# CENTER BUNCHING WITHOUT DYNAMICAL COHERENCE

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ABSTRACT. We answer a question of Burns and Wilkinson, showing that there are open families of volume-preserving partially hyperbolic diffeomorphisms which are accessible and center bunched and neither dynamically coherent nor Anosov. We also show in the volume-preserving setting that any diffeomorphism which is partially hyperbolic and Anosov may be isotoped to a diffeomorphism which is partially hyperbolic and not Anosov.

Many partially hyperbolic dynamical systems are ergodic. One of the most general results in this direction is the following theorem of K. Burns and A. Wilkinson [5].

> Any $C^2$ volume-preserving, accessible, center-bunched, partially hyperbolic system is ergodic.

We define these terms briefly and refer the reader to [5, 4] for further details. A $C^1$ diffeomorphism $f$ on a compact Riemannian manifold $M$ is *partially hyperbolic* if there is an integer $k \ge 1$, a non-trivial splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u$$

invariant under the derivative $Df$, and continuous positive functions $\nu, \hat\nu, \gamma, \hat\gamma$ such that $\nu, \hat\nu < 1$ and

$$\|Tf^k v^s\| < \nu(x) < \gamma(x) < \|Tf^k v^c\| < \hat\gamma^{-1}(x) < \hat\nu^{-1}(x) < \|Tf^k v^u\|$$

for all $x \in M$ and unit vectors $v^s \in E^s(x)$, $v^c \in E^c(x)$ and $v^u \in E^u(x)$. We say that $f$ is *center bunched* if the functions can be chosen so that $\max\{\nu, \hat\nu\} < \gamma\hat\gamma$. Further, $f$ is *accessible* if for any two points $x, y \in M$ there is a path from $x$ to $y$ which is a concatenation of $C^1$ subpaths, each tangent either to $E^s$ or $E^u$.

The above theorem is a generalization of an earlier result appearing in a unpublished preprint [3]. In that preprint, the partially hyperbolic system has an additional assumption of *dynamical coherence* meaning that there are invariant foliations $W^{cs}$ and $W^{cu}$ tangent to the subbundles $E^s \oplus E^c$ and $E^u \oplus E^c$. To avoid making this assumption, a considerable portion of the proof in [5] explains the definition and construction of "fake foliations" which fill the roles of $W^{cs}$ and $W^{cu}$ in cases where true foliations do not exist. This makes the exposition in [5] much longer and more complicated than the proof in [3].

At the time, however, Burns and Wilkinson did not know if such extra effort was necessary. There were no known non-dynamically coherent examples which could not be proven ergodic by simpler means. This current paper gives such an example showing that the fake foliations used in [5] are necessary to show ergodicity.

**Theorem 1.** *For $r \geq 1$, there is an open family $\mathscr{U}$ in the $C^1$ topology of $C^r$ volume-preserving diffeomorphisms such that each diffeomorphism in $\mathscr{U}$ is partially hyperbolic, accessible, and center bunched and is neither dynamically coherent nor Anosov.*

To show this, we first define a diffeomorphism $f$ which is Anosov, partially hyperbolic, center bunched, and not dynamical coherent. We then deform $f$ to produce a diffeomorphism $g$ which is not Anosov, but satisfies the other three properties. By [6], there is a open set of diffeomorphisms $\mathscr{U}$ which are $C^1$ close to $g$ and which also have these properties and are accessible. This will therefore prove the result.

Define a hyperbolic $3 \times 3$ matrix $A$ with integer entries such that the eigenvalues $\lambda_i$ satisfy

$$0 < \lambda_2^2 < \lambda_1 < \lambda_2 < 1 < \lambda_3.$$

For example,

$$A = \begin{pmatrix} 2 & -3 & 1 \\ -3 & 6 & -2 \\ 1 & -2 & 1 \end{pmatrix}.$$

As the characteristic polynomial of $A$ is irreducible over $\mathbb{Q}$, the splitting field $F/\mathbb{Q}$ has a Galois group with an order three subgroup $\{\mathrm{id}, \sigma, \sigma^2\}$ where $\sigma$ is a cyclic permutation such that $\sigma(\lambda_1) = \lambda_2$.

