MTH 1035 – Solution to Handout – Abstract Vector Spaces

Definition of an abstract vector space (taken from Elementary Linear Algebra by Anton and Rorres).

Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by scalars. By **addition** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object, called the **sum** of $\mathbf{u}$ and $\mathbf{v}$; by **scalar multiplication** we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the **scalar multiple** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all scalars k and m, then we call V a **vector space** and we call the objects in V **vectors**.

Rule 1 If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is also in V.

Rule 2 $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

Rule 3 $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

Rule 4 There is an object $\vec{\mathbf{0}}$ in V, called a **zero vector** for V, such that

$$\vec{\mathbf{0}}+\mathbf{u}=\mathbf{u}+\vec{\mathbf{0}}=\mathbf{u} \quad \text{for all } \mathbf{u} \text{ in } V.$$

Rule 5 For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ such that

$$\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\vec{\mathbf{0}}$$

Rule 6 If k is any scalar and $\mathbf{u}$ is an object in V, then $k\mathbf{u}$ is in V.

Rule 7 $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

Rule 8 $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$

Rule 9 $k(m\mathbf{u})=(km)\mathbf{u}$

Rule 10 $1\mathbf{u}=\mathbf{u}$

We sometimes write $t\cdot\mathbf{v}$ to emphasize the scaling operation.

Exercise. Do the following sets satisfy the definition of a vector space under the most sensible choices of addition and scalar multiplication?

(1) $\mathbb{R}^2$

(2) $\mathbb{R}^3$

(3) $\mathbb{R}$

(4) $\mathbb{C}$

(5) $\mathbb{N}$

(6) $\mathbb{Q}$

(7) $\mathbb{Z}$

(8) A set $X=\{\mathbf{v}\}$ with exactly one element.

(9) A set

$$\text{Beatles} = \{\text{John}, \text{Paul}, \text{George}, \text{Ringo}\}$$

with the operations that if t is a real number and $\mathbf{v}, \mathbf{w}$ are elements of Beatles, then $t\cdot\mathbf{v}=\text{John}$ and $\mathbf{v}+\mathbf{w}=\text{Ringo}$.

(10) The set of all sequences of real numbers.

(11) The set of all bounded sequences of real numbers.

(12) The set of all increasing sequences of real numbers.

(13) The set of all "Fibonacci-like" sequences.
That is sequences of real numbers of the form $\{x_n\}_{n=1}^{\infty}$ where $x_{n+2} = x_{n+1} + x_n$ for all $n \geq 1$.

(14) The set of polynomials.

(15) The set of all real-valued functions defined on $\mathbb{R}$.

(16) The set of all real-valued continuous functions defined on $\mathbb{R}$.

(17) The set of all real-valued functions defined on $\mathbb{R}$ with the property $f(0) = 7$.

(18) The set of all real-valued functions defined on $\mathbb{R}$ with the property $f(0) = 0$.

(19) The set of all real-valued functions defined on $\mathbb{R}$ with the property $f(0) = f(1)$.

(20) The set of all real-valued differentiable functions defined on $\mathbb{R}$ with the property $f'(0) = 0$.

(21) The set of all real-valued differentiable functions defined on $\mathbb{R}$ with the property $f'(0) = f(0)$.

(22) The set of all real-valued differentiable functions defined on $\mathbb{R}$ with the property $f'(0) = f(1)$.

(23) The set of all Lipschitz functions defined on $\mathbb{R}$.

(24) The set of all linear functions from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Answer.

(1-3) The set $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, or indeed any $\mathbb{R}^d$ for $d \geq 1$ is a vector space under the standard coordinate-wise rules for addition and scaling.

(4) The set $\mathbb{C}$ is a vector space under the usual complex addition and multiplication of a real number with a complex number.

(5-7) The sets $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{Z}$ are not vector spaces under the usual definition of scalar multiplication. For instance, 1 is an element of $\mathbb{N}$ but $\sqrt{2} \cdot 1$ is not an element of $\mathbb{N}$. The same holds for $\mathbb{Q}$ and $\mathbb{Z}$.

