MTH 1035 Handout – Abstract Vector Spaces

Definition of an abstract vector space (taken from Elementary Linear Algebra by Anton and Rorres).

Let $V$ be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by scalars. By **addition** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in $V$ an object, called the **sum** of $\mathbf{u}$ and $\mathbf{v}$; by **scalar multiplication** we mean a rule for associating with each scalar $k$ and each object $\mathbf{u}$ in $V$ an object $k\mathbf{u}$, called the **scalar multiple** of $\mathbf{u}$ by $k$. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $V$ and all scalars $k$ and $m$, then we call $V$ a **vector space** and we call the objects in $V$ **vectors**.

**Rule 1** If $\mathbf{u}$ and $\mathbf{v}$ are objects in $V$, then $\mathbf{u}+\mathbf{v}$ is also in $V$.

**Rule 2** $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

**Rule 3** $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

**Rule 4** There is an object $\vec{\mathbf{0}}$ in $V$, called a **zero vector** for $V$, such that

$$\vec{\mathbf{0}}+\mathbf{u}=\mathbf{u}+\vec{\mathbf{0}}=\mathbf{u} \quad \text{for all } \mathbf{u} \text{ in } V.$$

**Rule 5** For each $\mathbf{u}$ in $V$, there is an object $-\mathbf{u}$ such that

$$\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\vec{\mathbf{0}}$$

**Rule 6** If $k$ is any scalar and $\mathbf{u}$ is an object in $V$, then $k\mathbf{u}$ is in $V$.

**Rule 7** $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

**Rule 8** $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$

**Rule 9** $k(m\mathbf{u})=(km)\mathbf{u}$

**Rule 10** $1\mathbf{u}=\mathbf{u}$

We sometimes write $t\cdot\mathbf{v}$ to emphasize the scaling operation.

**Exercise.** Do the following sets satisfy the definition of a vector space under the most sensible choices of addition and scalar multiplication?

(1) $\mathbb{R}^2$

(2) $\mathbb{R}^3$

(3) $\mathbb{R}$

(4) $\mathbb{C}$

(5) $\mathbb{N}$

(6) $\mathbb{Q}$

(7) $\mathbb{Z}$

(8) A set $X=\{\mathbf{v}\}$ with exactly one element.

(9) A set

$$\text{Beatles}=\{\text{John},\text{Paul},\text{George},\text{Ringo}\}$$

with the operations that if $t$ is a real number and $\mathbf{v},\mathbf{w}$ are elements of Beatles, then $t\cdot\mathbf{v}=$ John and $\mathbf{v}+\mathbf{w}=$ Ringo.

(10) The set of all sequences of real numbers.

(11) The set of all bounded sequences of real numbers.

(12) The set of all increasing sequences of real numbers.

(13) The set of all "Fibonacci-like" sequences.
That is sequences of real numbers of the form $\{x_n\}_{n=1}^{\infty}$ where $x_{n+2} = x_{n+1} + x_n$ for all $n \geq 1$.

(14) The set of polynomials.

(15) The set of all real-valued functions defined on $\mathbb{R}$.

(16) The set of all real-valued continuous functions defined on $\mathbb{R}$.

(17) The set of all real-valued functions defined on $\mathbb{R}$ with the property $f(0) = 7$.

(18) The set of all real-valued functions defined on $\mathbb{R}$ with the property $f(0) = 0$.

(19) The set of all real-valued functions defined on $\mathbb{R}$ with the property $f(0) = f(1)$.

(20) The set of all real-valued differentiable functions defined on $\mathbb{R}$ with the property $f'(0) = 0$.

(21) The set of all real-valued differentiable functions defined on $\mathbb{R}$ with the property $f'(0) = f(0)$.

(22) The set of all real-valued differentiable functions defined on $\mathbb{R}$ with the property $f'(0) = f(1)$.

(23) The set of all Lipschitz functions defined on $\mathbb{R}$.

(24) The set of all linear functions from $\mathbb{R}^m$ to $\mathbb{R}^n$.

**Proof exercises for an abstract vector space.**

(1) Show that $\mathbf{v} + \mathbf{v} = 2\mathbf{v}$.

(2) Show that $\mathbf{v} + \mathbf{v} + \mathbf{v} = 3\mathbf{v}$ (and argue why $\mathbf{v} + \mathbf{v} + \mathbf{v}$ makes sense as an expression).

(3) Show that if $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

(4) Show that $0 \cdot \mathbf{v} = \vec{\mathbf{0}}$. (Note the two different "zeros".)

(5) Show that $t \cdot \vec{\mathbf{0}} = \vec{\mathbf{0}}$.

(6) Show that $(-1) \cdot \mathbf{v} = -\mathbf{v}$.

(7) Show that if $t$ is a non-zero real number and $\mathbf{v}$ is a non-zero vector, then $t \cdot \mathbf{v} \neq \vec{\mathbf{0}}$.

(8) Show that if $t \cdot \mathbf{v} = s \cdot \mathbf{v}$, then either $\mathbf{v} = 0$ or $t = s$.

(9) Show that if $t \cdot \mathbf{v} = t \cdot \mathbf{w}$, then either $t = 0$ or $\mathbf{v} = \mathbf{w}$.

(10) Show that if a vector space $V$ has (at least) two distinct elements, then $V$ has infinitely many elements.

(11) If $V$ and $W$ are vector spaces, show that the set $V \times W$ which consists of all pairs $(\mathbf{v}, \mathbf{w})$ with $\mathbf{v}$ in $V$ and $\mathbf{w}$ in $W$ is also a vector space. What are the operations?

(12) Suppose $V$ is a vector space, and consider the set $\text{Fib}(V)$ of all Fibonacci-like sequences of the form $\{\mathbf{v}_n\}_{n=1}^{\infty}$ where each $\mathbf{v}_n$ is an element of $V$ and $\mathbf{v}_{n+2} = \mathbf{v}_{n+1} + \mathbf{v}_n$ for all $n \geq 1$. Is $\text{Fib}(V)$ a vector space?

(13) Find a set which satisfies Rules 1 to 9, but does not satisfy Rule 10.