NOTES ON GLOBAL PRODUCT STRUCTURE

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1. Notes on global product structure

These notes give a careful explanation of parts of the proof of the following result.

Theorem 1.1 (Brin-Manning)**.** *If an Anosov diffeomorphism f has Global Product Structure, and the universal cover has polynomial growth of volume, then f is topologically conjugate to an infranilmanifold automorphism.*

The original proof of Brin and Manning [3] relies on an incorrect statement of Auslander regarding infranilmanifold automorphisms whereas these current notes avoid using this incorrect statement.

In this note, I go through in detail only the first part of the proof of the theorem, which is the construction of a semi-conjugacy. I do not include the proof that the semi-conjugacy is injective and therefore a true conjugacy since the original arguments for this step hold without modification. I have ordered the steps of the proof so that as much is proved as possible before introducing infranilmanifolds.

I wrote these notes mainly for myself, in order to convince myself of the proof. Almost all of the following repeats arguments already given by Brin, Manning, Franks, Dekimpe, and others [3][6][5][4].

Lemma 1.2. *If $f : M \to M$ has Global Product Structure, then for every $\epsilon > 0$, there is $k \geq 1$ such that*

$$f^{-k}(B_\epsilon(x)) \cap B_\epsilon(y) \neq \varnothing$$

for all $x, y \in M$. Here, $B_\epsilon(x) := \{y \in M : d(x, y) < \epsilon\}$.

Proof. For the lifted foliations W^u and W^s on the universal cover $\tilde{M}$, the intersection

$$[\tilde{x}, \tilde{y}] = W^s(\tilde{x}) \cap W^u(\tilde{y})$$

depends continuously on $\tilde{x}, \tilde{y} \in \tilde{M}$. As W^u and W^s are tangent to continuous subbundles E^u and E^s, $d_s(\tilde{x}, [\tilde{x}, \tilde{y}])$ and $d_u(\tilde{y}, [\tilde{x}, \tilde{y}])$ are continuous as well and are bounded for $(\tilde{x}, \tilde{y}) \in K \times K$ where K is a compact fundamental domain of the covering $\tilde{M} \to M$. Projecting down, there is $R > 0$ such that for all $x, y \in M$, there is

$$z \in W^s(x) \cap W^u(y) \subset M$$

with $d_s(x, z) < R$ and $d_u(y, z) < R$. One can then find $n \geq 1$, independent of x and y, such that

$$d_s(f^n(x), f^n(z)) < \epsilon \quad \text{and} \quad d_u(f^{-n}(y), f^{-n}(z)) < \epsilon.$$

This is enough to prove the lemma with $k = 2n$. ∎

Corollary 1.3. *With f, ϵ, k as in the last lemma, for any periodic point $f^m(x) = x$, there is a closed ϵ-pseudo orbit*

$$x, fx, \cdots, f^{m-1}x, y, fy, \cdots, f^{k-1}y, x.$$

Proof. Take $y \in B_\epsilon(x) \cap f^{-k}(B_\epsilon(x)) \neq \varnothing$. □

Lemma 1.4. *If f has Global Product Structure, there is k such that $\#\operatorname{Fix}(f^m) \leq \#\operatorname{Fix}(f^{m+k})$ for all $m \geq 1$.*

Proof. f is expansive; there is $\delta > 0$ such that if $d(f^n(x), f^n(x')) < \delta$ for all $n \in \mathbb{Z}$, then $x = x'$. It also has a periodic shadowing property; there is $\epsilon > 0$ such that every closed ϵ-pseudo orbit $x_0, x_1, \cdots x_i = x_0$ is $\frac{1}{3}\delta$-shadowed by a true orbit $x = f^i(x)$.