For $i = 1, 2, 3$, let $\mathfrak{h}_i = \langle X_i, Y_i, Z_i \rangle$ be a copy of the Heisenberg Lie algebra where $[X_i, Y_i] = Z_i$. Define

- $\mathfrak{g} = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$,
- $\tilde{\Gamma}_1 = \mathbb{Z}[\lambda_1] \times \mathbb{Z}[\lambda_1] \times \frac{1}{2}\mathbb{Z}[\lambda_1] \subset \mathfrak{h}_1$,
- $\tilde{\Gamma} = \{v \times \sigma(v) \times \sigma^2(v) : v \in \tilde{\Gamma}_1\} \subset \mathfrak{g}$,
- $B_i \in \mathrm{Aut}(\mathfrak{h}_i)$ by $B_i(X_i) = \lambda_i X_i$ and $B_i(Y_i) = \lambda_i Y_i$ (which implies $B_i(Z_i) = \lambda_i^2 Z_i$),
- $B = B_1 \times B_2 \times B_3 : \mathfrak{g} \to \mathfrak{g}$,
- $G = \exp(\mathfrak{g})$, and
- $\Gamma = \exp(\tilde{\Gamma})$.

Then $B$ defines an Anosov diffeomorphism $f$ of the nilmanifold $G/\Gamma$.

Define a partially hyperbolic splitting for $f$ by

$$E^s = \langle Z_1, Z_2, Y_1, X_1 \rangle, \quad E^c = \langle Y_2, X_2 \rangle, \quad \text{and} \quad E^u = \langle X_3, Y_3, Z_3 \rangle.$$

The inequalities on eigenvalues were chosen so that neither this nor any other partially hyperbolic splitting for $f$ is dynamically coherent.

We now set about deforming $f$ to make a non-Anosov example. The techniques are similar to those used to construct volume-preserving examples in [1, 2, 9]. Here we use the following lemma, proven in the appendix, which will allow us to ensure that center bunching is preserved after the deformation.

**Lemma 2.** *Suppose $f \in \mathrm{Diff}^r(M)$ $(r \geq 1)$ has a dominated splitting, i.e., a continuous invariant splitting $TM = E'_f \oplus E''_f$ with continuous functions $\alpha, \beta : M \to \mathbb{R}$ such that*

$$\|Df v'_x\| < \alpha(x) < \beta(x) < \|Df v''_x\|$$

*for all $x \in M$ and unit vectors $v'_x \in E'_f(x)$ and $v''_x \in E''_f(x)$.*

*Let $q \in M$ be a fixed point and $P \subset T_q M$ a $Df_q$-invariant plane. For $\theta \in \mathbb{R}$ define $R_\theta : T_q M \to T_q M$ as the rotation by angle $\theta$ in the plane $P$ and suppose for some $a > 0$ and all $\theta \in [0, a]$ that the linear map $R_\theta \circ Df_q$ has no eigenvalues in $\{z \in \mathbb{C} : \alpha(q) \leq |z| \leq \beta(q)\}$.*

*Then, there is $g \in \mathrm{Diff}^r(M)$ isotopic to $f$ such that $Dg_q = R_a \circ Df_q$ and $g$ has a dominated splitting $TM = E'_g \oplus E''_g$ which, for some $n \geq 1$, satisfies*

$$\|Dg^n v'_x\| < \prod_{k=0}^{n-1} \alpha(g^k(x)) < \prod_{k=0}^{n-1} \beta(g^k(x)) < \|Dg^n v''_x\|$$

*for all $x \in M$ and unit vectors $v'_x \in E'_g(x)$ and $v''_x \in E''_g(x)$. If $f$ preserves a smooth volume form, one may choose $g$ to preserve the same volume form.*

*Moreover, for any $\epsilon > 0$ and closed set $K \subset M$ with $q \notin K$, one may define $g$ such that $f|_K = g|_K$ and for all $x \in K$ the splittings $E'_f(x) \oplus E''_f(x)$ and $E'_g(x) \oplus E''_g(x)$ are $\epsilon$-close.*

**Remark.** Analogous results hold when the fixed point is replaced by a periodic point and when the domination is assumed to hold only on an invariant closed subset, instead of all of $M$. Also, if $f$ has more than one dominated splitting, as is the case for a partially hyperbolic diffeomorphism, then the deformation $g$ may taken as the same for each splitting. These properties can be seen from the proof of the lemma.