(8) If a set has exactly one element, $\mathbf{v}$, then the addition and scaling operations must be $\mathbf{v} + \mathbf{v} = \mathbf{v}$ and $t \cdot \mathbf{v} = \mathbf{v}$ for any $t \in \mathbb{R}$, so that Rules 1 and 6 hold. All of the other rules reduce down to the fact that $\mathbf{v} = \mathbf{v}$ and so they also hold. Hence, the set $V = \{\mathbf{v}\}$ is a vector space. Note, that the only element of this space is the zero vector, $\vec{\mathbf{0}}$ and so $V = \{\vec{\mathbf{0}}\}$.

(9) If $\mathbf{v} \in$ Beatles and $\mathbf{v} \neq$ John, then $1 \cdot \mathbf{v} \neq \mathbf{v}$ and so Rule 10 does not hold.

(10) If V is the set of all sequences of real numbers and $\{x_n\}, \{y_n\} \in V$ and $t \in \mathbb{R}$, then define vector addition by adding the sequences term by term, and scaling by scaling each term by t. That is, the addition of $\{x_n\}$ and $\{y_n\}$ is defined to be a sequence $\{z_n\}$ where $z_n = x_n + y_n$ for all n and the scaling of $\{x_n\}$ by $t \in \mathbb{R}$ is defined to be a sequence $\{s_n\}$ where $s_n = tx_n$ for all n. One can verify that this set is vector space by checking the rules term by term. For instance,

$$\big(\{a_n\} + \{b_n\}\big) + \{c_n\} = \{a_n\} + \big(\{b_n\} + \{c_n\}\big)$$

holds because

$$(a_n + b_n) + c_n = a_n + (b_n + c_n)$$

for each individual n.

(11) Assume addition and scaling are defined as in (10). Then most of the rules follow by the same reasoning as in (10). The only ones to check are Rules 1 and 6. Suppose $\{x_n\}$ and $\{y_n\}$ are bounded sequences. Say C and D are constants such that

$$|x_n| < C \quad \text{and} \quad |y_n| < D$$

for all n. Then, by the triangle inequality,

$$|x_n + y_n| \leq |x_n| + |y_n| < C + D$$

for all n. This shows that the sequence $\{x_n + y_n\}$ is also bounded. Hence the set satisfies Rule 1. If $t \in \mathbb{R}$, then

$$|tx_n| < |t|C$$

for all n, and the sequence $\{tx_n\}$ is bounded as well. This shows that the set satisfies Rule 6. Hence, the set of bounded sequences is a vector space.

Note: the vector space in (11) is a subset of the larger vector space given in (10). This is an example of a **subspace** of a vector space.

(12) The sequence $\{1, 2, 3, 4, 5, \cdots\}$ is an increasing sequence. If we scale this sequence by $-1 \in \mathbb{R}$, the result is $\{-1, -2, -3, -4, -5, \cdots\}$ which is not an increasing sequence. Hence, this set does not satisfy Rule 6.

(13) As in (10), the addition of $\{x_n\}$ and $\{y_n\}$ is defined to be a sequence $\{z_n\}$ where $z_n = x_n + y_n$ for all n and the scaling of $\{x_n\}$ by $t \in \mathbb{R}$ is defined to be a sequence $\{s_n\}$ where $s_n = tx_n$ for all n. If the sequences $\{x_n\}$ and $\{y_n\}$ are Fibonacci-like, then

$$\begin{aligned} z_{n+2} &= x_{n+2} + y_{n+2} \\ &= x_{n+1} + x_n + y_{n+1} + y_n \\ &= x_{n+1} + y_{n+1} + x_n + y_n \\ &= z_{n+1} + z_n, \end{aligned}$$

showing that $\{z_n\}$ is Fibonacci-like, and

$$\begin{aligned} s_{n+2} &= tx_{n+2} \\ &= t(x_{n+1} + x_n) \\ &= tx_{n+1} + tx_n \\ &= s_{n+1} + s_n \end{aligned}$$

showing that $\{s_n\}$ is Fibonacci-like. Hence, the set satisfies Rules 1 and 6. All other rules hold as in (10) and so the set is a vector space.

