Partial hyperbolicity on 3-dimensional nilmanifolds

Andy Hammerlindl

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Abstract

Every partially hyperbolic diffeomorphism on a 3-dimensional nilmanifold is leaf conjugate to a nilmanifold automorphism. Moreover, if the nilmanifold is not the 3-torus, the center foliation is an invariant circle bundle.

1 Introduction

A defining achievement of the study of dynamical systems was the classification of all Anosov systems on nilmanifolds due to J. Franks and A. Manning. A diffeomorphism $f : M \to M$ is *Anosov* if the tangent bundle of the manifold M splits into two invariant subbundles

$$TM = E^u \oplus E^s,$$

the unstable subbundle E^u which is strongly expanded by Tf and the stable subbundle E^s which is strongly contracted. A *nilmanifold* is a manifold constructed as the quotient space of a nilpotent Lie group. A *nilmanifold isomorphism* is a homeomorphism between nilmanifolds that is the quotient of a Lie group isomorphism on the covering spaces. The classification result is that every Anosov diffeomorphism on a compact nilmanifold is topologically conjugate to a nilmanifold isomorphism [5, 4, 9]. In other words, an infinitesimal condition on the derivative dictates the global behaviour of the diffeomorphism, and the algebraic structure of the manifold gives an algebraic structure to the maps acting on it.

It is conjectured that all Anosov diffeomorphisms occur on nilmanifolds or on spaces finitely covered by nilmanifolds, and thus that all of these systems can be understood in terms of this algebraic classification. To broaden our understanding of dynamics in general, we must look beyond the uniform hyperbolicity of Anosov systems. One such generalization is the notion of partial hyperbolicity. A diffeomorphism is *partially hyperbolic* if the tangent bundle splits into three subbundles

$$TM = E^u \oplus E^c \oplus E^s.$$

Here, the center subbundle E^c may contract or expand slightly, but it is dominated by the strong expansion and contraction of the unstable and stable subbundles. A precise definition will be given in Section 3.

In contrast to the stable and unstable directions, there is almost no restriction on the dynamics in the center direction, and thus, in general, the center is badly behaved. To attempt a reasonable classification in the style of Franks and Manning, we must ignore the "unhyperbolic" behaviour along the center direction. Suppose $f : M \to M$ and $g : M \to M$ are partially hyperbolic and there is a foliation tangent to the center subbundle of each of the two diffeomorphisms. A *(center) leaf conjugacy* between f and g is a homeomorphism $h : M \to M$, such that for every center leaf $\mathcal{L}$ of f, $h(\mathcal{L})$ is a center leaf of g and

$$hf(\mathcal{L}) = gh(\mathcal{L}).$$

A leaf conjugacy does not preserve the arbitrary dynamics which happens along center leaves, but it does preserve the hyperbolic dynamics which acts on the center leaves.

The most easily understood partially hyperbolic systems occur on three-dimensional manifolds, where each of the subbundles E^u, E^c and E^s is one-dimensional. In this paper, we prove the following classification result.

Theorem 1.1. *Any partially hyperbolic diffeomorphism on a compact, three-dimensional nilmanifold is leaf conjugate to a nilmanifold automorphism.*

There are two nilpotent, three-dimensional Lie groups, and so there are two families of compact, three-dimensional nilmanifolds. The first group consists only of the 3-torus and the nilmanifold automorphisms on the 3-torus are exactly the toral automorphisms given by 3×3 invertible matrices with integer entries. Theorem 1.1 for the specific case of the 3-torus was proven in [6]. Hence, this paper deals exclusively with nilmanifolds in the second family, given by quotients of the Heisenberg group, $\mathcal{H}$, the Lie group of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \quad (x, y, z \in \mathbb{R})$$

under the group operation of matrix multiplication. For purposes of brevity, we will often write the above triangular matrix as $(x, y, z) \in \mathcal{H}$. In this notation, the group operation is

$$(a, b, c) \cdot (x, y, z) = (a + x, b + y, z + c + ay).$$

The prototypical example of a nilmanifold is the quotient of $\mathcal{H}$ by a lattice consisting of matrices with integer entries. A fundamental domain of this quotient is the unit "cube"

$$\{(x, y, z) \in \mathcal{H} : 0 \le x, y, z \le 1\}.$$

To obtain the nilmanifold from the cube, identify the top face with the bottom

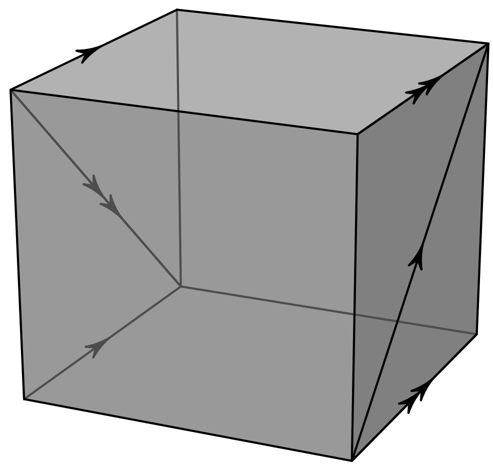


Figure 1: A nilmanifold can be constructed from a cube by identifying left and right faces in a slanted manner. The other faces are identified by simple translation.

face and the front face with the back face, as one would in constructing the 3-torus. The group multiplication

$$(1,0,0) \cdot (x,y,z) = (x+1, y, z+y)$$

tells us to join left and right faces of the cube by the identification

$$(0,y,z) \sim (1,y,z+y) \pmod 1.$$

This slanted identification in the x-direction frustrates analysis on the nilmanifold. To overcome this, we almost exclusively work on the universal cover, $\mathcal{H}$, and with sets which are bounded in the x-direction.

To construct an example of a nilmanifold automorphism which is partially hyperbolic, first consider the Lie group automorphism

$$\Phi : \mathcal{H} \to \mathcal{H}, \quad (x,y,z) \mapsto (2x+y, x+y, z + x^2 + \tfrac{1}{2}y^2 + xy).$$

This maps the lattice

$$\Gamma = \{(x,y,z) \in \mathcal{H} : x,y \in \mathbb{Z},\ z \in \tfrac{1}{2}\mathbb{Z}\}$$

onto itself, and so Φ quotients down to a nilmanifold automorphism $\Phi_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$. The derivative at $(0,0,0)$, and hence at any point, is given by the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This has eigenvalues $\lambda > 1 > \lambda^{-1}$. Associating the three corresponding eigenspaces with E^u, E^c, and E^s respectively, one sees that Φ_0 is partially hyperbolic. The E^c direction is given by the vector field $\frac{\partial}{\partial z}$. The center foliation is a vertical foliation in the cube where, due to the simple identification of the top face of the cube with the bottom, all leaves are circles.

For every linear automorphism, $L : T_0\mathcal{H} \to T_0\mathcal{H}$, of the tangent space of the identity, there is a group automorphism $\mathcal{H} \to \mathcal{H}$ of the form

$$\Phi(x, y, z) = (A(x, y),\ cz + p(x, y))$$

where A is a 2×2 matrix, $c \in \mathbb{R}$, and p is a quadratic polynomial, and such that the derivative $d\Phi$ at the identity is equal to L. Given L, the corresponding Φ can always be found by expanding the equations $d\Phi|_0 = L$ and $\Phi(u)\cdot\Phi(v) = \Phi(u\cdot v)$, then solving for the coefficients of p. Since every Lie group automorphism is uniquely determined by its derivative at the identity, every automorphism of the Heisenberg group must be of this form. Such an automorphism Φ is partially hyperbolic if and only if the matrix A is hyperbolic.

Any compact nilmanifold constructed from the Heisenberg group is diffeomorphic to $\mathcal{H}/\Gamma$ where the lattice Γ is of the form

$$\Gamma = \{(x, y, z) \in \mathcal{H} : x, y \in \mathbb{Z},\ z \in \tfrac{1}{k}\mathbb{Z}\}$$

for some k. In this light, Theorem 1.1 may be restated as follows.

Corollary 1.2. *Any partially hyperbolic diffeomorphism f on a compact, three-dimensional nilmanifold (other than the 3-torus) is topologically conjugate to a homeomorphism of the form*

$$N \to N, \quad (x, y, z) \mapsto (A(x, y), g(x, y, z))$$

where N is the unit cube $[0, 1]^3$ with some identification of faces, A is a hyperbolic toral automorphism (on the 2-torus), and g is a continuous function $N \to \mathbb{R}/\mathbb{Z}$.

Moreover, all of the center leaves of f are circles and the conjugacy maps the center foliation of f to the vertical foliation on $[0, 1]^3$.

Implicit in the statement of Theorem 1.1 is that any partially hyperbolic diffeomorphism on a compact, three-dimensional nilmanifold has a foliation tangent to the center subbundle, as is necessary in the definition of a leaf conjugacy. The author proved this condition while preparing this paper and later discovered that it was independently proved by K. Parwani using different techniques [10]. In the interests of keeping the paper as self contained as possible, we give a proof of this result in Section 4. It arises naturally in the proof of the main theorem, and does not significantly add to the length of the exposition.

The paper is structured as follows. Section 2 details the algebraic structure of the Heisenberg group and of functions on that space. Section 3 gives a precise definition of partial hyperbolicity and references known results in the study of three-dimensional partially hyperbolic systems that will be needed later in the proof. Section 4 studies the foliations of a partially hyperbolic systems on the Heisenberg group, culminating in proofs of two key properties referred to as Global Product Structure (Theorem 3.6) and the Central Shadowing Lemma (Lemma 3.7). Finally, Section 5 uses these two results to build a leaf conjugacy between an arbitrary partially hyperbolic system on a nilmanifold, and an algebraic nilmanifold automorphism.

2 Nilmanifolds

Due to the Bianchi classification, there are exactly two simply connected, nilpotent Lie groups of dimension three:

- the abelian Lie group, $\mathbb{R}^3$, where the group operation is addition, and
- the Heisenberg group, $\mathcal{H}$, consisting of matrices of the form
$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \qquad (x, y, z \in \mathbb{R})$$
under multiplication.

The only compact nilmanifold found as a quotient of $\mathbb{R}^3$ is the 3-torus, and the proof of Theorem 1.1 for this manifold is given in [6]. Therefore, from this point on, we take a three-dimensional nilmanifold to mean a quotient of the form $\mathcal{H}/\Gamma$ where $\mathcal{H}$ is the Heisenberg group and Γ is a discrete, cocompact subgroup, a lattice. To be specific, $\mathcal{H}/\Gamma$ is defined by the equivalence relation $\gamma \cdot p \sim p$ for $\gamma \in \Gamma$ and $p \in \mathcal{H}$.

The Lie algebra $\mathfrak{h}$ associated to the Heisenberg group is generated by elements

$$X = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$$

with $[X, Y] = Z$ and $[X, Z] = [Y, Z] = 0$. The algebra $\mathfrak{h}$ may be regarded as a three-dimensional vector space. Then, using X, Y, and Z as a basis, any Lie algebra endomorphism $\phi : \mathfrak{h} \to \mathfrak{h}$ may be written as a 3×3 matrix

$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ \hline \alpha & \beta & \det(A) \end{array} \right). \tag{1}$$

The entries of the last column are determined by the first two columns by the requirement $\phi([X, Y]) = [\phi(X), \phi(Y)]$. The Lie algebra endomorphism ϕ is invertible if and only if the associated 3×3 matrix is invertible if and only if the 2×2 submatrix A is invertible. Let G denote the group of all invertible matrices of the form given in (1).

