cu} is transverse to E^s , there is $r > 0$ such that the (incomplete) submanifold

$$\Sigma_p := \exp_p\{u \in E^{cu}(p) : \|u\| < r\}$$

is transverse to W^s . There is also $\eta > 0$ such if $d(p, q) < \eta$, then $W^s(q)$ intersects Σ_p in a point z which satisfies $d(f^n(q), f^n(z)) < \delta/2$ for all $n \geq 0$.

Write $x_n := f^n(x)$ and $v_n := F^n(v)$. Then $\|v_{n+1}\| < \lambda(x_n)\|v_n\|$, so that $\|v_n\| < 2^{-n}\delta$ for all $n \geq 0$. As $2^{-n}\delta < \eta$ for large n , there is $k \geq 0$ and a vector $w \in E^{cu}(x_k)$ such that $\exp_{x_k}(w)$ and $\exp_{x_k}(v_k)$ lie on the same stable leaf and $F^n(w) - v_{n+k} < \delta/2$ for all $n \geq 0$. Without loss of generality, assume $k = 0$ and write $w_n = F^n(w)$. If $w = 0$, the result is proved. Therefore, we assume $w \neq 0$. Then $w \in \mathcal{C}^*$ implies $w_n \in \mathcal{C}^*$ and therefore $\|w_{n+1}\| > 2\lambda(x_n)\|w_n\|$ for all $n \geq 0$. However, one can show that $\|v_{n+1} - w_{n+1}\| < \lambda(x_n)\|v_n - w_n\|$ for all large n , and this gives a contradiction. \square

Notation. For the rest of the section, if x_0 and y_0 are distinct points on Λ define $x_n = f^n(x_0)$ and $y_n = f^n(y_0)$ for all $n \in \mathbb{Z}$. For those indices where $d(x_n, y_n) < 1$, define $v_n \in T_{x_n} M$ such that $\|v_n\| < 1$ and $y_n = \exp_{x_n}(v_n)$.

Lemma 4.4. *There is a uniform constant $N > 0$ such that for any distinct $x_0, y_0 \in \Lambda$ either $d(x_n, y_n) > \delta$ or $v_n \in \mathcal{C}^*$ for some $0 \leq n < N$.*

Proof. First note that since $W^s(x_0) \cap \Lambda = \{x_0\}$ by assumption, lemma 4.3 implies that such an N exists for each pair (x_0, y_0) considered on its own. The goal is to find a uniform constant N which works for all pairs. Let $0 < \epsilon < \delta$ be such that $\|F(v)\| < \epsilon$ implies $\|v\| < \delta$. The set $\{(x, y) \in \Lambda \times \Lambda : \epsilon \leq d(x, y) \leq \delta\}$ is compact. One may then use an open cover to show that there is a uniform constant $N > 0$ such that if $\epsilon \leq d(x_0, y_0) \leq \delta$ then either $d(x_n, y_n) > \delta$ or $v_n \in \mathcal{C}^*$ for some $0 \leq n < N$.

Now suppose $0 < d(x_0, y_0) < \epsilon$. Let $m \geq 0$ be the smallest integer such that either $\|v_{-m}\| \geq \epsilon$ or $v_{-m} \in \mathcal{C}^*$. Such an m must exist as F^{-1} uniformly expands vectors in \mathcal{C} . If $v_{-m} \in \mathcal{C}^*$, then $v_{-m+1} \in \mathcal{C}^*$ and the minimality of m implies that $m = 0$. If $\|v_{-m}\| \geq \epsilon$, then $\|v_{-m}\| < \delta$ by the choice of ϵ and so there is $0 \leq n < N$ such that $v_{n-m} \in \mathcal{C}^*$. Since $m \geq 0$, this implies that $v_n \in \mathcal{C}^*$. \square

Corollary 4.5. *There is a sequence $\{\epsilon_n\}$ of positive numbers such that if $d(x_0, y_0) < \epsilon_n$ then $v_0 \in Df^n(\mathcal{C}^*)$.*