(14) The addition of two polynomials is a polynomial and scaling a polynomial by a real constant yields a polynomial. Hence, Rules 1 and 6 holds. All other rules can be checked pointwise. For instance, if f and g are polynomials then $f + g = g + f$ since

$$f(x) + g(x) = g(x) + f(x)$$

for all $x \in \mathbb{R}$. Hence, the set is a vector space.

(15) This is a vector space by similar reasoning as in (14).

(16) The sum of continuous functions is continuous and scaling a continuous function by constant results in a continuous function. Hence, one may verify that this set is a vector space.

(17) If f is an element of this set, then $f(0) = 7$. If $g = 2 \cdot f$, then $g(0) = 14 \neq 7$, so g is not an element of this set. Hence, Rule 6 fails.

(18-22) One may verify that all of these are examples of vector spaces. For instance, if f and g are functions as in (22) and $t \in \mathbb{R}$, then

$$(f+g)'(0) = f'(0) + g'(0) = f(1) + g(1) = (f+g)(1)$$

so $f + g$ is an element of the space and

$$(t \cdot f)'(0) = tf'(0) = tf(1) = (t \cdot f)(1)$$

so $t \cdot f$ is an element of the space.

(23) We assume here that all functions are defined on the same domain. We showed in a previous worksheet that if f and g are Lipschitz then $f + g$ is Lipschitz. If f is Lipschitz with constant L and $t \in \mathbb{R}$, then

$$|(t \cdot f)(x) - (t \cdot f)(y)| = |tf(x) - tf(y)| = |t||f(x) - f(y)| \leq |t|L$$

so $t \cdot f$ is Lipschitz with constant $|t|L$. This shows that Rules 1 and 6 hold. The other rules hold as in (14).

(24) Recall from a previous worksheet that a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is **linear** if

(i) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^m$, and

(ii) $f(tx) = tf(x)$ for all $x \in \mathbb{R}^m$ and $t \in \mathbb{R}$.

If f and g are linear, then

$$\begin{aligned}(f+g)(x+y) &= f(x+y) + g(x+y)\\ &= f(x) + f(y) + g(x) + g(y)\\ &= f(x) + g(x) + f(y) + g(y)\\ &= (f+g)(x) + (f+g)(y)\end{aligned}$$

and

$$\begin{aligned}(f+g)(tx) &= f(tx) + g(tx)\\ &= tf(x) + tg(x)\\ &= t(f(x) + g(x))\\ &= t(f+g)(x)\end{aligned}$$

which together show that $f + g$ is a linear function from $\mathbb{R}^m$ to $\mathbb{R}^n$ and so Rule 1 holds. Similar equations show that if $c \in \mathbb{R}$ is a constant, then $c \cdot f$ is a linear function and so Rule 6 holds. The other rules may be verified pointwise at each point $x \in \mathbb{R}^m$. Hence, this set is a vector space.

Proof exercises for an abstract vector space.

(1) Show that $\mathbf{v} + \mathbf{v} = 2\mathbf{v}$.

Answer.

$$\begin{aligned}
\mathbf{v}+\mathbf{v} &= 1\cdot\mathbf{v}+\mathbf{v} && \text{(by Rule 9)}\\
&= 1\cdot\mathbf{v}+1\cdot\mathbf{v} && \text{(by Rule 9 again)}\\
&= (1+1)\cdot\mathbf{v} && \text{(by Rule 8)}\\
&= 2\cdot\mathbf{v} && \text{(by real arithmetic)}
\end{aligned}$$