Each Lie algebra automorphism $\phi : \mathfrak{h} \to \mathfrak{h}$ faithfully corresponds to a Lie group automorphism $\Phi : \mathcal{H} \to \mathcal{H}$. Further, if there is a lattice Γ such that $\Phi(\Gamma) = \Gamma$, then Φ defines a quotient map $\mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ on the nilmanifold. Conversely, any endomorphism on the discrete group Γ extends uniquely to an endomorphism on $\mathcal{H}$ [1].

Suppose $f_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ is a continuous function, and $f : \mathcal{H} \to \mathcal{H}$ is a lift of f_0 to the universal cover. There is a function $f_* : \Gamma \to \Gamma$ such that

$$f(\gamma \cdot p) = f_*(\gamma) \cdot f(p).$$

for all $\gamma \in \Gamma$ and $p \in \mathcal{H}$. f_* is a group endomorphism and once the lift f is chosen, f_* is unique. There is a unique Lie group endomorphism $\Phi : \mathcal{H} \to \mathcal{H}$ such that $\Phi|_\Gamma = f_*$ which we call the *algebraic part* of f.

Proposition 2.1. *Suppose the Heisenberg group $\mathcal{H}$ is equipped with a metric $d(\cdot,\cdot)$ invariant under left-multiplication. If $f_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ is continuous, $f : \mathcal{H} \to \mathcal{H}$ is a lift of f_0 to the universal cover, and $\Phi : \mathcal{H} \to \mathcal{H}$ is its algebraic part, then the distance between f and Φ is bounded: there is $C > 0$ such that*

$$d\big(f(p), \Phi(p)\big) < C$$

for all $p \in \mathcal{H}$.

Proof. First note

$$d\big(f(\gamma \cdot p), \Phi(\gamma \cdot p)\big) = d\big(f(p), \Phi(p)\big)$$

for $p \in \mathcal{H}$ and $\gamma \in \Gamma$. Then

$$\sup_{p \in \mathcal{H}} d\big(f(p), \Phi(p)\big) = \sup_{p \in K} d\big(f(p), \Phi(p)\big)$$

where K is a fundamental domain of the covering map $\mathcal{H} \to \mathcal{H}/\Gamma$. As K can be taken as compact, the supremum is finite. □

If the map $f_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ is a homeomorphism, then so is a lift $f : \mathcal{H} \to \mathcal{H}$ and the algebraic part $\Phi : \mathcal{H} \to \mathcal{H}$ is invertible. It is a Lie group automorphism and the corresponding Lie algebra automorphism is represented by a matrix $T \in G$. Call T the matrix associated to f.

Proposition 2.2. *If $f_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ is a homeomorphism, then the matrix associated to a lift $f : \mathcal{H} \to \mathcal{H}$ is of the form*

$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & 0 \\ & & 0 \\ \hline \alpha & \beta & \pm 1 \end{array}\right).$$

That is, it is of the form given in (1) *with the additional property that* $\det(A) = \pm 1$.

Proof. Let $\Phi : \mathcal{H} \to \mathcal{H}$ be the algebraic part of f. As a Lie group automorphism, Φ acts as an isomorphism on the (group-theoretic) center of $\mathcal{H}$. The center

$$Z(\mathcal{H}) = \{(0, 0, z) : z \in \mathbb{R}\}$$

is isomorphic to the real line under addition. Thus, there is a non-zero factor $a \in \mathbb{R}$ such that Φ acts on $Z(\mathcal{H})$ as multiplication by a. The set $Z(\Gamma) = Z(\mathcal{H}) \cap \Gamma$ is a discrete, non-trivial subgroup and therefore of the form

$$Z(\Gamma) = \{(0, 0, bz) : z \in \mathbb{Z}\}$$

for some non-zero constant b. Here, $\Phi|_{Z(\Gamma)}$ acts as an isomorphism. This implies that $a = \pm 1$ and a is exactly the entry $\det(A)$ in equation (1). $\square$

Suppose $\Phi : \mathcal{H} \to \mathcal{H}$ is a Lie group automorphism and $\Gamma \subset \mathcal{H}$ is a lattice. Define Γ' as the image $\Phi(\Gamma)$. Then Φ quotients down to a homeomorphism Φ_0 between the compact nilmanifolds $\mathcal{H}/\Gamma$ and $\mathcal{H}/\Gamma'$. Call such a map Φ_0 a *nilmanifold isomorphism.* Call two homeomorphisms $f_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ and $g_0 : \mathcal{H}/\Gamma' \to \mathcal{H}/\Gamma'$ *algebraically conjugate* if there is a nilmanifold isomorphism $\Phi_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma'$ such that

$$\Phi_0 f_0 = g_0 \Phi_0.$$

Lemma 2.3. *Suppose $f_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ is a homeomorphism with a lift associated to a matrix $S \in G$. If S is conjugate to $T \in G$ (that is, there is $P \in G$ such that $T = PSP^{-1}$), then f_0 is algebraically conjugate to a homeomorphism $g_0 : \mathcal{H}/\Gamma' \to \mathcal{H}/\Gamma'$ with a lift associated to the matrix T.*

Proof. Say $P \in G$ is such that $PSP^{-1} = T$. Then, P induces a Lie group automorphism $\Psi : \mathcal{H} \to \mathcal{H}$. Set $\Gamma' = \Psi(\Gamma)$. It follows that Ψ descends to a nilmanifold isomorphism $\Psi_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma'$. Define g_0 as the composition $\Psi_0 f_0 \Psi_0^{-1}$.

Let $f : \mathcal{H} \to \mathcal{H}$ be the lift of f_0 associated to S, and let $\Phi : \mathcal{H} \to \mathcal{H}$ be its algebraic part. Define $g : \mathcal{H} \to \mathcal{H}$ by $g = \Psi f \Psi^{-1}$. It is a lift of g_0, its algebraic part is $\Psi\Phi\Psi^{-1}$, and its associated matrix is $PSP^{-1} = T$. $\square$

Call a matrix $T \in G$ *partially hyperbolic* if it has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ where $|\lambda_1| < 1$, $|\lambda_2| > 1$ and $|\lambda_3| = 1$. All eigenvalues are by necessity real, and because the structure of the matrix is of the form given in (1), it must be that the submatrix A is hyperbolic and $\lambda_1 \lambda_2 = \lambda_3$.

Proposition 2.4. *Suppose $f_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ is a homeomorphism with a lift associated to a partially hyperbolic matrix $T \in G$. Then f_0 is algebraically conjugate to a homeomorphism on a nilmanifold $\mathcal{H}/\Gamma'$ with a lift associated to a diagonal matrix*

$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

with entries given by the eigenvalues of T.

Proof. In light of the previous lemma, we need only show that a partially hyperbolic matrix in G is conjugate (in G) to a diagonal one. For simplicity, assume that 1 is an eigenvalue. The other case, with -1 as an eigenvalue, is handled similarly.

If

$$T = \left(\begin{array}{c|c} A & 0 \\ \hline u & det(A) \end{array}\right) \in G$$

is partially hyperbolic with eigenvalues λ_1, λ_2, and $\lambda_3 = 1$, it must be that $\det(A) = 1$ and that A is hyperbolic. Here, u is a 1×2 block. Let v be another block of the same dimensions, and let I denote the 2×2 identity matrix. Then

$$\left(\begin{array}{c|c} I & 0 \\ \hline v & 1 \end{array}\right) \in G$$

and

$$\left(\begin{array}{c|c} I & 0 \\ \hline v & 1 \end{array}\right) \left(\begin{array}{c|c} A & 0 \\ \hline u & 1 \end{array}\right) \left(\begin{array}{c|c} I & 0 \\ \hline v & 1 \end{array}\right)^{-1} = \left(\begin{array}{c|c} A & 0 \\ \hline vA + u - v & 1 \end{array}\right).$$

As A is hyperbolic, $A - I$ is invertible and there is a unique value of v such that $vA + u - v = 0$. Therefore, T is conjugate in G to

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right).$$

A is diagonalizable; there is $P \in SL(2, \mathbb{R})$ such that $PAP^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Then

$$\left(\begin{array}{c|c} P & 0 \\ \hline 0 & 1 \end{array}\right) \in G$$

and

$$\left(\begin{array}{c|c} P & 0 \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} P & 0 \\ \hline 0 & 1 \end{array}\right)^{-1} = \left(\begin{array}{cc|c} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

as desired. ☐

The Lie group automorphism associated to the matrix

$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & 1 \end{pmatrix}$$

is particularly tractable. It is of the form

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \lambda_1 x & z \\ & 1 & \lambda_2 y \\ & & 1 \end{pmatrix}$$

with only linear terms.

3 Definitions and Dependencies

A C^1 diffeomorphism $f : M \to M$ on a Riemannian manifold M is *partially hyperbolic* if there is a splitting of TM into three continuous, non-trivial, Tf-invariant sub-bundles $TM = E^u \oplus E^c \oplus E^s$ and constants $0 < \lambda < \hat{\gamma} < 1 < \gamma < \mu$ and $C_{\mathrm{ph}} > 1$ such that for $x \in M$ and $n \in \mathbb{Z}$

$$\begin{aligned} \frac{1}{C_{\mathrm{ph}}} \mu^n \|v\| \leq & \|Tf^n v\| & & \text{for } v \in E^u(x), \\ \frac{1}{C_{\mathrm{ph}}} \hat{\gamma}^n \|v\| \leq & \|Tf^n v\| \leq C_{\mathrm{ph}} \gamma^n \|v\| & & \text{for } v \in E^c(x), \\ & \|Tf^n v\| \leq C_{\mathrm{ph}} \lambda^n \|v\| & & \text{for } v \in E^s(x). \end{aligned}$$

Remark. The above definition is sometimes called *absolute* partial hyperbolicity in comparison to "relative" or "point-wise" partial hyperbolicity where the values $\lambda < \hat{\gamma} < 1 < \gamma < \mu$ are functions $M \to \mathbb{R}$.

On a compact manifold, any partially hyperbolic system under one metric will still be partially hyperbolic under a different choice of metric with at most a change in the constant C_{ph}. For nilmanifolds in particular, we work with one specific metric which we now define.

Recall that an element

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \in \mathcal{H}$$

is denoted by the shorthand (x, y, z). The Lie algebra elements X, Y, and Z may be thought of as vector fields on $\mathcal{H}$ invariant under left-multiplication by elements of the group. These vector fields are

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial x} + x \frac{\partial}{\partial z}, \quad \text{and} \quad Z = \frac{\partial}{\partial z}.$$

(The asymmetry between X and Y is due to the one-sidedness of left-invariance.) Fix a Riemannian metric on $\mathcal{H}$ such that at each point, the vectors from X, Y, and Z form an orthonormal basis of the tangent space. Call this the *left-invariant metric* on $\mathcal{H}$ and note that it descends to every nilmanifold $\mathcal{H}/\Gamma$. It is with respect to this metric that we consider partial hyperbolicity. For points (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathcal{H}$, we may also consider the more familiar the Euclidean metric

$$\|(x_1, y_1, z_1) - (x_2, y_2, z_2)\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

There is no meaningful way to project this metric down to a compact nilmanifold $\mathcal{H}/\Gamma$, but it will be helpful when studying systems lifted to the Heisenberg group. To avoid confusion, we always use $d(p, q)$ for distance in the left-invariant metric and $\|p - q\|$ for the Euclidean. Note that while the two metrics are different, they yield the exact same volume form on $\mathcal{H}$.