Suppose $x = f^m(x)$. By the previous corollary, there is an ϵ-pseudo orbit

$$x, fx, \cdots, f^{m-1}x, y, fy, \cdots, f^{k-1}y, x.$$

and this is $\frac{1}{3}\delta$-shadowed by some $z = f^{m+k}(z)$. In particular, $d(f^i(x), f^i(z)) < \frac{1}{3}\delta$ for $0 \leq i < m$. Say by the same process that $x' = f^m(x')$ leads to a point $z' = f^{m+k}(z')$. If $z = z'$, then $d(f^i(x), f^i(x')) < \frac{2}{3}\delta$ for $0 \leq i < m$. Hence, for all $i \in \mathbb{Z}$, and therefore $x = x'$. □

Remark. Brin and Manning use $\frac{1}{3}\delta$, but it seems that $\frac{1}{2}\delta$ would suffice.

Lemma 1.4 will later be used to establish hyperbolicity of a Lie group automorphism. For this, we will also need an elementary result about complex numbers.

Lemma 1.5. *Suppose that $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ are such that no λ_i is a root of unity. Define*

$$a_m := \prod_{i=1}^{n} |1 - \lambda_i^m|.$$

If there is $k \geq 1$ such that $a_m \leq a_{m+k}$ for all $m \geq 1$, then $|\lambda_i| \neq 1$ for all i.

For completeness, we include a proof at the end of the paper.

Lemma 1.6. *If A is a hyperbolic automorphism of a nilpotent Lie group L, then $\alpha_b : L \to L,\ x \mapsto A(x) \cdot b$ has a fixed point for all $b \in L$.*

Proof. By the Anosov closing lemma, if $g : M \to M$ is Anosov, there are constants $\delta, \epsilon > 0$ such that if $d(g(x), x) < \epsilon$, then there is $y = f(y)$ with $d(x, y) < \delta$. (This is just shadowing of a constant pseudo orbit.) The values of δ and ϵ depend on bounds on the angle between E^u and E^s and bounds related to expansion and contraction given in the definition of an Anosov diffeomorphism. The closing lemma is proven locally, so it holds for M non-compact so long as E^u and E^s are uniformly continuous and these bounds hold uniformly on M.

To prove the lemma, equip L with a metric such that $d(x \cdot y, x \cdot z) = d(y, z)$ for all $x, y, z \in L$. The closing lemma holds for all α_b and with δ and ϵ independent of b. If α_b has a fixed point x, then $d(\alpha_{b'}(x), x) < \epsilon$ for all $b' \in B_\epsilon(b)$, so every such b' has a fixed point. Since $\alpha_1 = A$ has a fixed point, and $B_\epsilon(B_{n\epsilon}(1)) = B_{(n+1)\epsilon}(1)$ we can show that any α_b has a fixed point. □

Assumption 1.7. *For the rest of this note, assume that $f : M \to M$ is an Anosov diffeomorphism with Global Product Structure and the universal cover $\tilde{M}$ has polynomial growth of volume.*

Lemma 1.8. *$\tilde{M}$ is homeomorphic to $\mathbb{R}^n$ for some n.*

Proof. By Global Product Structure, $\tilde{M}$ is homeomorphic to an unstable leaf direct product with a stable leaf. As each of these is homeomorphic to $\mathbb{R}^n$ for some n, so is the direct product. □

Lemma 1.9. *$\pi_1(M)$ is torsion free.*

Proof. Suppose not. Then there would be a deck transformation $\gamma : \tilde{M} \to \tilde{M}$ of finite period. As $\tilde{M}$ is homeomorphic to $\mathbb{R}^n$, this gives a fixed-point free homeomorphism of $\mathbb{R}^N$ which is periodic. This is ruled out by a classic result of P. A. Smith. (This is repeating an argument given by Franks [6]. Franks cites [2] as a reference.) □

Lemma 1.10. *The maximal normal nilpotent subgroup N of $\pi_1(M)$ has finite index.*

Proof. By Gromov, $\pi_1(M)$ has a nilpotent group H of finite index [7]. Then, there is a subgroup $K \leq H$ of finite index and normal (in $\pi_1(M)$). As it is a subgroup of H, it is nilpotent. The Hirsch-Plotkin radical N of $\pi_1(M)$ contains K and is therefore of finite index. Hence, it the desired maximal subgroup. □