Proof. Using δ , r_n , and N as above, take $\epsilon_n > 0$ small enough that

$$d(x_0, y_0) < \epsilon_n \Rightarrow d(x_{-k}, y_{-k}) < \min\{\delta, r_n\}$$

for all $0 \leq k \leq N + n + 1$. By lemma 4.4, $v_{-k} \in \mathcal{C}^*$ for some $k \geq n + 1$ which further implies that $v_{-n-1} \in \mathcal{C}^*$. The result then follows from lemma 4.2. \square

Since $\bigcap_{n \geq 0} Df^n(\mathcal{C}^*) = E^{cu}$, this shows that Λ is a C^1 submanifold tangent to E^{cu} .

5. CROSS SECTIONS

To prove theorem 2.5, we will combine theorem 2.4 with the following result, applied to a flow along the unstable direction.

Theorem 5.1. *Let M be a compact connected manifold with boundary, ψ a C^0 flow on M , $g : M \rightarrow [0, 1]$ a continuous function, and $\ell > 0$ a constant such that*

- (1) $g(\partial M) = \{0, 1\}$, and
- (2) *if $x \in M$, $t \in \mathbb{R}$, $0 < g(x) < 1$ and $|t| > \ell$, then $g(\psi_t(x)) \in \{0, 1\}$.*

Then, there is a compact codimension one submanifold S in the interior of M which intersects any orbit in at most one point.

Assume now that the hypotheses in the theorem 5.1 hold. Note that ψ is a global flow defined for all time. For each $t \in \mathbb{R}$, $\psi_t : M \rightarrow M$ is a homeomorphism, and so ∂M is invariant under the flow. For $i, j \in \{0, 1\}$, define

$$X_{i,j} = \{x \in M : \lim_{t \rightarrow -\infty} g(\psi_t(x)) = i \text{ and } \lim_{t \rightarrow +\infty} g(\psi_t(x)) = j\}.$$

Since $g(\partial M) = \{0, 1\}$, at least one boundary component is contained in $X_{0,0}$ and at least one is contained in $X_{1,1}$. The second item in the theorem implies that $M = X_{0,0} \cup X_{0,1} \cup X_{1,0} \cup X_{1,1}$.

Lemma 5.2. *The subsets $X_{0,0}$ and $X_{1,1}$ are closed.*

Proof. Suppose $\{x_k\}$ is a sequence in $X_{0,0}$, converging to $x \in M \setminus X_{0,0}$. Then there is $s \in \mathbb{R}$ such that $g(\psi_s(x)) \neq 0$. As ψ_s is continuous, $g(\psi_s(x_k)) \neq 0$ for all large k . Then $g(\psi_t(x_k)) = 0$ for all large k and all $t \in \mathbb{R}$ with $|t - s| > \ell$. By continuity, $g(\psi_t(x)) = 0$ for all t with $|t - s| > \ell$. \square

Corollary 5.3. *At least one of $X_{0,1}$ or $X_{1,0}$ is non-empty.*

Proof. Otherwise, $X_{0,0}$ and $X_{1,1}$ disconnect M into two clopen subsets. \square

Without loss of generality, assume $X_{0,1}$ is non-empty.

Also define $U_{0,0}$ as

$$U_{0,0} = \{x \in X_{0,0} : \text{there is } t \in \mathbb{R} \text{ such that } g(\psi_t(x)) \neq 0\}$$

and define $U_{1,1}$ similarly.

Lemma 5.4. *The subsets $X_{0,1}$, $X_{1,0}$, $U_{0,0}$, and $U_{1,1}$ are open.*

Proof. If $x \in X_{0,1}$, then there is $t \in \mathbb{R}$ such that

$$g(\psi_{t-\ell}(x)) < \frac{1}{3} < g(\psi_t(x)) < \frac{2}{3} < g(\psi_{t+\ell}(x)).$$