In a partially hyperbolic system, the unstable E^u and stable E^s distributions are always uniquely integrable; that is, there are unique foliations W^u and W^s tangent to E^u and E^s respectively. The center direction E^c is sometimes integrable and sometimes not. The system is called *dynamically coherent* if E^c, $E^{cu} = E^c \oplus E^u$ and $E^{cs} = E^c \oplus E^s$ are uniquely integrable.

In [3], M. Brin, D. Burago, and S. Ivanov show the remarkable result that all (absolutely) partially hyperbolic systems on the 3-torus are dynamically coherent. A key step in their analysis is showing the absence of so-called "transverse contractible cycles" for three-dimensional systems.

While this paper does not consider transverse contractible cycles directly, two consequences of their study will be invaluable. For the following, suppose M is a closed Riemannian 3-manifold with universal cover $\tilde{M}$ and $f_0 : M \to M$ is a partially hyperbolic diffeomorphism with lift $f : \tilde{M} \to \tilde{M}$. This lift is also partially hyperbolic, so we may consider its stable and unstable foliations.

Proposition 3.1 (Lemma 3.1 of [3])**.** *If f_0 (and therefore f) is dynamically coherent, then on the universal cover, a leaf of W^{cs} and a leaf of W^u intersect at most once.*

In fact, the result in [3] does not *a priori* assume the system is dynamically coherent, and is stated in a slightly different form. The uniqueness of intersection immediately gives a topological property of the leaves.

Corollary 3.2. *If f is dynamically coherent, each center-stable leaf is properly embedded on the universal cover.*

Proof. If a center-stable leaf $\mathcal{L}$ accumulated on a point $p \in \mathcal{H}$, then, as the foliations are transverse, $W^u(p)$ would intersect $\mathcal{L}$ an infinite number of times. $\square$

By the same logic, the unstable leaves are properly embedded as well, but we can say something further. For a finite-length subcurve J of an unstable leaf, let $U_1(J)$ denote the set of all points $p \in \tilde{M}$ such that $\text{dist}(p, J) < 1$.

Proposition 3.3 (Lemma 3.3 of [3])**.** *There is a constant C such that, for every unstable curve J on the universal cover, one has*

$$\text{volume } U_1(J) \;\geq\; C \cdot \text{length}(J).$$

Using μ as given in the definition of partial hyperbolicity, this result can be generalized to apply to iterates of the curve.

Corollary 3.4. *There is a constant C such that, for every unstable curve J on the universal cover and every $n \geq 0$, one has*

$$\text{volume } U_1(f^n(J)) \;\geq\; C\mu^n \cdot \text{length}(J).$$

In the case of the Heisenberg group, the neighbourhood U_1 can be defined where distance is measured with the left-invariant metric, but not with the Euclidean metric, since only the former quotients down to the nilmanifold.

Additionally, we need a result proven by F. Rodriguez Hertz, M. A. Rodriguez Hertz, and R. Ures when establishing ergodicity for measure-preserving partially hyperbolic systems on three-dimensional nilmanifolds.

Proposition 3.5 (Proposition 7.2 of [7]). *If $f_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ is partially hyperbolic with a lift $f : \mathcal{H} \to \mathcal{H}$, then the matrix associated to f is partially hyperbolic.*

As stated in their paper, the proposition only holds under certain additional assumptions which are now known to be unnecessary. To remove these assumptions, we adapt the proof slightly.

Proof. By Proposition 2.2, the matrix associated to f has at least one eigenvalue of modulus one. If the matrix is not partially hyperbolic, all three eigenvalues must have modulus one.

At the core of the proof given in [7] is the following fact:

> If all eigenvalues are of modulus at most one, and if $J \subset \mathcal{H}$ is an unstable curve of f, there is a polynomial p_1 such that for all positive integers n
> $$\text{diam } f^n(J) \le p_1(n).$$

Therefore, diam $U_1(f^n(J)) \le p_2(n)$ for another polynomial, p_2. As three-dimensional nilmanifolds have polynomial growth of volume (see, for instance, Lemma 6.4 again in [7]), there is a polynomial p_3 such that

$$\text{volume } U_1(f^n(J)) \le p_3(n).$$

This contradicts the exponential growth required by Corollary 3.4. ☐

Finally, we will need a result by J. Franks to establish a semi-conjugacy between a partially hyperbolic system on a nilmanifold and a two-dimensional hyperbolic toral automorphism. This will be discussed in more detail in Section 5 when the time comes for its use.

The proof of Theorem 1.1 proceeds by gradually establishing increasingly stronger properties about the foliations on the universal cover. One such property is Global Product Structure. An Anosov system, lifted to the universal cover, has Global Product Structure if every stable leaf intersects every unstable leaf exactly once. It is not entirely clear how best to extend this notion to partially hyperbolic systems. We use the following definition.

> A partially hyperbolic system $f : \tilde{M} \to \tilde{M}$ has *Global Product Structure* if it is dynamically coherent and the following four properties hold:
>
> 1. For $p, q \in \tilde{M}$, $W^u(p)$ and $W^{cs}(q)$ intersect exactly once.
> 2. For $p, q \in \tilde{M}$, $W^s(p)$ and $W^{cu}(q)$ intersect exactly once.

3. For p and q on the same center-unstable leaf, $W^u(p)$ and $W^c(q)$ intersect exactly once.
4. For p and q on the same center-stable leaf, $W^s(p)$ and $W^c(q)$ intersect exactly once.

This Global Product Structure allows us to understand the foliations at large scales in a similar way to transverse foliations in small neighbourhoods.

Theorem 3.6. *If $f_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ is partially hyperbolic, then a lift $f : \mathcal{H} \to \mathcal{H}$ has Global Product Structure.*

Anosov systems are expansive. For any such system $f : M \to M$, there is an $\epsilon > 0$ such that

$$d(f^n(p), f^n(q)) < \epsilon$$

for all $n \in \mathbb{Z}$ if and only if the points p and q coincide.

To generalize this notion to partially hyperbolic systems, we must somehow account for the possibly unexpansive behaviour along the center direction. One good candidate is the notion of "plaque expansiveness" first developed in [8]. In this paper, we instead exploit the algebraic structure of the Heisenberg group to give a specific generalization of expansiveness.

Lemma 3.7 (Central Shadowing Lemma)**.** *Let $\mathcal{H}$ be the Heisenberg group with the projection*

$$P : \mathcal{H} \to \mathbb{R}^2, \quad (x, y, z) \mapsto (x, y).$$

Suppose $f_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ is partially hyperbolic with lift $f : \mathcal{H} \to \mathcal{H}$. Then $p, q \in \mathcal{H}$ lie on the same center leaf if and only if $\|Pf^n(p) - Pf^n(q)\|$ is bounded for all $n \in \mathbb{Z}$.

This says that if the projected orbits of two points on the universal cover shadow each other, then the two points must lie on the same center leaf.

In Section 4, we establish Global Product Structure and prove the Central Shadowing Lemma for systems on three-dimensional nilmanifolds. Then in Section 5, using these two properties, we prove Theorem 1.1, the main classification result. These two remaining sections are structured to be as independent as possible, and may be read in either order.

4 Bounding Boxes

Throughout this section, assume $f_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ is partially hyperbolic and $f : \mathcal{H} \to \mathcal{H}$ is a lift of f_0 to the universal cover. By Proposition 3.5, the matrix associated to f, has eigenvalues $\pm\lambda^{-1}$, $\pm\lambda$, and ± 1 for some $\lambda > 1$. The two properties to be proved in this section, namely Global Product Structure and the Central Shadowing Lemma, hold true for f if and only if they hold true for f^2. Therefore, without loss of generality, assume that the above eigenvalues are positive.

We further wish to compare the system to a Lie group automorphism which is linear. To realize this, we will replace f_0 by an algebraically conjugate system (as defined in Section 2). First, we must show that such a conjugation does not affect the properties we wish to prove.

Proposition 4.1. *Suppose diffeomorphisms $f_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ and $g_0 : \mathcal{H}/\Gamma' \to \mathcal{H}/\Gamma'$ are algebraically conjugate. Each of the following properties holds for f_0 if and only if it holds for g_0:*

1. *partial hyperbolicity,*
2. *dynamical coherence,*
3. *Global Product Structure,*
4. *the Central Shadowing Lemma.*

Proof. For the first three items, the proof is immediate, as the subbundles of the partially hyperbolic splitting and any foliations tangent to these subbundles are unaffected when pushed forward by a smooth conjugacy.

The last item requires some explanation. Suppose f_0 is algebraically conjugate to g_0. Then on the universal cover, there are lifts f and g satisfying $g = \Phi f \Phi^{-1}$ for some Lie group automorphism $\Phi : \mathcal{H} \to \mathcal{H}$. As previously noted, Φ is of the form

$$\Phi(x, y, z) = \big(A(x, y), cz + p(x, y)\big)$$

for a linear map A on $\mathbb{R}^2$, a constant $c \in \mathbb{R}$, and some polynomial $p : \mathbb{R}^2 \to \mathbb{R}$. In particular, the projection $P(x, y, z) = (x, y)$ used in the statement of the Central Shadowing Lemma satisfies the relation $P\Phi = AP$. Then,

$$\begin{aligned}
\|Pg^n(p) - Pg^n(q)\| \quad &\text{is bounded for all } n \in \mathbb{Z}. \quad \Leftrightarrow \\
\|P\Phi f^n \Phi^{-1}(p) - P\Phi f^n \Phi^{-1}(q)\| \quad &\text{is bounded for all } n \in \mathbb{Z}. \quad \Leftrightarrow \\
\|APf^n \Phi^{-1}(p) - APf^n \Phi^{-1}(q)\| \quad &\text{is bounded for all } n \in \mathbb{Z}. \quad \Leftrightarrow \\
\|Pf^n \Phi^{-1}(p) - Pf^n \Phi^{-1}(q)\| \quad &\text{is bounded for all } n \in \mathbb{Z}.
\end{aligned}$$

As Φ^{-1} maps center leaves of g to center leaves of f, this shows that the Central Shadowing Lemma holds for f if and only if it holds for g. ☐

We are now free to algebraically conjugate the system. By Proposition 2.4, assume that

$$\begin{pmatrix} \lambda^{-1} & & \\ & \lambda & \\ & & 1 \end{pmatrix} \tag{2}$$

is the matrix associated to a lift $f : \mathcal{H} \to \mathcal{H}$ of f_0. This matrix corresponds to the Lie group automorphism

$$L : \mathcal{H} \to \mathcal{H}, \quad (x, y, z) \mapsto (\lambda^{-1}x, \lambda y, z). \tag{3}$$

By Proposition 2.1, the distance $d(f(p), L(p))$ is uniformly bounded for all $p \in \mathcal{H}$.

This bound holds when measured with the left-invariant metric, a natural metric to consider on Heisenberg space. As humans living in more-or-less Euclidean space, however, this metric is a pain to understand. Doubters of the difficulties involved are invited to calculate a formula for $d\big((0,0,0),(0,0,z)\big)$ from the definition of d. (Hint: It's not $|z|$.) As such, we would like to switch our analysis from the left-invariant metric to the Euclidean metric as quickly as possible. The next few steps show that this is indeed possible.