In this specific setting, choose the plane $P$ as the span of $X_2$ and $Z_3$ at a fixed point $q$. Since $\lambda_2 \lambda_3^2 > \lambda_3$, there is an angle $a > 0$ such that the map $R_a \circ Df_q$ when restricted to $P$ has two eigenvalues: one slightly greater than one and the other greater than $\lambda_3$.

Applying the lemma, there is a deformation $g$ of $f$ and an iterate $n \geq 1$ such that

- $q$ is a hyperbolic fixed point for $g$ with an unstable subspace equal to $\langle X_2, X_3, Y_3, Z_3 \rangle$,
- $\|Dg^n(v^s)\| < (\lambda_1 + \epsilon)^n$ for unit vectors $v^s \in E^s_g$,
- $(\lambda_2 - \epsilon)^n < \|Dg^n(v^c)\| < (1 + \epsilon)^n$ for unit vectors $v^c \in E^c_g$, and
- $(\lambda_3 - \epsilon)^n < \|Dg^n(v^u)\|$ for unit vectors $v^u \in E^u_g$.

If $\epsilon$ is sufficiently small, then $g$ is center bunched.

By the "moreover" part of the lemma, we may take a sequence of diffeomorphisms $g_k$ such that, except at the fixed point $q$, the splittings $E^u_{g_k} \oplus E^c_{g_k} \oplus E^s_{g_k}$ converge to the splitting for $f$ as $k \to \infty$. If each $g_k$ was dynamically coherent, then at a point $x \neq q$ there would be a sequence of submanifolds tangent to $E^{cu}_{g_k}$ converging to a submanifold tangent to $E^{cu}_f$. Since no such submanifold exists for $f$ at $x$, this is a contradiction. Therefore, we may assume $g$ is not dynamically coherent. By the same argument, no diffeomorphism $C^1$ close to $g$ is dynamically coherent. Since stable accessibility is $C^1$-dense [6], by perturbing $g$, one can find an open family of accessible examples as desired.

Using Lemma 2, we can also give a simple direct proof of the following.

**Theorem 3.** *Any volume-preserving Anosov diffeomorphism with a partially hyperbolic splitting may be deformed into a volume-preserving partially hyperbolic diffeomorphism which is not Anosov.*

*Proof.* Let $f$ be the Anosov diffeomorphism and assume $f$ has a fixed point $q$. (If $f$ has no fixed points, a similar proof will work for a periodic orbit.) If $Df_q$ has non-real eigenvalues, then there is an invariant plane $P \subset T_qM$ such that $Df_q|_P$ has complex conjugate eigenvalues $\lambda \neq \bar{\lambda}$. For each $\theta \in \mathbb{R}$, $R_\theta \circ Df_q|P$ has two eigenvalues whose product is $|\lambda|^2$. If $R_\theta \circ Df_q|P$ has non-real eigenvalues, they must have the same modulus as $\lambda$. Further, for some $a > 0$ the eigenvalues become real. Applying Lemma 2, replace $f$ by a diffeomorphism such that $Df_q$ has real eigenvalues on $P$. By induction, we may assume that all eigenvalues of $Df_q$ are real.

Suppose now that $Df_q$ is not diagonalizable. Then, there is a plane $P$ such that, with respect to some basis, $R_\theta|_P$ and $Df_q|_P$ are given respectively by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$$

where $\lambda, b \in \mathbb{R}$ and $b \neq 0$. Then, the trace of $R_\theta \circ Df_q|_P$ is given by $2\lambda\cos\theta + b\sin\theta$ and from this one sees that there is an arbitrary small $\theta$ (possibly negative) such that $R_\theta \circ Df_q|_P$ has distinct real eigenvalues. Thus, by Lemma 2 and induction, one may assume that $Df_q$ is diagonalizable with real eigenvalues.