This inequality also holds for all points in a neighbourhood V of x and implies that $V \subset X_{0,1}$. If $x \in U_{0,0}$, then there is $t \in \mathbb{R}$ and $\delta > 0$ such that

$$g(\psi_{t-\ell}(x)) < \delta < g(\psi_t(x)) > \delta > g(\psi_{t+\ell}(x)).$$

This also holds on a neighbourhood of x and shows that $U_{0,0}$ is open. The cases of $X_{1,0}$ and $U_{1,1}$ are analogous. \square

For the remainder of the proof, we assume $\ell < 1$. This can always be achieved by rescaling time for the flow, and makes the definitions simpler in what follows. We now adapt the averaging method of Fuller [Ful65] to this setting. For each integer $n \geq 1$, define $g_n : M \rightarrow [0, 1]$ by

$$g_n(x) = \frac{1}{2n} \int_{-n}^{+n} g(\psi_t(x)) dt.$$

Let D_ψ denote the derivative along the flow. That is, for a function $\alpha : M \rightarrow \mathbb{R}$, define

$$D_\psi \alpha(x) := \lim_{t \rightarrow 0} \frac{1}{t} (\alpha(\psi_t(x)) - \alpha(x)).$$

The Fundamental Theorem of Calculus implies that

$$D_\psi g_n(x) = \frac{1}{2n} [g(\psi_n(x)) - g(\psi_{-n}(x))].$$

The assumption $\ell < 1$ implies that if $x \in X_{0,1}$, then at least one of $g(\psi_{-n}(x)) = 0$ or $g(\psi_n(x)) = 1$ holds. Hence, $0 \leq D_\psi g_n(x) \leq \frac{1}{2n}$ for all n and $D_\psi g_n(x) = \frac{1}{2n}$ for a fixed $x \in X_{0,1}$ and large n . Define $p : M \rightarrow [0, 1]$ by

$$p(x) = \sum_{n=1}^{\infty} 2^{-n} g_n(x).$$

If $x \in X_{0,1}$, one can show that $\lim_{t \rightarrow -\infty} p(\psi_t(x)) = 0$, $\lim_{t \rightarrow +\infty} p(\psi_t(x)) = 1$, and

$$D_\psi p(x) = \sum_{n=1}^{\infty} 2^{-n} D_\psi g_n(x) > 0.$$

Hence, any orbit in $X_{0,1}$ intersects $p^{-1}(\frac{1}{2})$ in exactly one point. As in [Ful65], one can then show that locally $X_{0,1} \cap p^{-1}(\frac{1}{2})$ has the structure of a codimension one C^0 submanifold.

Lemma 5.5. *For $i, j \in \{0, 1\}$, the subset $X_{i,j} \cap p^{-1}(\frac{1}{2})$ is open in the topology of $p^{-1}(\frac{1}{2})$.*

Proof. The cases of $X_{0,1}$ and $X_{1,0}$ follow immediately from lemma 5.4. If $x \in X_{0,0} \cap p^{-1}(\frac{1}{2})$, then $p(x) = \frac{1}{2}$ implies that $g(\psi_t(x))$ cannot be zero for all t . Thus, $X_{0,0} \cap p^{-1}(\frac{1}{2}) = U_{0,0} \cap p^{-1}(\frac{1}{2})$ and is therefore open. Similarly for $X_{1,1} \cap p^{-1}(\frac{1}{2})$. \square

Corollary 5.6. *The set $X_{0,1} \cap p^{-1}(\frac{1}{2})$ is a finite disjoint union of compact connected codimension one C^0 submanifolds.*

Proof. As noted above, $X_{0,1} \cap p^{-1}(\frac{1}{2})$ locally has the structure of a C^0 submanifold. By lemma 5.5 and the fact that M splits into the disjoint union $M = X_{0,0} \cup X_{0,1} \cup X_{1,0} \cup X_{1,1}$, the subset $X_{0,1} \cap p^{-1}(\frac{1}{2})$ is clopen in the topology of $p^{-1}(\frac{1}{2})$. In particular, this subset is compact and therefore consists of a finite number of compact connected submanifolds. \square

To complete the proof of theorem 5.1, take S to be one of the components of $X_{0,1} \cap p^{-1}(\frac{1}{2})$.