Define projections $\pi^s(x,y,z) = x$ and $\pi^u(x,y,z) = y$ and note the following.

Lemma 4.2. *For $p, q \in \mathcal{H}$,*

$$|\pi^s(p) - \pi^s(q)| \leq d(p,q) \quad \textit{and}$$
$$|\pi^u(p) - \pi^u(q)| \leq d(p,q).$$

Proof. We prove the first inequality. The proof for the second equality is essentially the same. Let $\alpha : [0,1] \to \mathcal{H}$ be a C^1 path from p to q. At each point, $\alpha'(t)$ splits into components

$$\alpha'_X(t), \quad \alpha'_Y(t), \quad \text{and} \quad \alpha'_Z(t)$$

such that

$$\alpha'(t) = \alpha'_X(t)X_{\alpha(t)} + \alpha'_Y(t)Y_{\alpha(t)} + \alpha'_Z(t)Z_{\alpha(t)}.$$

By the Fundamental Theorem of Calculus,

$$q - p = \alpha(1) - \alpha(0) = \int_0^1 \alpha'(t)dt$$

where the subtraction is coordinate-wise. When projecting to the first component, this yields

$$\pi^s(q) - \pi^s(p) = \int_0^1 \alpha'_X(t)dt.$$

The length of α with respect to the left-invariant metric is

$$\text{length}(\alpha) = \int_0^1 \sqrt{(\alpha'_X(t))^2 + (\alpha'_Y(t))^2 + (\alpha'_Z(t))^2}dt.$$

Therefore,

$$\text{length}(\alpha) \geq \int_0^1 |\alpha'_X(t)|dt \geq \left|\int_0^1 \alpha'_X(t)dt\right| = \big|\pi^s(q) - \pi^s(p)\big|.$$

As $d(p,q)$ is the infimum of such lengths of paths, the lemma is proved. $\square$

Corollary 4.3. *There is $x_0 > 0$ such that for $p \in \mathcal{H}$*

$$|\pi^s(p)| \leq x_0 \quad \Rightarrow \quad |\pi^s f(p)| \leq x_0.$$

Proof. Recall that for $L : \mathcal{H} \to \mathcal{H}$ as defined in (3), the distance between $f(p)$ and $L(p)$ is uniformly bounded, say by $C > 0$. The definition also implies that $\pi^s L(p) = \lambda^{-1}\pi^s(p)$ and as $\lambda > 1$, there is $x_0 > 0$ such that $\lambda^{-1}x_0 + C < x_0$. Then,

$$\begin{aligned}
\big|\pi^s(p)\big| \le x_0 \quad &\Rightarrow \\
\big|\pi^s f(p)\big| &\le \big|\pi^s L(p)\big| + \big|\pi^s f(p) - \pi^s L(p)\big| \\
&\le \big|\pi^s L(p)\big| + d\big(f(p), L(p)\big) \\
&\le \lambda^{-1}x_0 + C < x_0.
\end{aligned}$$

□

For $0 < x_0, y_0, z_0 \le \infty$, define the set

$$B[x_0, y_0, z_0] = \{(x, y, z) \in \mathcal{H} : |x| \le x_0, |y| \le y_0, |z| \le z_0\}.$$

The last corollary may be restated as

$$f(B[x_0, \infty, \infty]) \subset B[x_0, \infty, \infty].$$

Note that if we restrict our analysis to $B[x_0, \infty, \infty]$, the "Euclidean length" of the basis vector $Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ is bounded and therefore distances in the left-invariant metric and the Euclidean metric are comparable. Fixing $x_0 > 0$, for every $a > 0$ there is $b > 0$ such that

$$d(p, q) < a \quad \Rightarrow \quad \|p - q\| < b \tag{4}$$

for all $p, q \in B[x_0, \infty, \infty]$. Restricted to this region, f and the linear map L are a bounded distance apart, not just when measured with the left-invariant metric, but with the Euclidean metric as well. This gives a means to measure how quickly subsets of Heisenberg space grow under iterates of f.

Lemma 4.4. *There is $x_0 > 0$ and $c > 0$ such that*

$$f(B[x_0, y, z]) \subset B[x_0, \lambda y + c, z + c]$$

for all $y, z > 0$. Moreover, if $\beta > \lambda$ then there is $y_0 > 0$ such that

$$f^n(B[x_0, y, z]) \subset B[x_0, \beta^n y, z + nc]$$

for all $y > y_0$, $z > 0$ and positive integers n.

Proof. As $d(L(p), f(p))$ is bounded on all of $\mathcal{H}$, it follows from (4) that there is $c > 0$ such that

$$\|f(p) - L(p)\| < c$$

for $p \in B[x_0, \infty, \infty]$, where x_0 is as in Corollary 4.3. Then, for $y, z > 0$,

$$\begin{aligned}
L(B[x_0, y, z]) = B[\lambda^{-1}x_0, \lambda y, z] \quad &\Rightarrow \\
f(B[x_0, y, z]) \subset B[\lambda^{-1}x_0 + c, \lambda y + c, z + c]&
\end{aligned}$$

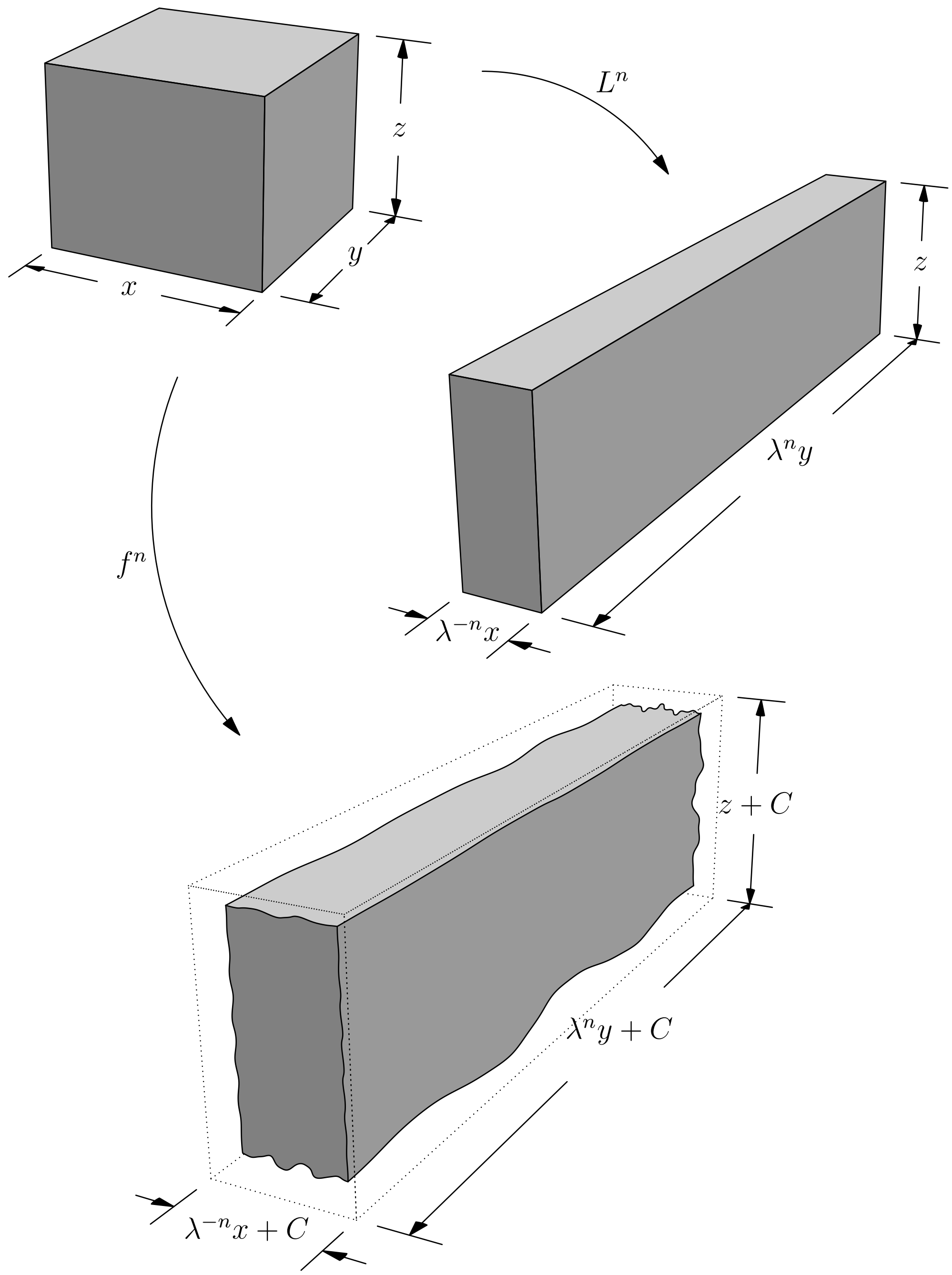


Figure 2: The images of a box under iterates of the linear map L and the diffeomorphism f.

Since we already know $f(B[x_0,\infty,\infty]) \subset B[x_0,\infty,\infty]$, this establishes the first statement of the lemma. To prove the second statement, just choose y_0 large enough that

$$\beta y_0 > \lambda y_0 + c$$

and apply induction. $\square$

We now know enough to compare λ, the expanding eigenvalue determined by the algebraic part of the diffeomorphism, with μ, the lower bound on the growth rate of the unstable direction in the definition of partial hyperbolicity.

Lemma 4.5. $\mu \leq \lambda$

Proof. Take β such that $\lambda < \beta$. Let J be a small unstable curve inside a region $B[x_0, y, z]$. By the previous lemma,

$$\begin{aligned} f^n(J) \subset B[x_0, \beta^n y, z + nc] \quad &\Rightarrow \\ U_1(f^n(J)) \subset B[x_0 + d, \beta^n y + d, z + nc + d]& \end{aligned}$$

for constants $c, d > 0$. Volume in Heisenberg space is computed exactly as in Euclidean space, and therefore

$$\text{volume } U_1(f^n(J)) < \beta^n p(n)$$

for some polynomial p. By Corollary 3.4,

$$C\mu^n < \text{volume } U_1(f^n(J))$$

for some constant $C > 0$. The only way this can hold for all $n \geq 0$ is if $\mu \leq \beta$. Since β is any constant greater than λ, this means $\mu \leq \lambda$ as well. $\square$

Proposition 4.6. *For every $\alpha < \lambda$, there is $M > 0$ such that*

$$\big|\pi^u(p) - \pi^u(q)\big| \geq M \quad \Rightarrow \quad \big|\pi^u f^n(p) - \pi^u f^n(q)\big| > \alpha^n$$

for all $p, q \in \mathcal{H}$ and positive integers n.

Proof. Let $C > 0$ be such that $d\big(f(p), L(p)\big) < C$ for all $p \in \mathcal{H}$, and choose $M > 1$ large enough that $\lambda M - 2C > \alpha M$.