(2) Show that $\mathbf{v}+\mathbf{v}+\mathbf{v}=3\mathbf{v}$ (and argue why $\mathbf{v}+\mathbf{v}+\mathbf{v}$ makes sense as an expression).
Answer. For an abstract vector space, addition is only defined in terms of two elements. Therefore if we wish to add three vectors together, we must first add two of them and then add the third to the result. That is, if V is an abstract vector space and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are elements, then we may either add them as

$$(\mathbf{u}+\mathbf{v})+\mathbf{w} \quad \text{or} \quad \mathbf{u}+(\mathbf{v}+\mathbf{w}).$$

Fortunately, Rule 3 tells us that the result is the same in either case. Therefore, we write $\mathbf{u}+\mathbf{v}+\mathbf{w}$ as an abbreviation for $(\mathbf{u}+\mathbf{v})+\mathbf{w}$.

Now,

$$\begin{aligned}
(\mathbf{v}+\mathbf{v})+\mathbf{v} &= (\mathbf{v}+\mathbf{v})+1\cdot\mathbf{v} && \text{(by Rule 10)}\\
&= 2\cdot\mathbf{v}+1\cdot\mathbf{v} && \text{(by Exercise 1)}\\
&= (2+1)\cdot\mathbf{v} && \text{(by Rule 8)}\\
&= 3\cdot\mathbf{v} && \text{(by real arithmetic)}
\end{aligned}$$

(3) Show that if $\mathbf{u}+\mathbf{v}=\mathbf{u}+\mathbf{w}$, then $\mathbf{v}=\mathbf{w}$.
Answer. The formulas $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}+\mathbf{w}$ both represent the same element of the vector space V. Since addition is defined in terms of the objects of V (and not the formulas representing them), it follows that if we add another vector to this element we get the same result regardless of which formula we use. This is a long-winded way of explaining that if

$$\mathbf{u}+\mathbf{v}=\mathbf{u}+\mathbf{w},$$

then

$$-\mathbf{u}+(\mathbf{u}+\mathbf{v}) = -\mathbf{u}+(\mathbf{u}+\mathbf{w})$$

where $-\mathbf{u}$ is given by Rule 5. Applying Rule 3 to both sides gives

$$(-\mathbf{u}+\mathbf{u})+\mathbf{v} = (-\mathbf{u}+\mathbf{u})+\mathbf{w}.$$

Applying Rule 5 to both sides gives

$$\vec{\mathbf{0}}+\mathbf{v}=\vec{\mathbf{0}}+\mathbf{w}.$$

Applying Rule 4 to both sides gives

$$\mathbf{v}=\mathbf{w}$$

as desired.

(4) Show that $0\cdot\mathbf{v}=\vec{\mathbf{0}}$. (Note the two different "zeros".)

Answer. Since $0+0=0$ for real numbers,

$$\begin{aligned} (0+0)\cdot\mathbf{v} &= 0\cdot\mathbf{v} \quad \Rightarrow & \\ 0\cdot\mathbf{v}+0\cdot\mathbf{v} &= 0\cdot\mathbf{v} \quad \Rightarrow & \text{(by Rule 8)} \\ 0\cdot\mathbf{v}+0\cdot\mathbf{v} &= 0\cdot\mathbf{v}+\vec{\mathbf{0}} \quad \Rightarrow & \text{(by Rule 4)} \\ 0\cdot\mathbf{v} &= \vec{\mathbf{0}} & \text{(by Exercise 3)} \end{aligned}$$

(5) Show that $t\cdot\vec{\mathbf{0}}=\vec{\mathbf{0}}$.
Answer. By Rule 4, $\vec{\mathbf{0}}+\vec{\mathbf{0}}=\vec{\mathbf{0}}$ in V. Then, scaling each side by t,

$$\begin{aligned} t\cdot(\vec{\mathbf{0}}+\vec{\mathbf{0}}) &= t\cdot\vec{\mathbf{0}} \quad \Rightarrow & \\ t\cdot\vec{\mathbf{0}}+t\cdot\vec{\mathbf{0}} &= t\cdot\vec{\mathbf{0}} \quad \Rightarrow & \text{(by Rule 7)} \\ t\cdot\vec{\mathbf{0}}+t\cdot\vec{\mathbf{0}} &= t\cdot\vec{\mathbf{0}}+\vec{\mathbf{0}} \quad \Rightarrow & \text{(by Rule 4)} \\ t\cdot\vec{\mathbf{0}} &= \vec{\mathbf{0}} & \text{(by Exercise 3)} \end{aligned}$$