Suppose the eigenvalues for $Df_q$ are $\lambda^s_1, \cdots, \lambda^s_k$, $\lambda^c_1, \cdots, \lambda^c_\ell$, $\lambda^u_1, \cdots, \lambda^u_m$ in order of increasing modulus and with superscripts denoting the bundle in the partially hyperbolic splitting to which they are associated. If $|\lambda^c_j| < 1 < |\lambda^c_{j+1}|$ for some $j$, Lemma 2 may be applied to the span of the two corresponding eigenvectors to produce a diffeomorphism $g$ where $Dg_q$ has an eigenvalue of modulus one. Therefore, we may assume either $|\lambda^c_1| > 1$ or $|\lambda^c_\ell| < 1$. Without loss of generality, assume the latter. Since the product of all of the eigenvalues is equal to one, it holds that $|\lambda^c_\ell \lambda^u_1 \cdots \lambda^u_m| > 1$ and there is $\mu > 1$ such that

$$\mu < |\lambda^u_1| \quad \text{and} \quad \mu^{-m}|\lambda^c_\ell \lambda^u_1 \cdots \lambda^u_m| > \mu^m.$$

Now, by applying Lemma 2 using the plane associated to $\lambda_\ell^c$ and $\lambda_i^u$, those two eigenvalues may be replaced eigenvalues of modulus $|\lambda_\ell^c \lambda_1^u|\mu$ and $\mu$ respectively. By applying similar rotations to replace, in turn, each eigenvalue $\lambda_2^u, \ldots, \lambda_m^u$ with $\pm\mu$, one produces a new diffeomorphism which has a hyperbolic fixed point at $q$ with an $(m+1)$-dimensional unstable direction. □

## APPENDIX

We give several technical lemmas before proving Lemma 2.

**Lemma 4.** *For any $a, \epsilon > 0$ there is a smooth decreasing function $\psi : [0,\infty) \to [0,a]$ such that*

- *$\psi(t) = a$ for $t$ in a neighbourhood of zero,*
- *$\psi(t) = 0$ for $t > \epsilon$, and*
- *$|t \cdot \psi'(t)| < \epsilon$ for all $t$.*

*Proof.* Define

$$\psi_0(t) = \begin{cases} a, & \text{for } t \in [0,b] \\ -\frac{\epsilon}{2}\log(t) + c, & \text{for } t \in [b, \epsilon/2] \\ 0, & \text{for } t \in [\epsilon/2, \infty) \end{cases}$$

where $b$ and $c$ are chosen so that $\psi_0$ is well-defined and continuous. By smoothing out $\psi_0$ near $b$ and $\epsilon/2$, one may define a function $\psi$ as desired. □

For the next two lemmas, let $P$ be a two-dimensional subspace of $\mathbb{R}^d$ and define $R_\theta$ to be the rotation by angle $\theta$ in $P$.

**Lemma 5.** *For any $a, \epsilon > 0$ there is a smooth function $h : \mathbb{R}^d \to \mathbb{R}^d$ such that at the origin the derivative $Dh_0$ equals $R_a$, and for all $x \in \mathbb{R}^d$*

- *there is $\theta \in [0,a]$ such that $\|Dh_x - R_\theta\| < \epsilon$,*
- *if $y \in \mathbb{R}^d \setminus \{0\}$ then $\|Dh_x - Dh_y\| < (2 + \frac{\|x\|}{\|y\|})\epsilon$, and*
- *if $\|x\| > \epsilon$ then $h(x) = x$.*

**Remark.** In the proof, the norm of a linear map $L : \mathbb{R}^d \to \mathbb{R}^d$ is taken to be $\sup|\ell_{ij}|$ where $\ell_{ij}$ is the $(i,j)$ entry of the matrix representing $L$ with respect to the standard basis. It is easy to see that the same results hold for any choice of norm and any Riemannian metric on $\mathbb{R}^d$.