Corollary 1.11. *N is characteristic; it is preserved by every automorphism.*

Lift $f : M \to M$ to a map $\tilde{f} : \tilde{M} \to \tilde{M}$. Viewing $\pi_1(M)$ as the group of all deck transformations, the choice of lift defines a group automorphism f_* of $\pi_1(M)$ by

$$\tilde{f}(\gamma(x)) = f_*(\gamma)(\tilde{f}(x))$$

for all $x \in \tilde{M}$ and $\gamma \in \pi_1(M)$. By the above corollary, $f_*(N) = N$.

We use the following results on nilmanifolds as given by Malcev [9].

Theorem 1.12 (Malcev)**.** *If N is a torsion-free nilpotent finitely-generated group, there is*

- *a connected, simply connected, nilpotent Lie group L,*
- *a subgroup $\Lambda < L$, and*
- *a group isomorphism $T : \Lambda \to N$,*

such that L/Λ is a compact manifold (a nilmanifold).

Any such L is unique up to Lie group isomorphism. In general, Λ and T are not unique.

Every group automorphism of Λ extends to a unique Lie group automorphism of L.

Lemma 1.13. *There is a nilpotent Lie group L, a subgroup Λ, an isomorphism $T : \Lambda \to N$, and a Lie group automorphism $A : L \to L$ such that*

- *$A(\Lambda) = \Lambda$, and*
- *$A(x) = T^{-1} f_* T(x)$ for $x \in \Lambda$.*

Proof. This is the theorem of Malcev applied to our specific situation. □

Lemma 1.14. *A is hyperbolic.*

Remark. As the choice of T in lemma 1.13 was arbitrary, this will show that A is hyperbolic for any choice of T.

In order to prove lemma 1.14, we will temporarily make an additional assumption.

Assumption 1.15. *For now, assume that on the universal cover, $\tilde{f}$ preserves the orientation of E^u. If need be, replace f and A with f^2 and A^2.*

As A is hyperbolic if and only if A^2 is hyperbolic, this assumption can be made freely for the purposes of proving lemma 1.14.

Viewing $\gamma \in \pi_1(M)$ as a diffeomorphism $\tilde{M} \to \tilde{M}$ whose derivative maps the lifted unstable bundle E^u to itself, define a group homomorphism $\theta : \pi_1(M) \to \{+1, -1\}$ by whether γ preserves or reverses the orientation of E^u. Then $N^+ := \ker(\theta) \cap N$ defines a finite-index normal nilpotent subgroup of $\pi_1(M)$.

By assumption 1.15, f_* restricts to an automorphism of N^+.

Define a manifold $\hat{M}$ as a quotient $\tilde{M}/N^+$. That is, $x, y \in \tilde{M}$ are identified if there is $\gamma \in N^+ \subset \pi_1(M)$ such that $\gamma(x) = y$. Since $\gamma(x) = y$ implies $f_*(\gamma)(\tilde{f}(x)) = \tilde{f}(y)$, $\tilde{f}$ quotients to a map $\hat{f} : \hat{M} \to \hat{M}$. Further, $\pi_1(\hat{M})$ is isomorphic to N^+ and $\tilde{f}$ can be viewed as a lift of $\hat{f}$ to $\tilde{M}$. As such, the induced group automorphism $\hat{f}_*$ of $\pi_1(\hat{M})$ can be identified with the restriction of f_* to N^+.

Lemma 1.16. *There is a subgroup $\Lambda^+ \subset \Lambda$ such that*

- $T(\Lambda^+) = N^+$,
- $A(\Lambda^+) = \Lambda^+$,
- $A(x) = T^{-1}\hat{f}_* T(x)$ *for* $x \in \Lambda^+$.