We now look at how these components interact with a diffeomorphism which preserves the orbits of the flow. In what follows, let S_1, \dots, S_m be the connected components of $X_{0,1} \cap p^{-1}(\frac{1}{2})$.

Proposition 5.7. *Suppose $f : M \rightarrow M$ is a homeomorphism which preserves the orbits of ψ and such that*

$$\lim_{t \rightarrow +\infty} g\psi_t(x) = \lim_{t \rightarrow +\infty} gf\psi_t(x) \quad \text{and} \quad \lim_{t \rightarrow -\infty} g\psi_t(x) = \lim_{t \rightarrow -\infty} gf\psi_t(x)$$

for all $x \in M_0$. Then there is $k \geq 1$ such that an orbit of ψ intersects a component S_i if and only if the orbit intersects $f^k(S_i)$.

Proof. The hypotheses imply that $f(X_{0,1}) = X_{0,1}$. For each point $x \in f(S_i)$, there is a unique point \hat{x} on the orbit of x such that $p(\hat{x}) = \frac{1}{2}$. Moreover, \hat{x} depends continuously on x . The image $\{\hat{x} : x \in f(S_i)\}$ is a compact manifold in $X_{0,1} \cap p^{-1}(\frac{1}{2})$ and is therefore equal to one of the S_j . This shows that, up to flowing along the orbits of ψ , the homeomorphism f permutes the components S_i . Up to replacing f again by an iterate, we may assume this is the identity permutation. \square

Proof of theorem 2.5. First, consider the case where E^u is orientable. Define a C^0 flow ψ such that the orbits of ψ are exactly the unstable leaves of f . This flow satisfies the hypotheses of theorem 5.1 (with M_0 in place of M). Consequently, there is a compact C^0 submanifold S in the interior of M_0 which intersects each unstable leaf in at most one point. By proposition 5.7, there is an iterate f^k such that an unstable leaf intersects $f^k(S)$ if and only if it intersects S . Then, $\text{dist}_u(S, f^k(S)) < \infty$ and theorem 2.4 implies that there is a compact periodic center-stable submanifold Λ as desired. This concludes the orientable case.

Instead of handling the non-orientable case directly, we assume now that there is an involution $\tau : M \rightarrow M$ which commutes with f , preserves the unstable foliation, and reverses the orientation of E^u . If $W^u(\Lambda)$ and $W^u(\tau(\Lambda))$ are disjoint, then Λ and $\tau(\Lambda)$ are disjoint. If, instead, $W^u(\Lambda)$ intersects $W^u(\tau(\Lambda))$, then the argument in proof of proposition 5.7 shows that $\text{dist}_u(\Lambda, \tau(\Lambda)) < \infty$ and the fact that Λ is f -periodic implies that $\Lambda = \tau(\Lambda)$. In either case, Λ quotients down to a compact submanifold embedded in M/τ . \square

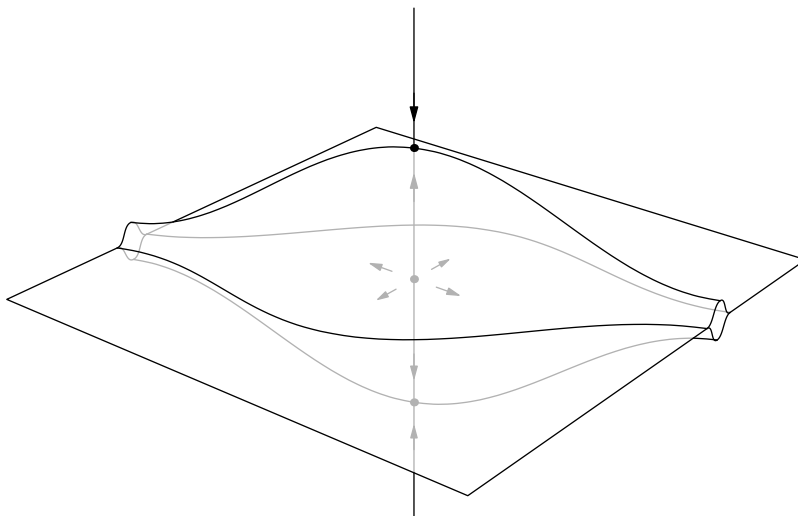


FIGURE 1. Two intersecting center-unstable tori.