(6) Show that $(-1)\cdot\mathbf{v}=-\mathbf{v}$.
Answer. Note that

$$\begin{aligned} \mathbf{v}+(-1)\cdot\mathbf{v} &= 1\cdot\mathbf{v}+(-1)\cdot\mathbf{v} & \text{(by Rule 10)} \\ &= (1+(-1))\cdot\mathbf{v} & \text{(by Rule 8)} \\ &= 0\cdot\mathbf{v} & \text{(by real arithmetic)} \\ &= \vec{\mathbf{0}} & \text{(by Exercise 5)} \end{aligned}$$

and $\mathbf{v}+(-\mathbf{v})=\vec{\mathbf{0}}$ by Rule 4. Hence,

$$\mathbf{v}+(-1)\cdot\mathbf{v}=\mathbf{v}+(-\mathbf{v})$$

since each side is equal to $\vec{\mathbf{0}}$. Exercise 3 implies that $(-1)\cdot\mathbf{v}=-\mathbf{v}$.

(7) Show that if t is a non-zero real number and $\mathbf{v}$ is a non-zero vector, then $t\cdot\mathbf{v}\neq\vec{\mathbf{0}}$.
Answer. We assume t is non-zero. Our goal is then to show that

$$\mathbf{v}\neq\vec{\mathbf{0}} \quad \text{implies} \quad t\cdot\mathbf{v}\neq\vec{\mathbf{0}}.$$

Equivalently, we may prove the contrapositive:

$$t\cdot\mathbf{v}=\vec{\mathbf{0}} \quad \text{implies} \quad \mathbf{v}=\vec{\mathbf{0}}.$$

Since, t is non-zero, $\frac{1}{t}$ is a real number and

$$
\begin{aligned}
t\cdot\mathbf{v} = \vec{\mathbf{0}} \quad &\Rightarrow \\
\frac{1}{t}\cdot(t\cdot\mathbf{v}) = \frac{1}{t}\cdot\vec{\mathbf{0}} \quad &\Rightarrow && \text{(scaling both sides)}\\
\Big(\frac{1}{t}\cdot t\Big)\cdot\mathbf{v} = \frac{1}{t}\cdot\vec{\mathbf{0}} \quad &\Rightarrow && \text{(by Rule 9)}\\
1\cdot\mathbf{v} = \frac{1}{t}\cdot\vec{\mathbf{0}} \quad &\Rightarrow && \text{(by real arithmetic)}\\
\mathbf{v} = \frac{1}{t}\cdot\vec{\mathbf{0}} \quad &\Rightarrow && \text{(by Rule 10)}\\
\mathbf{v} = \vec{\mathbf{0}} & && \text{(by Exercise 4)}
\end{aligned}
$$

(8) Show that if $t\cdot\mathbf{v} = s\cdot\mathbf{v}$, then either $\mathbf{v} = 0$ or $t = s$.
Answer. We first show that $(t-s)\cdot\mathbf{v} = \vec{\mathbf{0}}$. Indeed,

$$
\begin{aligned}
t\cdot\mathbf{v} = s\cdot\mathbf{v} \quad &\Rightarrow \\
t\cdot\mathbf{v} + (-s)\cdot\mathbf{v} = s\cdot\mathbf{v} + (-s)\cdot\mathbf{v} \quad &\Rightarrow && \text{(adding a vector to both sides)}\\
(t+(-s))\cdot\mathbf{v} = (s+(-s))\cdot\mathbf{v} \quad &\Rightarrow && \text{(by Rule 8)}\\
(t-s)\cdot\mathbf{v} = 0\cdot\mathbf{v} \quad &\Rightarrow && \text{(by real arithmetic)}\\
(t-s)\cdot\mathbf{v} = \vec{\mathbf{0}} & && \text{(by Exercise 4)}
\end{aligned}
$$

By the last exercise, if both $t-s$ and $\mathbf{v}$ are non-zero, then $(t-s)\cdot\mathbf{v}$ is non-zero as well, a contradiction. Hence, one of $t-s$ or $\mathbf{v}$ must be zero.