*Proof.* Define $r(x) = \|x\|$ as the usual Euclidean distance from the origin. Then define $h(x) = R_{\psi(r(x))}(x)$ where $\psi$ is from the previous lemma. From this, one can verify the desired properties. For instance, if $P$ is the span of the first two coordinates of $\mathbb{R}^d$ so that

$$R_\theta(x_1, x_2, x_3, \ldots, x_d) = (x_1 \cos\theta - x_2 \sin\theta,\ x_1 \sin\theta + x_2 \cos\theta,\ x_3, \ldots, x_d)$$

then, writing $r(x)$ simply as $r$,

$$\frac{\partial h_1}{\partial x_1} = \cos\psi(r) + [x_1 \sin\psi(r) + x_2 \cos\psi(r)]\ \frac{d\psi}{dr}\ \frac{\partial r}{\partial x_1}.$$

As $|x_1 \ \sin\psi(r) + x_2 \ \cos\psi(r)| \le r$ and $|\frac{\partial r}{\partial x_1}| \le 1$, it follows that

$$\left|\frac{\partial h_1}{\partial x_1} - \cos\psi(r)\right| < r\left|\frac{d\psi}{dr}\right| < \epsilon.$$

Similar inequalities hold for the other partial derivatives, showing that the Jacobian of $h$ at a point $x$ is $\epsilon$-close to the linear map $R_\theta$ where $\theta = \psi(r(x))$. By the mean value theorem, $|\psi(s) - \psi(t)| < \epsilon\frac{s}{t}$ for all $s, t \in (0,\infty)$. Therefore,

$$\begin{aligned}\|Dh_x - Dh_y\| &\le \|Dh_x - R_{\psi(r(x))}\| + \|R_{\psi(r(x))} - R_{\psi(r(y))}\| + \|R_{\psi(r(y))} - Dh_y\| \\ &< \epsilon + \epsilon\frac{\|x\|}{\|y\|} + \epsilon\end{aligned}$$

for all non-zero $x, y \in \mathbb{R}^d$. □

**Lemma 6.** *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism such that $f(0) = 0$. Then for $a, \delta > 0$ and $n \ge 1$, there is a diffeomorphism $g : \mathbb{R}^d \to \mathbb{R}^d$ such that at the origin $Dg_0$ equals $R_a \circ Df_0$ and if $x \in \mathbb{R}^d$ and $j \in \{1,\dots,2n\}$ are such that $g^j(x) \ne f^j(x)$ then $\|x\| < \delta$ and there is $\theta \in [0, a]$ such that $\|Dg^n_y - (R_\theta \circ Df_0)^n\| < \delta$ for all $y \in \{x, g(x), \dots, g^n(x)\}$.*

*Proof.* By continuity, there is a constant $\eta > 0$ such that any linear maps $F_k, H_k : \mathbb{R}^d \to \mathbb{R}^d$ and values $\theta \in [0, a]$ and $j \in \{1,\dots,2n\}$ which satisfy

- $\|F_k - Df_0\| < \eta$ for all $k \in \{1,\dots,2n\}$,
- $\|H_{k+1} - H_k\| < \eta$ for all $k \in \{1,\dots 2n-1\}$, and
- $\|H_j - R_\theta\| < \eta$,

must also satisfy $\|(H_{k+n-1} \circ F_{k+n-1} \circ \dots \circ H_k \circ F_k) - (R_\theta \circ Df_0)^n\| < \delta$ for all $k \in \{1,\dots,n\}$.

Let $U$ be a neighbourhood of the origin such that $\|Df_x - Df_0\| < \eta$ and $\|x\| < \delta$ for all $x \in U$. Define $K > 1$ such that $K^{-1}\|y\| \le \|R_\theta(f(y))\| \le K\|y\|$ for all $\theta \in [0, a]$ and $y \in U$. Then, there is $\epsilon > 0$ such that $(2 + K)\epsilon < \eta$ and such that $U$ includes the ball of radius $\epsilon K^{2n}$ centered at the origin. With this $\epsilon$, take $h$ as in the Lemma 5 and define $g = h \circ f$. □

Note that the diffeomorphism $h$ in Lemma 5 preserves the standard volume form on $\mathbb{R}^d$. Therefore, if $f$ is volume preserving in Lemma 6, then so is $g = h \circ f$.