6. MAKING A CALZONE

In this section, we construct the example in theorem 2.8. As in section 4, it is slightly easier from the notational viewpoint to switch the roles of E^s and E^u in the construction. Therefore, we will actually build a system with a normally repelling fixed point and two intersecting center-unstable tori.

First, we build a partially hyperbolic subset of $\mathbb{T}^2 \times \mathbb{R}$ which is the union of two non-disjoint cu -tori. Then, we explain how this partially hyperbolic subset can be glued into the 3-torus in such a way to produce a global dominated splitting.

The two cu -tori each have the same derived-from-Anosov dynamics with a repelling fixed point. They are glued together on the complement of the basin of repulsion of this fixed point. The cu -tori are, of course, tangent along this intersection and the construction vaguely resembles the type of food called a calzone, where two pieces of dough are pressed together to enclose a region which is full of other ingredients. A depiction of this construction is given in figure 1.

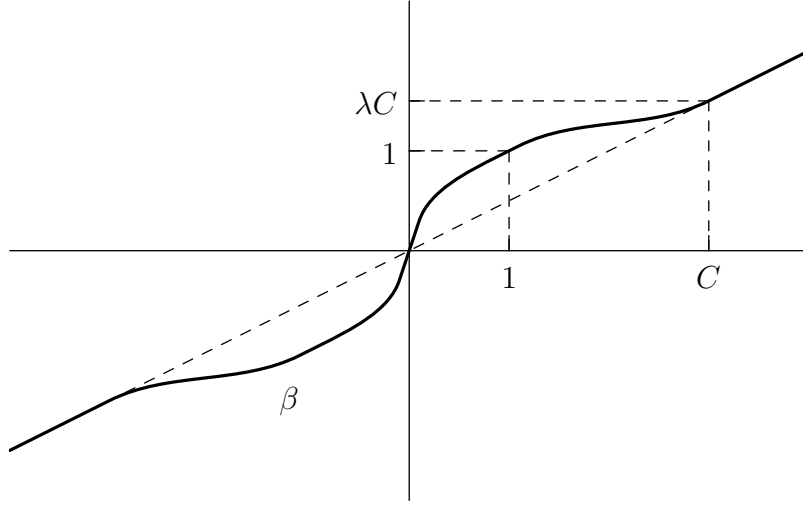
Let $g : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a weakly partially hyperbolic diffeomorphism with a splitting of the form $E^u \oplus E^c$. That is,

$$\|Df v^c\| < \|Df v^u\| \quad \text{and} \quad 1 < \|Df v^u\|$$

hold for all $x \in M$ and unit vectors $v^c \in E^c(x)$, and $v^u \in E^u(x)$. Further suppose that $q \in \mathbb{T}^2$ is a repelling fixed point for g . Let

$$B(q) = \{x \in \mathbb{T}^2 : \lim_{n \rightarrow \infty} g^{-n}(x) = q\}$$

be the basin of repulsion of q and define $K = \mathbb{T}^2 \setminus B(q)$. Define a constant $0 < \lambda < 1$ small enough that $\|Dg v\| > 2\lambda$ for all unit vectors $v \in \mathbb{T}^2$. Define a smooth function $\beta : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties:

FIGURE 2. The graph of the function β .

- (1) β is an odd function with fixed points exactly at $-1, 0$, and $+1$;
- (2) the fixed point at zero is expanding with $1 < \beta'(0) < \|Dg v\|$ for any unit vector $v \in T_q \mathbb{T}^2$;
- (3) the fixed points -1 and $+1$ are contracting with $\beta'(-1) = \beta'(1) = \lambda$; and
- (4) there is a constant $C > 1$ such that $\beta(s) = \lambda s$ for all $s \in \mathbb{R}$ with $|s| > C$.