(9) Show that if $t\cdot\mathbf{v} = t\cdot\mathbf{w}$, then either $t = 0$ or $\mathbf{v} = \mathbf{w}$.
Answer. If $t = 0$, the conclusion holds. Therefore, we may assume that $t \neq 0$. Then,

$$
\begin{aligned}
t\cdot\mathbf{v} = t\cdot\mathbf{w} \quad &\Rightarrow \\
\frac{1}{t}\cdot(t\cdot\mathbf{v}) = \frac{1}{t}\cdot(t\cdot\mathbf{w}) \quad &\Rightarrow && \text{(scaling both sides)}\\
\Big(\frac{1}{t}\cdot t\Big)\cdot\mathbf{v} = \Big(\frac{1}{t}\cdot t\Big)\cdot\mathbf{w} \quad &\Rightarrow && \text{(by Rule 9)}\\
1\cdot\mathbf{v} = 1\cdot\mathbf{w} \quad &\Rightarrow && \text{(by real arithmetic)}\\
\mathbf{v} = \mathbf{w} & && \text{(by Rule 10)}
\end{aligned}
$$

(10) Show that if a vector space V has two distinct elements, then V has infinitely many elements.
Answer. If V has two elements, at least one of the elements is not the zero vector. Call this element $\mathbf{v}$. Every real number t yields a vector $t\cdot\mathbf{v}$, and as $\mathbf{v} \neq 0$, Exercise 8 shows that all of these vectors are distinct from each other. Hence, there are at least as many elements of V as there are elements of $\mathbb{R}$. As $\mathbb{R}$ is infinite, this shows that V has infinitely many elements.

Bonus expert commentary: The real numbers are not just infinite, they are **uncountable** meaning that they cannot be listed out as a sequence. The above reasoning shows that any vector space V with at least two elements must also be uncountable. Since the sets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are all countable, they cannot be made into a (real) vector space under any definition of addition or scaling.

(11) If V and W are vector spaces, show that the set $V \times W$ which consists of all pairs $(\mathbf{v}, \mathbf{w})$ with $\mathbf{v}$ in V and $\mathbf{w}$ in W is also a vector space. What are the operations?
Answer. For elements $(\mathbf{v}_1, \mathbf{w}_1)$ and $(\mathbf{v}_2, \mathbf{w}_2)$ of $V \times W$ define addition by

$$(\mathbf{v}_1, \mathbf{w}_1) + (\mathbf{v}_2, \mathbf{w}_2) = (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 + \mathbf{w}_2)$$

where the + operations on the right-hand side are the addition operations in V and W. Similarly, define scaling by

$$t \cdot (\mathbf{v}, \mathbf{w}) = (t \cdot \mathbf{v}, t \cdot \mathbf{w})$$

where the $\cdot$ operations on the right-hand side are the scaling operations in V and W. One can show that $V \times W$ satisfies all of the rules for a vector space by using the corresponding rules for V and W. For instance,

$$\begin{aligned} k \cdot (m \cdot (\mathbf{v}, \mathbf{w})) &= k \cdot (m \cdot \mathbf{v}, m \cdot \mathbf{w}) && \text{(by definition)} \\ &= (k \cdot (m \cdot \mathbf{v}), k \cdot (m \cdot \mathbf{w})) && \text{(by definition)} \\ &= ((k \cdot m) \cdot \mathbf{v}, k \cdot (m \cdot \mathbf{w})) && \text{(by Rule 9 for } V) \\ &= ((k \cdot m) \cdot \mathbf{v}, (k \cdot m) \cdot \mathbf{w}) && \text{(by Rule 9 for } W) \\ &= (k \cdot m) \cdot (\mathbf{v}, \mathbf{w}), && \text{(by definition)} \end{aligned}$$

and this shows that $V \times W$ satisfies Rule 9.