Using the definition of L and Lemma 4.2,

$$\begin{aligned} \big|\pi^u(p) - \pi^u(q)\big| \geq M \quad &\Rightarrow \\ \big|\pi^u f(p) - \pi^u f(q)\big| &\geq \big|\pi^u L(p) - \pi^u L(q)\big| - \big|\pi^u f(p) - \pi^u L(p)\big| \\ &\qquad\qquad - \big|\pi^u f(q) - \pi^u L(q)\big| \\ &\geq \lambda\big|\pi^u(p) - \pi^u(q)\big| - d(f(p), L(p)) - d(f(q), L(q)) \\ &\geq \lambda\big|\pi^u(p) - \pi^u(q)\big| - 2C \\ &> \alpha\big|\pi^u(p) - \pi^u(q)\big|. \end{aligned}$$

By induction,

$$\big|\pi^u f^n(p) - \pi^u f^n(q)\big| > \alpha^n\big|\pi^u(p) - \pi^u(q)\big| > \alpha^n.$$

$\square$

Lemma 4.7. *For $M > 0$ there is $\ell > 0$ such that any unstable curve J of length greater than ℓ contains points p, q with the property*

$$|\pi^u(p) - \pi^u(q)| > M.$$

Proof. We first prove that for an unstable curve J of length exactly one, some iterate $f^n(J)$ satisfies the above property, and that n is independent of the choice of J.

Let J be an unstable curve of length exactly one. By applying a deck transformation, we assume without loss of generality that J is within some fixed distance of the origin and therefore that $U_1(J)$, the neighbourhood of radius one of J, is contained in $B[x_0, \infty, z_0]$ where x_0 is as in Lemma 4.4 and z_0 is independent of J.

Let $y_n^- = \inf_{p \in J} \pi^u(f^n(p))$ and $y_n^+ = \sup_{p \in J} \pi^u(f^n(p))$. Then, using Lemma 4.4,

$$\begin{aligned} U_1(f^n(J)) \subset \{(x, y, z) \in \mathcal{H} : & \ |x| \le x_0, \\ & \ y_n^- - 1 \le y \le y_n^+ - 1, \\ & \ |z| \le z_0 + nc\} \end{aligned}$$

and therefore the volume is bounded by

$$2x_0 \ (y_n^+ - y_n^- + 2) \ 2(z_0 + cn).$$

From Corollary 3.4,

$$\text{volume } U_1(f^n(J)) \ge C\mu^n$$

for some constant $C > 0$. Thus, $y_n^+ - y_n^- > M$ for large n, and $f^n(J)$ has points p, q such that

$$|\pi^u(p) - \pi^u(q)| > M.$$

Specifically, take n to be the smallest integer such that

$$2x_0 \ (M + 2) \ 2(z_0 + cn) \le C\mu^n.$$

This choice of n is independent of the unstable curve J, so long as it has unit length.

To complete the proof of the lemma as stated above, take $\ell > 0$ large enough that the length of $f^{-n}(J)$ is greater than one for any unstable curve J of length greater than ℓ. □

Corollary 4.8. *For $\alpha < \lambda$ and an unstable curve J, there is $C > 0$ such that*

$$\operatorname{diam} f^n(J) \ > \ C\alpha^n$$

for all $n \ge 0$.

Proof. This is a combination of Proposition 4.6 and Lemma 4.7. □

We are at a point now where we need to use that f is dynamically coherent. Fortunately, we are also at a point where we can prove it.

Theorem 4.9. *f is dynamically coherent.*

Proof. For a Riemannian manifold M, suppose $E \subset TM$ is a continuous distribution of codimension one. For $p \in M$ and $\epsilon > 0$ let $R_\epsilon(p)$ denote the set of all points reachable from p by a path tangent to E and of length less than ϵ. If E is uniquely integrable, then the set $R_\epsilon(p)$ is a small plaque of the leaf through p. If E is not uniquely integrable, there is a point p such that for any $\epsilon > 0$, the set $R_\epsilon(p) \subset M$ has non-empty interior.

In our case, suppose $E^{cs} \subset T\mathcal{H}$ is not uniquely integrable. Then, for any $\epsilon > 0$ there is a point $p \in \mathcal{H}$ and a small unstable curve J such that every point on J is reachable from p by a cs-path of length less than ϵ. Using the definition of partial hyperbolicity, for $q \in J$

$$d\big(f^n(p), f^n(q)\big) \le C_{\mathrm{ph}}\, \gamma^n\, \epsilon$$

and therefore

$$\operatorname{diam} f^n(J) \;\le\; 2\, C_{\mathrm{ph}}\, \gamma^n\, \epsilon.$$

This contradicts Corollary 4.8 if we use a value α such that

$$\gamma < \alpha < \mu \le \lambda.$$

Thus E^{cs} is uniquely integrable.

This shows that the center-stable bundle of any partially hyperbolic diffeomorphism on the nilmanifold is uniquely integrable. Since E^{cu}, the center-unstable subbundle of f, is equal to the center-stable subbundle of f^{-1} (also partially hyperbolic), it is uniquely integrable as well, as is E^c, the intersection of E^{cu} and E^{cs}. ☐

Lemma 4.10. *Center-stable leaves are bounded in the π^u direction: there is $R > 0$ such that for all $p \in \mathcal{H}$ and $q \in W^{cs}(p)$*

$$|\pi^u(p) - \pi^u(q)| < R.$$

Proof. Set α as in the previous proof and let $R = M > 0$ be as in Proposition 4.6. Then an analysis of the growth rate of $|\pi^u f^n(p) - \pi^u f^n(q)|$ similar in form to the last proof gives the result. ☐

Remark. The last two proofs crucially rely on f being absolutely partially hyperbolic in order to find a constant α between γ and μ. These are the only places we use absolute partial hyperbolicity instead of the more general pointwise definition.

By Corollary 3.2, each center-stable leaf is a properly embedded surface which divides $\mathcal{H}$ into two components. These components can be thought of as half-spaces in the following sense.

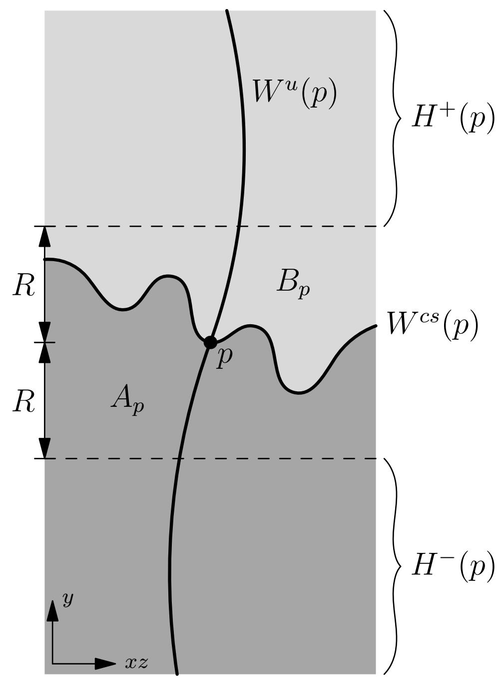


Figure 3: A depiction of the half-spaces defined in Lemma 4.11.

Lemma 4.11. *Let R be as in Lemma 4.10. For any $W^{cs}(p)$, the complement $\mathcal{H} \setminus W^{cs}(p)$ has connected components A_p and B_p satisfying the following set inclusions:*

$$H^-(p) \overset{def}{=} \{q \in \mathcal{H} : \pi^u(q) < \pi^u(p) - R\} \subset A_p$$

and

$$H^+(p) \overset{def}{=} \{q \in \mathcal{H} : \pi^u(q) > \pi^u(p) + R\} \subset B_p.$$

Proof. Let A and B denote the two connected components of $\mathcal{H} \setminus W^{cs}(p)$.

By Proposition 3.1, $W^u(p)$ intersects $W^{cs}(p)$ only at the point p and the intersection is transverse. Therefore, $W^u(p) \setminus \{p\}$ consists of two infinitely long curves, one of which lies entirely in A and the other entirely in B.

Each curve contains unstable sub-curves of arbitrarily long length, and so by Lemma 4.7, the image of each curve under π^u is unbounded. This shows that both $\pi^u(A)$ and $\pi^u(B)$ are unbounded.

By Lemma 4.10, each of $H^-(p)$ and $H^+(p)$ is contained in $\mathcal{H} \setminus W^{cs}(p)$ and so each is contained wholly in either A or B. Suppose both are contained in the same component, say A. Then

$$B \subset \mathcal{H} \setminus \big(H^-(p) \cup H^+(p)\big) = \{q \in \mathcal{H} : |\pi^u(p) - \pi^u(q)| \le R\}.$$

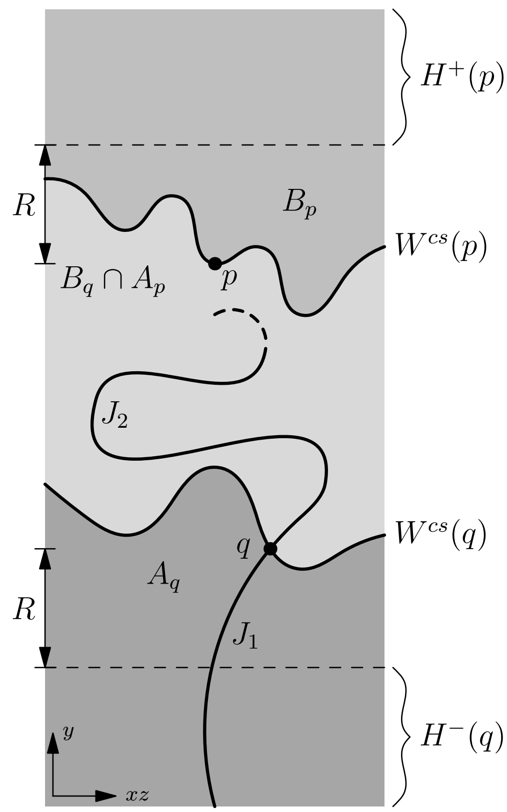


Figure 4: The unstable leaf though q must intersect the center-stable leaf through p. Otherwise, the arc J_2 would be bounded in the y-direction.

This contradicts the fact that $\pi^u(B)$ is unbounded. Therefore, either A or B contains $H^+(p)$ and the other contains $H^-(p)$. $\square$

Proposition 4.12. *For $p, q \in \mathcal{H}$, $W^{cs}(p)$ and $W^u(q)$ intersect exactly once.*

Proof. Uniqueness of the intersection has already been established in Proposition 3.1, so we need only prove existence.

Let A_p, B_p, $H^+(p)$, and $H^-(p)$ be as in the previous lemma. Adopt similar notation for the point q. Assume $W^u(q)$ does not intersect $W^{cs}(p)$. Then it lies in one of the connected components of the complement, and either

$$W^u(q) \subset A_p \subset \mathcal{H} \setminus H^+(p) \quad \text{or} \quad W^u(q) \subset B_p \subset \mathcal{H} \setminus H^-(p)$$

Assume, without loss of generality, that the first inclusion holds, as depicted in Figure 4.

As in the proof of the last lemma, $W^u(q) \setminus \{q\}$ consists of two unbounded curves lying in the two connected components of $\mathcal{H} \setminus W^{cs}(q)$. Label these curves as J_1 and J_2 where

$$J_1 \subset A_q \subset \mathcal{H} \setminus H^+(q) \quad \text{and} \quad J_2 \subset B_q \subset \mathcal{H} \setminus H^-(q).$$

Then,

$$\begin{aligned} J_2 &\subset \mathcal{H} \setminus \big(H^+(p) \cup H^-(q)\big) \\ &\subset \{s \in \mathcal{H} : \pi^u(q) - R \le \pi^u(s) \le \pi^u(p) + R\}, \end{aligned}$$

but, as J_2 is an infinitely long unstable curve, $\pi^u(J_2)$ is unbounded, a contradiction.