Proof. This follows from lemma 1.13 and the fact that $f_*(N^+) = N^+$ (under assumption 1.15). □

Let $\hat{g} : L/\Lambda^+ \to L/\Lambda^+$ be the quotient of A to the nilmanifold L/Λ^+.

Lemma 1.17. *There is a homotopy equivalence $\hat{h} : \hat{M} \to L/\Lambda^+$ with lift $\tilde{h} : \tilde{M} \to L$ such that the maps induced by $\hat{f}$, $\hat{g}$, and $\hat{h}$ on the fundamental groups satisfy $\hat{h}_* \hat{f}_* = \hat{g}_* \hat{h}_*$.*

Proof. By the construction of $\hat{g}$, there is an isomorphism between $\pi_1(\hat{M})$ and $\pi_1(L/\Lambda^+)$ that conjugates $\hat{f}_*$ and $\hat{g}_*$. Since $\tilde{M}$ is homeomorphic to $\mathbb{R}^n$ and L/Λ^+ is a nilmanifold, both spaces are of type $K(\pi, 1)$ the lemma follows from standard results in algebraic topology. □

Remark. Many treatments of Eilenberg-MacLane spaces assume that the fundamental group is defined with regard to a pointed space (X, x_0). This complicates matters, as we have not yet proved that $\hat{f}$ has a fixed point, and so may not be a based map. However, one can isotope $\hat{f}$ to a map $\hat{f}_1$ which does have a fixed point, prove the lemma for $\hat{f}_1$, and then use the isotopy to prove it for $\hat{f}$.

Lemma 1.18. *Let $L(\cdot)$ denote the Lefschetz number of a diffeomorphism of a manifold. Then,*

$$\#\operatorname{Fix}(\hat{f}^m) = |L(\hat{f}^m)| = |L(\hat{g}^m)| = \prod_{i=1}^{n} |1 - \lambda_i^m|$$

where λ_i are the eigenvalues of A.

Proof. The left-most equality is a standard result for Anosov diffeomorphisms which preserve the orientation of E^u. (See [11].) By the previous lemma, the maps induced by $\hat{f}$ and $\hat{g}$ on the homology groups are conjugate. Therefore, their traces are the same, and the resulting Lefschetz numbers are the same. The last equality is proved in [10]. □

Proof of lemma 1.14. None of the λ_i in lemma 1.18 can be a root of unity, for then some iterate of $\hat{f}$ would be an Anosov map without periodic points. The result then follows as a combination of lemmas 1.4, 1.5, and 1.18. □

End of assumption 1.15. As we have proved lemma 1.14, we no longer need the assumption.

Lemma 1.19. *f has a fixed point.*

Proof. Suppose that instead of using N^+ and Λ^+, we had used N and Λ to define maps $\breve{f} : \tilde{M}/N \to \tilde{M}/N$ and $\breve{g} : L/\Lambda \to L/\Lambda$. As $\breve{f}$ may not preserve an orientation of E^u, we cannot compare $\#\operatorname{Fix}(\breve{f}^m)$ to $|L(\breve{f}^m)|$. However, the other two equalities given by lemma 1.18 still hold in this case, and using $m = 1$ and that A is hyperbolic, we get that $L(\breve{f})$ is non-zero. Then $\breve{f}$ has a fixed point and this projects to a fixed point for f. □

Let $\operatorname{Aff}(L)$ denote the affine transformations of L, those functions of the form $x \mapsto b \cdot \beta(x)$, where β is an automorphism of the Lie group and $b \in L$.

Theorem 1.20 (Auslander-Schenkman)**.** *If Γ is a torsion-free finitely-generated group with nilpotent Hirsch-Plotkin radical N of finite index, then Γ can be viewed as a subgroup of $\operatorname{Aff}(L)$, where L is the Lie group given by theorem 1.12 corresponding to N. Moreover, L/Γ is a compact manifold.*

This follows from section 2 of [1].