See figure 2. Define a smooth function $\alpha : \mathbb{T}^2 \rightarrow [0, 1]$ which equals 0 on a neighbourhood of K and equals 1 on a neighbourhood of q . Then define $f : \mathbb{T}^2 \times \mathbb{R} \rightarrow \mathbb{T}^2 \times \mathbb{R}$ by

$$f(x, s) = (g(x), (1 - \alpha(x))\lambda s + \alpha(x)\beta(s)).$$

We now look at the behaviour of tangent vectors under the action of the derivative. If $p = (x, s) \in \mathbb{T}^2 \times \mathbb{R}$, a tangent vector $u \in T_p(\mathbb{T}^2 \times \mathbb{R})$ may be decomposed as $u = (v, w)$ with horizontal component $v \in T_x \mathbb{T}^2$ and vertical component $w \in T_s \mathbb{R}$.

Lemma 6.1. *For any point $p = (x, s) \in \mathbb{T}^2 \times \mathbb{R}$ and any tangent vector*

$$u = (v, w) \in T_p(\mathbb{T}^2 \times \mathbb{R}),$$

define $u_n = Df^n(u)$ and let v_n and w_n be its horizontal and vertical components respectively.

- (1) *If v is non-zero, then the ratio $\frac{\|w_n\|}{\|v_n\|}$ tends to 0 as $n \rightarrow +\infty$.*
- (2) *If $v \notin E_g^c(x)$, then the angle between v_n and $E_g^u(g^n(x))$ tends to 0 as $n \rightarrow +\infty$.*

Proof. The non-wandering set of f is

$$NW(f) = (K \times \{0\}) \cup (\{q\} \times \{-1, 0, +1\}).$$

At $(q, 0)$ the condition on $\beta'(0)$ implies that vectors in the horizontal direction are expanded more strongly than vectors in the vertical direction. At all other points

in $NW(f)$, the condition on λ implies that a vector in the vertical direction is contracted more strongly than any vector in the horizontal direction. Hence, there is a neighbourhood U of $NW(f)$ and a constant $\sigma < 1$ such that if $Df^n(p) \in U$, then

$$\frac{\|w_{n+1}\|}{\|v_{n+1}\|} \leq \sigma \frac{\|w_n\|}{\|v_n\|}.$$

Since $Df^n(p) \in U$ for all large n , this implies item (1).

From the definition of f , note that $v_n = Dg^n(v)$ for all n , and item (2) follows directly from fact that g is weakly partially hyperbolic. \square

From lemma 6.1, one may show that on the invariant subset

$$X := \bigcap_{n \geq 0} f^n(\mathbb{T}^2 \times [-C, C])$$

there is a dominated splitting with three one-dimensional subbundles. We will use $E^u \oplus E^c \oplus E^s$ to denote this splitting, even though the E^s direction is not uniformly contracting.

The fixed point $(q, 1)$ is hyperbolic with a two-dimensional unstable direction. Let $W^{cu}(q, 1)$ denote the two-dimensional unstable manifold through this point. This manifold may be expressed as the graph of a C^1 function from $B(q)$ to \mathbb{R} . Let T_+ be the closure of $W^{cu}(q, 1)$. Then T_+ may be expressed as the graph of a continuous function from \mathbb{T}^2 to \mathbb{R} which is zero on all points in K . One may show, either directly or by a variant of theorem 2.3, that T_+ is a C^1 submanifold tangent to $E^c \oplus E^u$. Since E^s is uniformly attracting on T_+ , this implies that the tangent bundle restricted to T_+ has a strongly partially hyperbolic splitting. By symmetry, the closure of the unstable manifold through the point $(q, -1)$ is also a surface, denoted T_- , with similar properties. Thus, the union $T_+ \cup T_-$ is a partially hyperbolic set and the intersection $T_+ \cap T_- = K \times 0$ is non-empty.