*Proof of Lemma 2.* By a result of Moser [7], there is a neighbourhood $U$ of $q$ and a volume preserving embedding $\phi : U \to \mathbb{R}^d$ such that $q$ is mapped to the origin. By abuse of notation, we simply assume that $U$ *is* a subset of $\mathbb{R}^d$ and identify the tangent space $T_xM$ with $\mathbb{R}^d$ for all $x \in U$. We further assume, by changing the embedding if necessary, that $R_\theta$ for $\theta \in [0, a]$ is a rotation with respect to the standard metric on $\mathbb{R}^d$. Without loss of generality, assume the function $\beta$ in the statement of the lemma is constant in a neighbourhood of $q$.

By considering the spectral radius, one can show that for every $\theta \in [0, a]$ there is $n \ge 1$ such that the cone

$$C_\theta = \{v \in \mathbb{R}^d : \|(R_\theta \circ Df_q)^n v\| \ge \beta(q)^n\|v\|\}$$

satisfies the property that $(R_\theta \circ Df_q)^n(C_\theta)$ is compactly contained in $C_\theta$. If this inclusion holds for some $n$ and $\theta$, then it also holds for the same $n$ and all nearby $\theta$. Therefore, as $[0,a]$ is compact, a single value of $n$ may be used.

Define the cone field $C_f$ by

$$C_f(x) = \{v \in T_xM : \|Df^n_x v\| \ge \beta_n(x)\|v\|\}$$

where $\beta_n$ is the cocycle $\beta_n(x) := \beta(f^{n-1}(x))\cdots\beta(f(x))\beta(x)$. By the properties of cone fields and dominated splittings, if $n$ is sufficiently large, then $Df^n(C_f)$ is compactly contained in $C_f$ [8].

For an arbitrary linear map $L : \mathbb{R}^d \to \mathbb{R}^d$, define a cone $C_L = \{v \in \mathbb{R}^d : \|Lv\| \ge \beta(q)^n\|v\|\}$. Suppose $\delta > 0$, $\theta \in [0,a]$, and that $L_1$ and $L_2$ are two linear maps which satisfy

$$\|L_i - (R_\theta \circ Df_q)^n\| < \delta.$$

Since $(R_\theta \circ Df_q)^n(C_\theta)$ is compactly contained in $C_\theta$, a continuity argument shows that if $\delta$ is sufficiently small, then $L_1(C_{L_1}) \subset C_{L_2}$. Moreover, $\delta$ may be chosen independently of $\theta \in [0,a]$. Using $a$, $\delta$, and $n$, define $g$ as in Lemma 6 and define a cone field $C_g$ for $g$ by

$$C_g(x) = \{v \in T_xM : \|Dg^n_x v\| \ge \beta_n(x)\|v\|\}.$$

Now consider $x \in M$. If $f^k(x) = g^k(x)$ for all $k \in \{1, \cdots, 2N\}$, then

$$Dg^n(C_g(x)) = Df^n(C_f(x)) \subset C_f(f^n(x)) = C_g(g^n(x)).$$

Otherwise, define $L_1 = Dg^n_x$ and $L_2 = Dg^n_y$ where $y = g^n(x)$. Then Lemma 6 implies that $\|L_i - (R_\theta \circ Df_q)^n\| < \delta$ and therefore

$$Dg^n(C_g(x)) = L_1(C_{L_1}) \subset C_{L_2} = C_g(g^n(x)).$$

This is enough to establish a dominated splitting $TM = E'_g \oplus E''_g$.

For any $\epsilon > 0$, one can show that there is $N_\epsilon$ such that if $f^k(x) = g^k(x)$ for all $k \in \{1, \dots, N_\epsilon\}$ then the splittings $E'_f(x) \oplus E''_f(x)$ and $E'_g(x) \oplus E''_g(x)$ are $\epsilon$-close. This can be used to prove the "moreover" part of the lemma. □

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