Remark. There is a subtle issue here. When regarded as a subgroup of Γ, N consists of affine maps. These maps are of the form $L \to L,\ x \mapsto a \cdot x$ for some $a \in L$. If N is regarded as a subgroup of L, its elements are no longer maps, but simply elements $a \in L$. These two distinct interpretations may have contributed to Auslander providing an incorrect proof for the extension of an automorphism of Γ. To try to avoid confusion, we will not regard N as a subgroup of L, and instead use the symbol $\Lambda \subset L$ and say that they are identified by an isomorphism $T : \Lambda \to N$.

We now follow the proof given in the paper of Lee and Raymond [8], but in regards to our specific situation. By the previous theorem, $\pi_1(M)$ can be identified with a subgroup Γ of $\operatorname{Aff}(L)$ and f_* then defines an automorphism $\psi : \Gamma \to \Gamma$ for which $\psi(N) = N$. This restriction to N defines an automorphism $T^{-1}\psi T$ on Λ which extends to $A : L \to L$ by the result of Malcev. Continuing on, the proof of Lee and Raymond shows that ψ is conjugation by an element of $\operatorname{Aff}(L)$. They give a formula in the proof: using their notation, ψ is conjugation by $(b, \mu(b^{-1})A)$ [8, page 75]. This can be re-written as $\psi(\gamma) = \alpha\gamma\alpha^{-1}$ where $\alpha(x) = A(x) \cdot b$.

Lemma 1.21. *α is hyperbolic.*

Proof. The above construction of A using the results of Malcev is exactly the same as in lemma 1.13, and therefore A is hyperbolic by lemma 1.14. □

Corollary 1.22. *α has a fixed point.*

Proof. This is a specific case of lemma 1.6. □

If $x_0 \in L$ is a fixed point of α, then $\alpha(x) = A(x \cdot x_0^{-1}) \cdot x_0$, so $\alpha = \beta A \beta^{-1}$, where $\beta(x) = x \cdot x_0$. Note that $\beta \in \mathrm{Aff}(L)$. For $\gamma \in \Gamma$, the formula $\psi(\gamma) = \alpha\gamma\alpha^{-1}$ expands to

$$\psi(\gamma) = \beta A \beta^{-1} \gamma \beta A^{-1} \beta^{-1} \quad \Rightarrow \quad \beta^{-1}\psi(\gamma)\beta A = A\beta^{-1}\gamma\beta.$$

Define $\bar{\Gamma} = \{\beta^{-1}\gamma\beta : \gamma \in \Gamma\}$ and $\bar{\psi} : \bar{\Gamma} \to \bar{\Gamma}$ by $\bar{\psi}(\beta^{-1}\gamma\beta) = \beta^{-1}\psi(\gamma)\beta$, so that the above formula can be rewritten simply as

$$\bar{\psi}(\bar{\gamma})A = A\bar{\gamma}.$$

The manifold $P = L/\bar{\Gamma}$ is a compact manifold, and the Lie group automorphism A quotients down to an Anosov diffeomorphism $g : P \to P$. Identifying $\pi_1(P)$ with $\bar{\Gamma}$, there is a commutative diagram

$$\begin{array}{ccccccc}
\pi_1(P) & \longrightarrow & \bar{\Gamma} & \longrightarrow & \Gamma & \longrightarrow & \pi_1(M) \\
\big\downarrow g_* & & \big\downarrow \bar{\psi} & & \big\downarrow \psi & & \big\downarrow f_* \\
\pi_1(P) & \longrightarrow & \bar{\Gamma} & \longrightarrow & \Gamma & \longrightarrow & \pi_1(M)
\end{array}$$

where all arrows are isomorphisms and the top and bottom rows are the same. Thus, f_* and g_* are conjugate.

As M and P are $K(\pi, 1)$ and f and g have fixed points, we can apply the results of Franks to find a semi-conjugacy $h : M \to P$ such that $hf = gh$ [6]. At this point, the proof given by Brin and Manning is fairly easy to follow (see also [5]), and there are no subtlies with respect to infranilness. Therefore, I will stop here.