We now describe how this example may be embedded into \mathbb{T}^3 . The constant $C > 1$ was defined so that the equality $f(x, s) = (g(x), \lambda s)$ holds for all (x, s) with $|s| > C$. By rescaling the vertical \mathbb{R} direction of $\mathbb{T}^2 \times \mathbb{R}$, one may, for any given $\epsilon > 0$, define a similar example such that this equality holds for all (x, s) with $|s| > \frac{\epsilon}{4}$. Then, take the construction of f given in the proof of [Ham16, Theorem 1.2] and replace the dynamics on $\mathbb{T}^2 \times [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$ defined there with that of the f defined here. Using lemma 6.1 and the techniques in [Ham16] one may show that this new system has a global dominated splitting and that, outside a basin of repulsion, this dominated splitting is partially hyperbolic. This establishes all of the properties listed in theorem 2.8.

As a final note, it is possible to define a variation on this example by composing f with the reflection $(x, s) \mapsto (x, -s)$. This new system has two compact center-unstable tori which intersect and which are the images of each other.

7. NO CALZONES

The last section constructed an example which was only partially hyperbolic on a subset of \mathbb{T}^3 . Here we prove theorem 2.7, showing that the example cannot be improved to a global partially hyperbolic splitting. The basic idea of the proof is that the region between the two tori must have finite volume, even after lifting to the universal cover. This region also has unstable curves of infinite length. The “length-versus-volume” argument of [BBI09] then gives a contradiction.

Assume f is a partially hyperbolic diffeomorphism of a 3-manifold M , and that T_0 and T_1 are two intersecting compact cs -submanifolds. Up to replacing f by an iterate, assume each T_i is f -invariant. Up to replacing M by a double cover, assume M is orientable. The results in [RHRHU11] then imply that M is either

- (1) the 3-torus,
- (2) the suspension of “minus the identity” on \mathbb{T}^2 , or
- (3) the suspension of a hyperbolic toral automorphism on \mathbb{T}^2 .

We only consider the case $M = \mathbb{T}^3 = \mathbb{R}^3 / \mathbb{Z}^3$. The other two cases have analogous proofs. Further, after applying a C^1 change of coordinates to the system, we assume without loss of generality that $T_0 = \mathbb{T}^2 \times 0$.

The lifted partially hyperbolic map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ on the universal cover is a finite distance from a map of the form $A \times \text{id}$ where $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is linear and hyperbolic. The subset $S_0 := \mathbb{R}^2 \times 0$ covers T_0 and is invariant under the lifted dynamics. By a slight abuse of notation, if $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $z = (z_1, z_2) \in \mathbb{Z}^2$, then write

$$x + z = (x_1 + z_1, x_2 + z_2, x_3).$$

Let $H : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the Franks semiconjugacy [Fra70]. That is, H is a uniformly continuous surjection such that $Hf(x) = AH(x)$ and $H(x + z) = H(x) + z$ for all $x \in \mathbb{R}^3$ and $z \in \mathbb{Z}^2$.

Up to replacing f by an iterate, assume the eigenvalues of A are positive. Let $\lambda > 1$ be the unstable eigenvalue. There is a non-zero linear map $\pi^u : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\pi^u(Av) = \lambda\pi^u(v)$ for any $v \in \mathbb{R}^2$. We will also consider π^u as a map from \mathbb{R}^3 to \mathbb{R} which depends only on the first two coordinates of \mathbb{R}^3 . Define $H^u = \pi^u \circ H$. One may then verify the following properties of H^u hold for any $x, y \in \mathbb{R}^3$ and $z \in \mathbb{Z}^2$.

- (1) $H^u(f(x)) = \lambda H^u(x)$;
- (2) $H^u(x + z) = H^u(x) + \pi^u(z)$;
- (3) if x and y are on the same stable leaf of f , then $H^u(x) = H^u(y)$; and
- (4) there is a uniform constant $R > 0$ such that $|H^u(x) - \pi^u(x)| < R$.