□

This proves the first of the four axioms required to establish Global Product Structure (Theorem 3.6). To prove the second axiom, consider Proposition 4.12 when applied to f^{-1} in place of f. To establish the last two axioms, we must show that on a *cu*-leaf each center leaf intersects each unstable leaf exactly once. To do this, repeat the preceding steps of this section, but restricted to a single center-unstable leaf. Specifically, if $\mathcal{L} \subset \mathcal{H}$ is a center-unstable leaf, show:

1. If $p \in \mathcal{L}$ then $W^c(p)$ is properly embedded in $\mathcal{L}$, akin to Corollary 3.2.
2. If $p \in \mathcal{L}$ then $\mathcal{L} \setminus W^c(p)$ has connected components A and B such that

$$\begin{aligned} &\{q \in \mathcal{L} : \pi^u(q) < \pi^u(p) - R\} \subset A \quad \text{and} \\ &\{q \in \mathcal{L} : \pi^u(q) > \pi^u(p) + R\} \subset B \end{aligned}$$

   akin to Lemma 4.11.
3. If $p, q \in \mathcal{L}$ then $W^c(p)$ and $W^u(q)$ intersect exactly once, akin to Lemma 4.12.

With Global Product Structure established, we proceed to prove the Central Shadowing Lemma (Lemma 3.7).

Lemma 4.13. *For $M > 0$ there is $\ell > 0$ such that for any unstable curve J of length greater than ℓ, the endpoints p and q satisfy*

$$|\pi^u(p) - \pi^u(q)| > M.$$

Remark. This claim is stronger than Lemma 4.7 because it concerns the endpoints specifically, instead of two points somewhere along the curve.

Proof. Let $\alpha : [0, 1] \to \mathcal{H}$ be a parametrization of a sufficiently long unstable curve. Then, by Lemma 4.7, there are $s, t \in [0, 1]$ such that

$$\pi^u\alpha(t) - \pi^u\alpha(s) > M + 2R$$

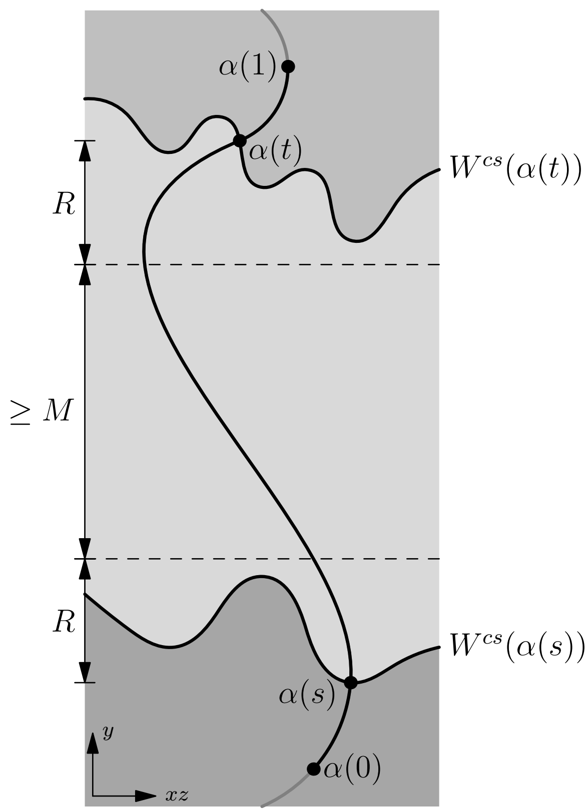


Figure 5: The curve α considered in the proof of Lemma 4.13.

where M is given in the hypothesis of this lemma, and R is as in Lemma 4.10. Without loss of generality, $0 < s < t < 1$.

The sub-curve $\alpha((t,1])$ must lie in one connected component of $\mathcal{H} \setminus W^{cs}(\alpha(t))$ while $\alpha(s)$ lies in the other. As such, $\alpha(s) \in H^-(\alpha(t))$ where H^- is as defined in Lemma 4.11 and therefore $\alpha((t,1]) \subset \mathcal{H} \setminus H^-(\alpha(t))$. Similarly, $\alpha([0,s)) \subset \mathcal{H} \setminus H^+(\alpha(s))$. Then,

$$\pi^u \alpha(1) - \pi^u \alpha(0) > M$$

due to the definitions of H^+ and H^- and the triangle inequality.

□

Remark. With a little more work, one could find constants $a, b > 0$ such that

$$|\pi^u(p) - \pi^u(q)| > a \cdot \ell + b$$

for any unstable curve of length ℓ and endpoints p and q. Then, by Lemma 4.2,

$$d(p,q) > a \cdot \ell + b$$

which is precisely what it means for the unstable foliation to be *quasi-isometric*. Due to Brin, quasi-isometry of the stable and unstable foliations is a sufficient condition for a system to be dynamically coherent [2]. Brin, Burago, and Ivanov used this property to show that all partially hyperbolic systems on the 3-torus were dynamically coherent [3], and Parwani extended this to all 3-nilmanifolds [10].

In contrast, the proof of dynamical coherence given in this paper does not use quasi-isometry, and may be adaptable to systems where quasi-isometry is not known to hold.

Lemma 4.13 is sufficient for the final goal of this section, proving the Central Shadowing Lemma, and so we do not further pursue the idea of quasi-isometry.

We first prove a baby version of the Central Shadowing Lemma for center-stable leaves.

Lemma 4.14. *For $p, q \in \mathcal{H}$, the following are equivalent:*

- $p \in W^{cs}(q)$.
- $|\pi^u f^n(p) - \pi^u f^n(q)|$ *is bounded for all* $n \in \mathbb{Z}$.

Proof. By the Global Product Structure, there is a unique point $r \in \mathcal{H}$ such that $r \in W^{cs}(p)$ and $r \in W^u(q)$. By Lemma 4.10, $|\pi^u f^n(p) - \pi^u f^n(r)|$ is bounded for all $n \in \mathbb{Z}$. Then, (using Lemma 4.13 for the second "$\Leftrightarrow$")

$$\begin{aligned} p \in W^{cs}(q) \quad & \Leftrightarrow \\ r = q \quad & \Leftrightarrow \\ |\pi^u f^n(r) - \pi^u f^n(q)| \quad & \text{is bounded for all } n \in \mathbb{Z} \quad \Leftrightarrow \\ |\pi^u f^n(p) - \pi^u f^n(q)| \quad & \text{is bounded for all } n \in \mathbb{Z}. \end{aligned}$$

□

By considering f^{-1}, we prove a similar statement for the center-unstable leaves, but because our coordinate system is adapted to f and not to f^{-1}, the proof requires some care.

Lemma 4.15. *For $p, q \in \mathcal{H}$, the following are equivalent:*

- $p \in W^{cu}(q)$.
- $|\pi^s f^n(p) - \pi^s f^n(q)|$ *is bounded for all* $n \in \mathbb{Z}$.

Proof. The standing assumption of this section is that $f : \mathcal{H} \to \mathcal{H}$ is a partially hyperbolic diffeomorphism with the associated matrix given in (2). Consider the Lie group automorphism

$$\Phi : \mathcal{H} \to \mathcal{H}, \quad (x, y, z) \mapsto (-y, x, z - xy)$$

with associated matrix

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Further consider $\Phi f^{-1}\Phi^{-1}$, the conjugation of the inverse of f by Φ. This also has the associated matrix (2) and so Lemma 4.14 applies just as well to $\Phi f^{-1}\Phi^{-1}$ as it does to f.

If $\mathcal{L}$ is a *cu*-leaf of f, then $\Phi(\mathcal{L})$ is a *cs*-leaf of $\Phi f^{-1}\Phi^{-1}$ and, therefore, the following are equivalent:

- p and q lie on the same *cu*-leaf of f.
- $\Phi(p)$ and $\Phi(q)$ lie on the same *cs*-leaf of $\Phi f^{-1}\Phi^{-1}$.
- $|\pi^u \Phi f^{-n}(p) - \pi^u \Phi f^{-n}(q)|$ is bounded for all $n \in \mathbb{Z}$.

The definition of Φ combined with $\pi^s(x,y,z) = x$ and $\pi^u(x,y,z) = y$ shows that $\pi^u\Phi = \pi^s$, implying that the last item above is equivalent to

- $|\pi^s f^{-n}(p) - \pi^s f^{-n}(q)|$ is bounded for all $n \in \mathbb{Z}$

and the lemma is proved. □

Let $P : \mathcal{H} \to \mathbb{R}^2$ be the projection $(x,y,z) \mapsto (x,y)$. Due to the Global Product Structure, two points lie on the same center leaf if and only if they lie on the same *cu*-leaf and the same *cs*-leaf. As such, the last two lemmas combine to show the following are equivalent:

- $p \in W^c(q)$.
- $\|Pf^n(p) - Pf^n(q)\|$ is bounded for all $n \in \mathbb{Z}$.

This is precisely the Central Shadowing Lemma.

5 The Leaf Conjugacy

In the previous section, we viewed the partially hyperbolic system in such a way that it could be closely compared to a Lie group automorphism of the form $(x,y,z) \mapsto (\lambda^{-1}x, \lambda y, z)$. This was achieved by a change of coordinates that put the lattice Γ defining the nilmanifold $\mathcal{H}/\Gamma$ into an unknown state. Since the proofs of the previous section did not involve the lattice, the change was benign. In this section, however, to construct a leaf conjugacy on $\mathcal{H}/\Gamma$, it is important that Γ be as simple as possible.

Any lattice $\Gamma \subset \mathcal{H}$ defining a nilmanifold $\mathcal{H}/\Gamma$ is of the form $\Gamma = \langle a,b,c\rangle$ where $[a,b] = aba^{-1}b^{-1} = c^k$ for some positive integer k. In fact, k is the index of $[\Gamma,\Gamma]$ as a subgroup of $[\mathcal{H},\mathcal{H}] \cap \Gamma$ and depends only on Γ and not the choice of generators. If $\Gamma' = \langle a',b',c'\rangle$ has the same index k, there is a unique Lie group automorphism $\Phi : \mathcal{H} \to \mathcal{H}$ mapping a, b, c to a', b', c' respectively. This is most easily seen by pulling the lattices back by the surjective exponential map $\exp : \mathfrak{h} \mapsto \mathcal{H}$ and first defining the automorphism on the Lie algebra.

Once defined, the Lie group automorphism descends to a nilmanifold isomorphism $\mathcal{H}/\Gamma \to \mathcal{H}/\Gamma'$ which is a smooth diffeomorphism. Therefore, assume without loss of generality that the partially hyperbolic system under study is defined on a nilmanifold $\mathcal{H}/\Gamma$ where the lattice is given by

$$\Gamma = \left\langle\ (1,0,0),\ (0,1,0),\ (0,0,\tfrac{1}{k})\ \right\rangle$$

for some positive integer k.

Recall that there is a group isomorphism $f_* : \Gamma \to \Gamma$ such that

$$f(\gamma \cdot p) = f_*(\gamma) \cdot f(p)$$

for $\gamma \in \Gamma$ and $p \in \mathcal{H}$. This restricts to an isomorphism on the (group-theoretic) center $Z(\Gamma) = \{(0,0,\frac{i}{k}) : i \in \mathbb{Z}\}$, which implies that

$$f(\gamma \cdot p) = \pm\gamma \cdot f(p)$$

for all $\gamma \in Z(\Gamma)$.