(12) Suppose V is a vector space, and consider the set $\text{Fib}(V)$ of all Fibonacci-like sequences of the form $\{\mathbf{v}_n\}_{n=1}^{\infty}$ where each $\mathbf{v}_n$ is an element of V and $\mathbf{v}_{n+2} = \mathbf{v}_{n+1} + \mathbf{v}_n$ for all $n \geq 1$. Is $\text{Fib}(V)$ a vector space?
Answer. $\text{Fib}(V)$ is a vector space. The idea is to adapt the techniques used for Fibonacci-like sequences of real numbers as introduced earlier in this worksheet. The only properties of real numbers that were used there are ones that hold for any vector space V.

(13) Find a set which satisfies Rules 1 to 9, but does not satisfy Rule 10.
Answer. Let V be a vector space with at least one non-zero element. (You can use $V = \mathbb{R}$ if you like.) Keep the addition operation as is, but replace the scaling operation by

$$t \cdot \mathbf{v} = \vec{\mathbf{0}}$$

for all $t \in \mathbb{R}$ and $\mathbf{v} \in V$. Then Rules 1 though 5 hold since they only concern the addition operation, which was unchanged from before. Rules 6 holds as $t \cdot \mathbf{v} = \vec{\mathbf{0}}$ is an element of V. Rules 7 and 8 hold as $\vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\mathbf{0}}$. Rule 9 holds because $\vec{\mathbf{0}} = \vec{\mathbf{0}}$. Finally, Rule 10 fails because if $\mathbf{v}$ is a non-zero element, then $1 \cdot \mathbf{v} = \vec{\mathbf{0}} \neq \mathbf{v}$.

Bonus – Inner Product Spaces. Abstract vector spaces are an extension of the ideas of vectors in $\mathbb{R}^d$. Whereas $\mathbb{R}^d$ has a dot product which takes two vectors $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^d$ and returns a real number, no such operation exists in general for an abstract vector space. Informally speaking, an **inner product** on an abstract vector space V is an operation which takes two vectors $\mathbf{v}, \mathbf{w} \in V$ and returns a real number $\langle \mathbf{v}, \mathbf{w} \rangle$ and which behaves similarly to the dot product on $\mathbb{R}^d$.

Bonus Exercises.

(1) Just as with abstract vector spaces, inner products should follow a few rules in order to be "sensible". What should those rules be?

(2) Adapt the proof of the Cauchy–Schwarz inequality (as given in Burkard's course notes) to this general setting. That is, if V is an abstract vector space equipped with an inner product, show that

$$\langle \mathbf{v}, \mathbf{w} \rangle \leq \|\mathbf{v}\| \, \|\mathbf{w}\|$$

where $\|\mathbf{v}\|$ is defined as the square root of $\langle \mathbf{v}, \mathbf{v} \rangle$.

Note that a detailed analysis of the proof of the Cauchy–Schwarz inequality will help in determining what rules an inner product should have.

(3) Let V be the vector space of continuous function from $[0,1]$ to $\mathbb{R}$. Show that

$$\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$$

defines an inner product on this space.

(4) Show that if f is any continuous function from $[0,1]$ to $\mathbb{R}$ with

$$\int_0^1 f^2(t)\, dt \leq 1 \quad \text{then} \quad \int_0^1 t^2 f(t)\, dt \leq \frac{1}{\sqrt{5}}.$$

(5) Find a function f such that both

$$\int_0^1 f^2(t)\, dt = 1 \quad \text{and} \quad \int_0^1 t^2 f(t)\, dt = \frac{1}{\sqrt{5}}.$$