Lemma 7.1. *Suppose U is a non-empty proper subset of S_0 which is saturated by stable leaves and which is invariant under translations by \mathbb{Z}^2 . Then H^u is constant on any connected component of U .*

Proof. Fix some non-zero element $z \in \mathbb{Z}^2$ with $\pi^u(z) > 0$ and consider any point $p \in S_0$. The set

$$S_0 \setminus (W^s(p) \cup W^s(p+z))$$

consists of three connected components, V_1, V_2, V_3 . Up to relabelling these components, one may show using items (3) and (4) of the list above that

$$\{x \in S_0 : \pi^u(x) < \pi^u(p) - R\} \subset V_1$$

and

$$\{x \in S_0 : \pi^u(x) > \pi^u(p+z) + R\} \subset V_3.$$

The remaining component V_2 is then bounded in the sense that

$$|\pi^u(x) - \pi^u(y)| < 2R + \pi^u(z)$$

for all $x, y \in V_2$. Note that this bound is independent of the choice of p .

Choose a point q in $S_0 \setminus U$. Then U is a subset of

$$S_0 \setminus \bigcup_{k \in \mathbb{Z}} W^s(q + kz).$$

If x and y are points on the same connected component U_0 of U , then one may find a point p of the form $p = q + kz$ so that V_2 as defined above satisfies $U_0 \subset V_2$. Then, the above bound holds for x and y . For any $n \in \mathbb{Z}$, the set $f^n(U)$ also satisfies the hypotheses of this lemma, and so

$$\left| \pi^u f^n(x) - \pi^u f^n(y) \right| < 2R + \pi^u(z).$$

Then

$$\lambda^n \left| H^u(x) - H^u(y) \right| = \left| H^u f^n(x) - H^u f^n(y) \right| < 4R + \pi^u(z)$$

for all $n \geq 0$, and this shows that $|H^u(x) - H^u(y)| = 0$. \square

Since the tori T_0 and T_1 intersect, we may lift T_1 to a surface S_1 which intersects S_0 . Since S_1 is invariant under translation by \mathbb{Z}^2 , there is $N > 0$ such that $S_1 \subset \mathbb{R}^2 \times [-N, N]$.

Note that $S_0 \setminus S_1$ is saturated by stable leaves and is invariant under translations by \mathbb{Z}^2 . Let X_0 be the closure of a connected component of $S_0 \setminus S_1$. Using lemma 7.1, one can show that H^u is constant on X_0 . There is a corresponding set X_1 which is the closure of a connected component of $S_1 \setminus S_0$ and such that the intersection $X_0 \cap X_1$ consists of two stable manifolds. The union $X := X_0 \cup X_1$ may be regarded as two infinitely long strips glued together along their boundaries. Hence, X is a properly embedded topological cylinder cutting \mathbb{R}^3 into two pieces.

By adapting the proof of lemma 7.1, one may show that H^u is constant on X_1 . Then H^u is constant on all of X and $\pi^u(X)$ is a bounded subset of \mathbb{R} . Thus, one of the connected components of $\mathbb{R}^3 \setminus X$ is a region Y with $\pi^u(Y)$ bounded. Note that H^u is constant on each $f^n(\partial Y)$ for $n \in \mathbb{Z}$ and therefore the length of $\pi^u f^n(Y)$ is uniformly bounded for all n .

Similarly to π^u , define a linear map π^s such that the composition $H^s = \pi^s \circ H$ satisfies $H^s(f(x)) = \lambda^{-1}H^s(x)$ and $|H^s(x) - \pi^s(x)| < R$ for all $x \in \mathbb{R}^3$. Let J be an unstable segment inside Y . Then, the lengths of both $\pi^s f^n(J)$ and $\pi^u f^n(J)$ are uniformly bounded for all $n \geq 0$. Further, $f^n(J) \subset \mathbb{R}^2 \times [-N, N]$. Thus, the diameters of $f^n(J)$ are uniformly bounded for all $n \geq 0$. The results in [BBI09] imply that since the lengths of the unstable curves $f^n(J)$ grow without bound, their diameters as subsets of \mathbb{R}^3 must grow without bound as well. This gives a contradiction.

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