Proposition 5.1. *For all $p \in \mathcal{H}$, the point $(0,0,\frac{1}{k}) \cdot p$ lies on the same center leaf as p.*

Proof. Let $\gamma = (0,0,\frac{1}{k})$. For $p \in \mathcal{H}$, $f(\gamma \cdot p) = \pm\gamma \cdot f(p)$ and by induction, $f^n(\gamma \cdot p) = \pm\gamma \cdot f^n(p)$ for all $n \in \mathbb{Z}$. The distance $d(\pm\gamma \cdot f^n(p), f^n(p))$ is bounded independently of n, and therefore, by the Central Shadowing Lemma, p and $\gamma \cdot p$ lie on the same center leaf. $\square$

Multiplication by $(0,0,\frac{1}{k})$ is given by

$$(0,0,\tfrac{1}{k}) \cdot (x,y,z) = (x,y,z+\tfrac{1}{k}).$$

This shows that the center foliation of f is more-or-less vertical, and when we quotient down to the nilmanifold $\mathcal{H}/\Gamma$, all of the center leaves are circles.

Consider the projection $P : \mathcal{H} \to \mathbb{R}^2$, $(x,y,z) \mapsto (x,y)$. As $P(\Gamma) = \mathbb{Z}^2$, the projection quotients down to a map from the nilmanifold $\mathcal{H}/\Gamma$ to the torus $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$. This map, by abuse of notation, is also called P.

The group isomorphism f_* on Γ quotients down to an isomorphism of $\Gamma/Z(\Gamma) \cong \mathbb{Z}^2$. This new isomorphism $A : \mathbb{Z}^2 \to \mathbb{Z}^2$ can be regarded as a 2×2 matrix. Further, the diagram

$$\begin{tikzcd} \Gamma \arrow[r, "f_*"] \arrow[d, "P"] & \Gamma \arrow[d, "P"] \\ \mathbb{Z}^2 \arrow[r, "A"] & \mathbb{Z}^2 \end{tikzcd}$$

commutes. In fact, the matrix A is equal to the 2×2 matrix, also called A, studied in Section 2. By Proposition 3.5, we know this matrix is hyperbolic. It gives a hyperbolic toral automorphism $A : \mathbb{T}^2 \to \mathbb{T}^2$ and the above diagram can be re-written as

$$\begin{tikzcd} \pi_1(\mathcal{H}/\Gamma) \arrow[r, "f_*"] \arrow[d, "P_*"] & \pi_1(\mathcal{H}/\Gamma) \arrow[d, "P_*"] \\ \pi_1(\mathbb{T}^2) \arrow[r, "A"] & \pi_1(\mathbb{T}^2) \end{tikzcd}$$

By the results of J. Franks [5], there is a semi-conjugacy $H : \mathcal{H}/\Gamma \to \mathbb{T}^2$ such that H and P induce the same action on the fundamental group, and such that the diagram

$$\begin{array}{ccc} \mathcal{H}/\Gamma & \xrightarrow{f_0} & \mathcal{H}/\Gamma \\ \Big\downarrow{\scriptstyle H} & & \Big\downarrow{\scriptstyle H} \\ \mathbb{T}^2 & \xrightarrow{A} & \mathbb{T}^2 \end{array}$$

commutes. As stated in [5], f_0 must have a fixed point x_0 in order to find a unique semi-conjugacy such that $H(x_0) = 0$. If, as is the case here, uniqueness is not required, we may adapt the proof of Franks to find a semi-conjugacy H, even though f_0 does not necessarily have a fixed point.

Lift H to a map $\mathcal{H} \to \mathbb{R}^2$ which, by more abuse of notation, will also be called H. After conjugating f with some multiplication, assume without loss of generality that $H(0,0,0) = (0,0)$. Then $H|_\Gamma = P|_\Gamma$ by the definition of H.

Let $g : \mathcal{H} \to \mathcal{H}$ be the algebraic part of f_0 as defined in Section 2. g quotients down to $g_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ and by Proposition 3.5, both g and g_0 are partially hyperbolic diffeomorphisms. Moreover, the diagram

$$\begin{array}{ccc} \mathcal{H}/\Gamma & \xrightarrow{g_0} & \mathcal{H}/\Gamma \\ \Big\downarrow{\scriptstyle P} & & \Big\downarrow{\scriptstyle P} \\ \mathbb{T}^2 & \xrightarrow{A} & \mathbb{T}^2 \end{array}$$

commutes. The center bundle of g is given by the vector field $Z = \frac{\partial}{\partial z}$ and the center leaves are vertical lines. These quotient down to compact center leaves on the nilmanifold.

Our goal is to produce a leaf conjugacy between f_0 and g_0, a homeomorphism $h_0 : \mathcal{H}/\Gamma \to \mathcal{H}/\Gamma$ which takes center leaves of g_0 to center leaves of f_0 and such that

$$h_0 g_0(\mathcal{L}) = f_0 h_0(\mathcal{L})$$

for every center leaf $\mathcal{L}$ of g_0. We achieve this by constructing a leaf conjugacy $h : \mathcal{H} \to \mathcal{H}$ between f and g on the universal cover such that

$$h(\gamma \cdot p) = \gamma \cdot h(p)$$

for all $\gamma \in \Gamma$ and $p \in \mathcal{H}$. h_0 is then the quotient of h.

The first step is to better understand the semi-conjugacy H.

Proposition 5.2. *The fibers of H are the center leaves of f. That is, for $p, q \in \mathcal{H}$, $H(p) = H(q)$ if and only if $p \in W^c(q)$.*

Proof. Note that as P and H have the same action on the fundamental group, the distance between $P(q)$ and $H(q)$ is uniformly bounded for all points $q \in \mathcal{H}$.

Then,

$$\begin{aligned}
p \in W^c(q) &\quad \Leftrightarrow \\
\|Pf^n(p) - Pf^n(q)\| &\quad \text{is bounded for all } n \in \mathbb{Z} \quad \Leftrightarrow \\
\|Hf^n(p) - Hf^n(q)\| &\quad \text{is bounded for all } n \in \mathbb{Z} \quad \Leftrightarrow \\
\|A^n H(p) - A^n H(q)\| &\quad \text{is bounded for all } n \in \mathbb{Z}
\end{aligned}$$

where the first equivalence is due to the Central Shadowing Lemma and the last equivalence is due to the definition of H. As A is a hyperbolic linear map, the last condition can be satisfied if and only if $H(p) = H(q)$. $\square$

Proposition 5.3. *There is a continuous function $\sigma : \mathbb{R}^2 \to \mathcal{H}$ such that $H \circ \sigma$ is the identity on $\mathbb{R}^2$.*

Proof. Fix any point $p_0 \in \mathcal{H}$. The Global Product Structure of the invariant foliations of f shows that for any $q \in \mathcal{H}$, there are unique points $x, y \in \mathcal{H}$ such that

$$p_0 \overset{u}{\rightsquigarrow} x \overset{s}{\rightsquigarrow} y \overset{c}{\rightsquigarrow} q.$$

That is, $x \in W^u(p_0) \cap W^{cs}(q)$ and $y \in W^s(x) \cap W^c(q)$. Define a map $\omega : \mathcal{H} \to \mathcal{H}$ by $\omega(q) = y$. The image $\omega(\mathcal{H})$ is a topological surface through the point p_0 which intersects every center leaf of f exactly once. In fact, it is a *us-pseudoleaf* as described in [6]. ω is continuous as the stable, center, and unstable foliations of f are continuous and transverse. Further, due to the uniqueness of the intersections x and y above, points $q, q' \in \mathcal{H}$ lie on the same center leaf of f if and only if $\omega(q) = \omega(q')$.

As a semi-conjugacy, H is surjective. For $v \in \mathbb{R}^2$ take any point $q \in H^{-1}(v)$ and define $\sigma : \mathbb{R}^2 \to \mathcal{H}$ by $\sigma(v) = \omega(q)$. This is well-defined as by the previous proposition the fibers of H are center leaves, and therefore

$$q, q' \in H^{-1}(v) \quad \Rightarrow \quad \omega(q) = \omega(q').$$

Moreover, $\omega(q)$ and q lie on the same center leaf by the definition of ω, and so $H\sigma(v) = H\omega(q) = H(q) = v$. That is, σ is a one-sided inverse of H as desired. All that remains is to show that σ is continuous.

For $v \in \mathbb{R}^2$, fix a point a $q \in H^{-1}(v)$ and find a small compact topological disk D through q transversal to the center foliation such that v lies in the interior of $H(D)$. $H|_D$ is continuous and injective and is therefore a homeomorphism onto its image. Then $\sigma|_{H(D)}$ is continuous as the composition of ω and the inverse of $H|_D$. $\square$

With the function σ in hand, it is easy to construct leaf conjugacies on the universal cover $\mathcal{H}$.

Lemma 5.4. *Let φ_t be a flow generated by a non-zero vector field tangent to E^c. The homeomorphism*

$$\mathcal{H} \to \mathcal{H}, \quad (x, y, z) \mapsto \varphi_z(\sigma(x, y)) \tag{5}$$

defines a leaf conjugacy between f and g. More generally, for any continuous function, $\rho : \mathbb{R}^2 \to \mathbb{R}$, the homeomorphism

$$\mathcal{H} \to \mathcal{H}, \quad (x, y, z) \mapsto \varphi_{\rho(x,y)+z}(\sigma(x, y)) \tag{6}$$

defines a leaf conjugacy.

Proof. Let h be the map defined in (5). The orbits of φ_t are exactly the center leaves of f.

Let $\mathcal{L}$ be a center leaf of g. It is of the form $(x, y) \times \mathbb{R} \subset \mathcal{H}$ for real numbers x and y. Then, $g(\mathcal{L}) = A(x, y) \times \mathbb{R}$ and $hg(\mathcal{L})$ is the center leaf of f through the point $\sigma A(x, y)$.

On the other hand, $fh(\mathcal{L})$ is the center leaf of f through the point $f\sigma(x, y)$ and, by the definition of H and of σ,

$$Hf\sigma(x, y) = AH\sigma(x, y) = A(x, y) = H\sigma A(x, y).$$

Both $fh(\mathcal{L})$ and $hg(\mathcal{L})$ are the center leaf given by the pre-image $H^{-1}A(x, y)$, establishing the leaf conjugacy.

The addition of a function ρ only serves to slide the homeomorphism along center leaves. It does not affect the images of complete center leaves and therefore the more general definition given by (6) is also a leaf conjugacy. $\square$

This gives a large family of leaf conjugacies on the universal covering space $\mathcal{H}$. To find a leaf conjugacy which will quotient down to the nilmanifold $\mathcal{H}/\Gamma$ requires a careful choice of both the flow φ_t and the offset function ρ.