Appendix A. Proof of lemma 1.5

Assumption A.1. *Assume for the next two lemmas that $\lambda_1, \cdots \lambda_n \in \mathbb{C}$ are not roots of unity, and $|\lambda_i| = 1$ for all i. The λ_i need not be distinct.*

Lemma A.2. *If $\{n_j\}$ is a subsequence of $\mathbb{N}$ such that $\lambda_1^{n_j} \to 1$, then there is $p \in \mathbb{N}$ such that $\lambda_1^{pn_j} \to 1$ and $\lambda_i^{pn_j} \not\to \lambda_i$ for all i.*

Proof. Clearly, $(\lambda_1^{n_j})^p \to 1$ for any p. Suppose for some i and distinct p, q, that $\lambda_i^{pn_j} \to \lambda_i$ and $\lambda_i^{qn_j} \to \lambda_i$. Then $(\lambda_i^{pn_j})^q \to \lambda_i^q$ and $(\lambda_i^{qn_j})^p \to \lambda_i^p$, and so $\lambda_i^p = \lambda_i^q$. This contradicts the fact that λ_i is not a root of unity. Therefore, for all but finitely many p, the lemma is satisfied. □

Lemma A.3. *There is a subsequence $\{m_j\}$ of $\mathbb{N}$ such that for all i, $L_i = \lim_{j\to\infty} \lambda_i^{m_j}$ exists and $L_i \neq 1$. Further, $\lim_{j\to\infty} \lambda_1^{m_j+1} = 1$.*

Proof. As λ_1 is not a root of unity, there is a subsequence n_j such that $\lambda_1^{n_j} \to 1$. Since each sequence $\lambda_i^{n_j}$ lies in a compact subset of $\mathbb{C}$, by replacing n_j with a further subsequence, we may assume $\lambda_i^{n_j}$ converges for all i. Choosing p as in the previous lemma, $m_j = pn_j - 1$ is the desired subsequence in the statement of this lemma. □

Lemma A.4. *Suppose that $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ are such that no λ_i is a root of unity. Further suppose $|\lambda_1| = 1$. Define*

$$a_m := \prod_{i=1}^{n} |1 - \lambda_i^m|.$$

Then there is a subsequence $\{a_{m_j}\}$ *such that* $\frac{a_{m_j+1}}{a_{m_j}} \to 0$.

Proof. If $|\lambda_i| < 1$, then

$$\frac{|1 - \lambda_i^{m+1}|}{|1 - \lambda_i^m|} \to 1.$$

If $|\lambda_i| > 1$, then

$$\frac{|1 - \lambda_i^{m+1}|}{|1 - \lambda_i^m|} \to |\lambda_i|.$$

By the previous lemma, there is $\{m_j\}$ such that

$$\lim_{j\to\infty} \frac{|1 - \lambda_i^{m_j+1}|}{|1 - \lambda_i^{m_j}|}$$

exists for those i with $|\lambda_i| = 1$, and for $i = 1$ in particular, the limit is zero. Then,

$$\lim_{j\to\infty} \frac{a_{m_j+1}}{a_{m_j}} = \prod_i \lim_{j\to\infty} \frac{|1 - \lambda_i^{m_j+1}|}{|1 - \lambda_i^{m_j}|} = 0.$$

□

Proof of lemma 1.5. If we define $\tilde{a}_m$ similarly to a_m, but using $\tilde{\lambda}_i = \lambda_i^k$ in place of λ_i, then $\tilde{a}_m = a_{km}$ and the hypothesis $a_m \le a_{m+k}$ for all m implies $\tilde{a}_m \le \tilde{a}_{m+1}$ for all m. Therefore, it is enough to prove the lemma in the case $k = 1$. Suppose that $|\lambda_i| = 1$ for some i. Without loss of generality, $i = 1$. By the previous lemma, there is a sequence of terms $a_{m_j+1}/a_{m_j} \ge 1$ which tends to zero, a contradiction. □

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