Define φ_t such that the flow is of constant speed along each individual leaf and that the time-one map takes a point $p \in \mathcal{H}$ to $(0, 0, 1) \cdot p$ which, by Proposition 5.1, lies on the same center leaf. More precisely, define a length function ℓ by

$$\ell : \mathcal{H} \to \mathbb{R}, \quad \ell(p) = d_c\big(p, (0, 0, 1) \cdot p\big)$$

where d_c denotes distance as measured along a center leaf. This function is continuous due to the continuity of the foliation. Let V_1 be a continuous unit vector field tangent to E^c and consider the re-scaled vector field

$$V(p) = \ell(p)V_1(p).$$

If φ_t is the flow induced by V, then

$$\begin{aligned} d_c\big(p, \varphi_t(p)\big) &= \ell(p) \cdot |t| \quad \Rightarrow \\ d_c\big(p, \varphi_1(p)\big) &= \ell(p) \quad \Rightarrow \\ \varphi_1(p) &= (0, 0, \pm 1) \cdot p \end{aligned}$$

By reversing the flow if necessary, we can ensure that $\varphi_1(p) = (0, 0, 1) \cdot p$ as desired.

With σ and φ_t now fixed, define, for any continuous $\rho : \mathbb{R}^2 \to \mathbb{R}$, the leaf conjugacy

$$h_\rho : \mathcal{H} \to \mathcal{H}, \quad (x, y, z) \mapsto \varphi_{\rho(x,y)+z}(\sigma(x, y)).$$

The definition of φ_t immediately gives the following.

Lemma 5.5. *For any $\rho : \mathbb{R}^2 \to \mathbb{R}$ and all $q \in \mathcal{H}$,*

$$h_\rho\big((0,0,1) \cdot q\big) = (0,0,1) \cdot h_\rho(q).$$

To complete the proof of Theorem 1.1, we need only find ρ such that the above relation holds for all $\gamma \in \Gamma$, not just for $\gamma = (0,0,1)$. This special ρ will be constructed piece-by-piece.

First, the assumption that $H(0,0,0) = (0,0)$ implies that there is a unique value $r \in \mathbb{R}$ such that $\varphi_r(\sigma(0,0)) = (0,0,0)$. Set $\rho(0,0) = r$ so that $h_\rho(0,0,0) = (0,0,0)$. The semi-conjugacy H was defined to satisfy the relation

$$H\big((1,0,0) \cdot p\big) = H(p) + (1,0).$$

In particular, $H(1,0,0) = (1,0)$ and there is a unique value $\rho(1,0)$ such that $h_\rho(1,0,0) = (1,0,0)$. Fix $\rho(1,0)$ as such and then extend ρ to all of the line segment $[0,1] \times \{0\}$ in some continuous way. The simplest extension would be

$$\rho(t,0) = (1-t)\rho(0,0) + t\rho(1,0).$$

With ρ now defined on $[0,1] \times \{0\} \subset \mathbb{R}^2$, h_ρ is defined on $[0,1] \times \{0\} \times \mathbb{R} \subset \mathcal{H}$ and it is a homeomorphism of that set onto $H^{-1}([0,1] \times \{0\})$.

Set $\gamma = (0,1,0) \in \Gamma$ and define a homeomorphism $h : [0,1] \times \{1\} \times \mathbb{R} \to H^{-1}([0,1] \times \{1\})$ by the relation

$$h(p) = \gamma \cdot h_\rho(\gamma^{-1} \cdot p).$$

Note that the flow φ_t commutes with multiplication by γ and therefore that

$$h(x,1,t) = \varphi_t\big(h(x,1,0)\big)$$

for all $x \in [0,1]$ and $t \in \mathbb{R}$. Both $h(x,1,0)$ and $\sigma(x,1)$ lie on the center leaf $H^{-1}(x,1)$. Define $\rho(x,1)$ as the unique value satisfying

$$\varphi_{\rho(x,1)}\big(\sigma(x,1)\big) = h(x,1,0).$$

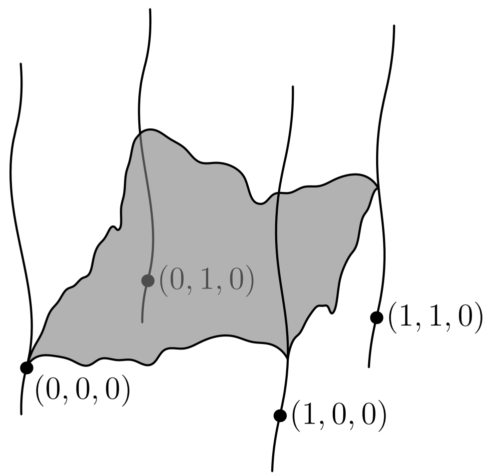
Then, h_ρ is equal to h on the domain of the latter, and as h is continuous, ρ is continuous. Thus, we have extended ρ to the set $[0,1] \times \{0,1\} \subset \mathbb{R}^2$ and

$$h_\rho(\gamma \cdot p) = \gamma \cdot h_\rho(p)$$

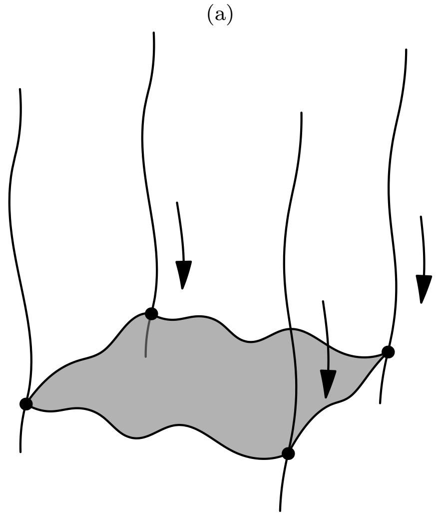
when $p \in [0,1] \times \{0\}$ and $\gamma = (0,1,0)$. In particular, $h_\rho(0,1,0) = (0,1,0)$ and $h_\rho(1,1,0) = (1,1,0)$.

Continuously extend ρ so that its domain includes the line segment $\{0\} \times [0,1]$ (again by linear interpolation, if you like). Then, using the same reasoning as above, there is a unique extension of ρ to $\{1\} \times [0,1]$ such that

$$h_\rho\big((1,0,0) \cdot p\big) = (1,0,0) \cdot h(p)$$

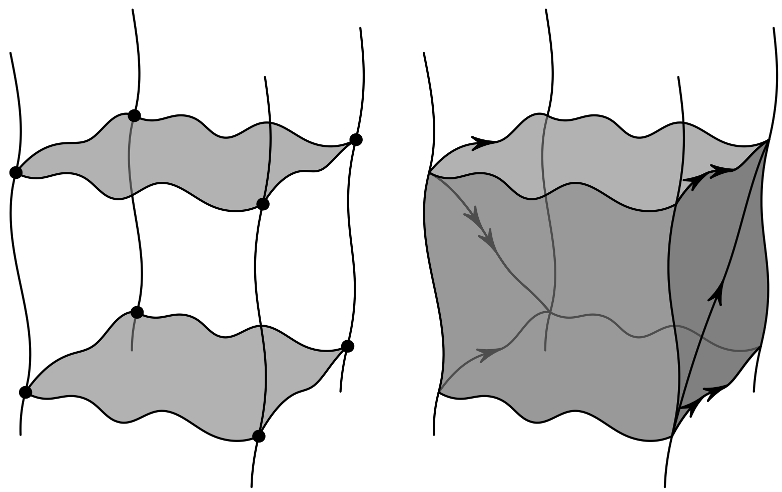


(a)



(b)

Figure 6: The section $\sigma : \mathbb{R}^2 \to \mathcal{H}$ maps the square $S = [0,1] \times [0,1] \subset \mathbb{R}^2$ to a surface in $\mathcal{H}$, and the four corners of S to the center leaves of the four points labelled in (a). The effect of ρ is to slide the image along center leaves until it meets these four lattice points, as in (b).



(c) (d)

Figure 6: *(continued)* Applying the deck transformation $p \mapsto (0,0,1) \cdot p$ moves the surface to one which intersects the same center leaves, (c), and the region found by taking all of the segments of center leaves between the two surfaces, (d), gives a fundamental domain for the nilmanifold.

for $p \in \{0\} \times [0,1] \times \mathbb{R} \subset \mathcal{H}$. One must check that this extension of ρ agrees with the previous definition of $\rho(1,0)$ and $\rho(1,1)$, but either definition implies that

$$h_\rho(1,0,z) = (1,0,z)$$

and

$$h_\rho(1,1,z) = (1,1,z)$$

for all integers z. This uniquely determines the value of ρ at these two points, so the map ρ is well-defined and continuous on the boundary ∂S of the square $S = [0,1] \times [0,1]$. Extend ρ continuously to the interior of S. Then, ρ is defined on S and h_ρ is defined on $[0,1] \times [0,1] \times \mathbb{R} \subset \mathcal{H}$. We now extend this definition to all of $\mathcal{H}$.

Any point $p \in \mathcal{H}$ can be written as

$$(a,b,0) \cdot (x,y,z)$$

where $(a,b,0) \in \Gamma$ and $(x,y) \in S$. Using this, define $h : \mathcal{H} \to \mathcal{H}$ as

$$h\big((a,b,0) \cdot (x,y,z)\big) = (a,b,0) \cdot h_\rho(x,y,z).$$

The decomposition is unique except for points of the form

$$(a+1,b,0) \cdot (0,y,z) = (a,b,0) \cdot (1,0,0) \cdot (0,y,z) = (a,b,0) \cdot (1,y,z+y)$$

or

$$(a, b+1, 0) \cdot (x, 0, z+a) = (a, b, 0) \cdot (0, 1, 0) \cdot (x, 0, z) = (a, b, 0) \cdot (x, 0, z+a).$$

The careful definition of ρ on ∂S ensures that in either of the above cases, the two choices of decomposition give the same value for $h(p)$ and therefore h is well-defined and continuous on all of the Heisenberg group.

As multiplication by $(a, b, 0) \in \Gamma$ commutes with the flow φ_t, one has that $\varphi_t \circ h(x, y, z) = h(z, y, z+t)$ so that h is equal to h_ρ for some ρ now defined on all of $\mathbb{R}^2$. Thus, h is a leaf-conjugacy and

$$h(\gamma \cdot p) = \gamma \cdot h(p) \tag{7}$$

where $\gamma = (0, 0, 1)$. Further, for points $(a, b, c) \in \Gamma$ and $(x, y, z) \in S \times \mathbb{R}$,

$$\begin{aligned} h\big((a, b, c) \cdot (x, y, z)\big) &= h\big((a, b, 0) \cdot (0, 0, c) \cdot (x, y, z)\big) \\ &= (a, b, 0) \cdot h_\rho\big((0, 0, c) \cdot (x, y, z)\big) \\ &= (a, b, 0) \cdot (0, 0, c) \cdot h_\rho(x, y, z) \\ &= (a, b, c) \cdot h(x, y, z). \end{aligned}$$

For any lattice point $\gamma \in \Gamma$ and any point $p \in \mathcal{H}$, let $p = \eta \cdot q$ be such that $\eta \in \Gamma$ and $q \in S \times \mathbb{R}$. Then,

$$h(\gamma \cdot p) = h(\gamma \cdot \eta \cdot q) = \gamma \cdot \eta \cdot h(q) = \gamma \cdot h(\eta \cdot q) = \gamma \cdot h(p).$$

This shows that the leaf conjugacy h on the universal cover $\mathcal{H}$ descends to the original nilmanifold $\mathcal{H}/\Gamma$. It is a conjugacy between the partially hyperbolic system f_0 and the nilmanifold automorphism g_0 and the proof of Theorem 1.1 is complete.

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