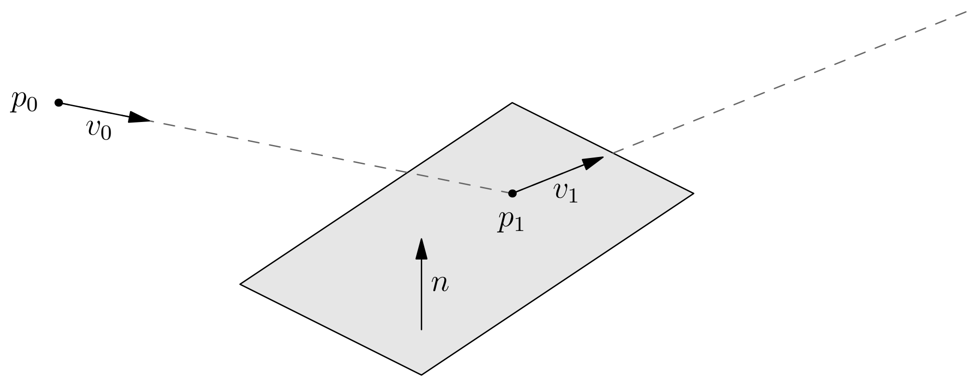
MTH 1035 — Solutions to the Computer Graphics Worksheet

Exercises.

(1) Consider a plane $Q \subset \mathbb{R}^3$ passing through the origin and with normal vector $n \in \mathbb{R}^3$. That is,

$$Q = \{q \in \mathbb{R}^3 : q \bullet n = 0\}.$$

Also consider a ray of light starting from a point $p_0 \in \mathbb{R}^3$ and pointing in the direction of $v_0 \in \mathbb{R}^3$. Describe a step-by-step procedure (an algorithm) which determines whether or not the ray of light hits the plane. If the ray hits the plane, also determine the position p_1 and direction v_1 of the ray of light when it is reflected off of the plane.



Answer. For simplicity and since we are only concerned with the direction of the ray of light, we assume in all questions that v_0 is a unit vector. We also assume the normal n is a unit vector.

The ray of light points only in the direction of v_0 and not backwards in the direction of $-v_0$ and so the ray is given by the set $\{p_0 + t v_0 : t > 0\}$. Any intersection of this set with the plane Q will satisfy

$$\begin{aligned}(p_0 + t v_0) \bullet n = 0 \quad &\Leftrightarrow \\ p_0 \bullet n + t v_0 \bullet n = 0 \quad &\Leftrightarrow \\ t v_0 \bullet n = -p_0 \bullet n.&\end{aligned}$$

We disregard the degenerate case where $v_0 \bullet n$ is zero and the ray is tangent to the plane.[1] Then, we may calculate t by the formula

$$t = \frac{-p_0 \bullet n}{v_0 \bullet n}.$$

If t is positive, the ray hits the plane. Otherwise, it does not. Moreover, the point of intersection is given by

$$p_1 = p_0 + t v_0 = p_0 - \frac{p_0 \bullet n}{v_0 \bullet n} v_0.$$

To calculate the direction of the reflected ray, split v_0 into components parallel and perpendicular to the normal n. That is $v_0 = v_\parallel + v_\perp$ where

$$v_\parallel = (v_0 \bullet n)\, n \quad \text{and} \quad v_\perp = v_0 - v_\parallel.$$

[1]When implementing this on a computer, one would have to detect and handle small values of $v_0 \bullet n$ as a special case.

Then, the component parallel to n is negated by the reflection and the other component stays the same, so $v_1 = v_\perp - v_\parallel$. One can show that this is equal to

$$v_1 = v_0 - 2(v_0 \bullet n)\, n.$$

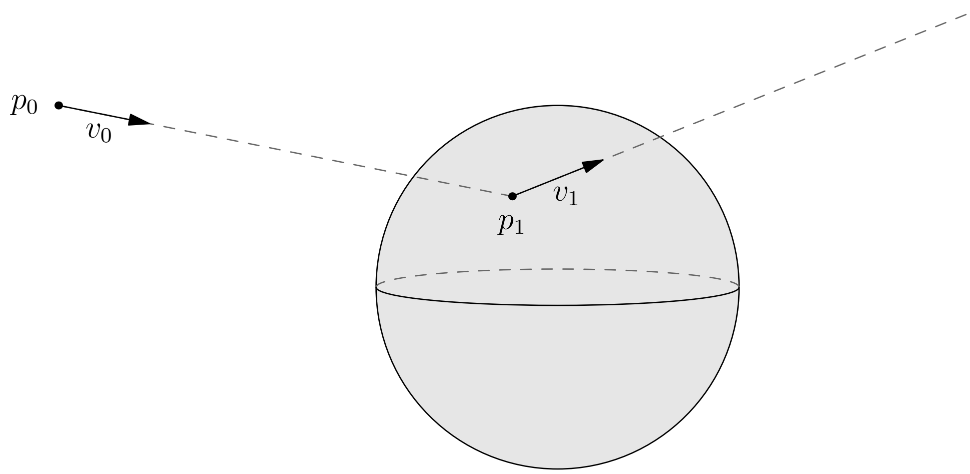
Note that this is slightly different to the formula we determined in the workshop. The main difference is whether the v_0 is pointing towards or away from the light source.

Also, this assumes the ray hits the plane on the same side from which the normal vector is pointing. If it hits the other side, we should use $-n$ in place of n. However, one can see that since n appears twice in

$$v_1 = v_0 - 2(v_0 \bullet n)\, n,$$

that v_1 will be the same even if $-n$ is used in place of n.

(2) Consider a sphere of radius $r > 0$ centered at the origin in $\mathbb{R}^3$. As before, consider a ray of light starting from a point $p_0 \in \mathbb{R}^3$ and pointing in the direction of $v_0 \in \mathbb{R}^3$. Describe an algorithm which determines whether or not the ray of light hits the sphere. If the ray hits the sphere, also determine the position p_1 and direction v_1 of the ray of light when it is reflected off of the sphere.



Answer. The sphere is given by all points q with norm $\|q\|$ equal to r. This is equivalent to $\|q\|^2 = q \bullet q = r^2$, and so we solve here for t where

$$(p_0 + t v_0) \bullet (p_0 + t v_0) = r^2.$$

Using properties of dot products, one may rewrite this as

$$(p_0 \bullet p_0) \;+\; t\,(2v_0 \bullet p_0) \;+\; t^2\,(v_0 \bullet v_0) \;=\; r^2$$

or $at^2 + bt + c = 0$ where

$$a = v_0 \bullet v_0, \quad b = 2v_0 \bullet p_0, \quad \text{and} \quad c = p_0 \bullet p_0 - r^2.$$

This is a quadratic equation and we may solve for its real roots using, say, the quadratic formula. If there are no real roots, then the ray does not hit the sphere. If there is exactly one real root, the ray is tangent to the sphere when it hits the sphere. We ignore this degenerate case.

This leaves the case where there are two distinct real roots, say $t_1 < t_2$. If both roots are negative, the ray does not hit the sphere. If both roots are positive, the ray will hit the sphere first at the smaller value t_1. Hence, the point of intersection will be $p_1 = p_0 + t_1 v_0$. Since the sphere is centered at the origin, the normal vectors point directly away from the origin and the normal vector at the point p_1 will be

$$n = \frac{1}{\|p_1\|} p_1.$$

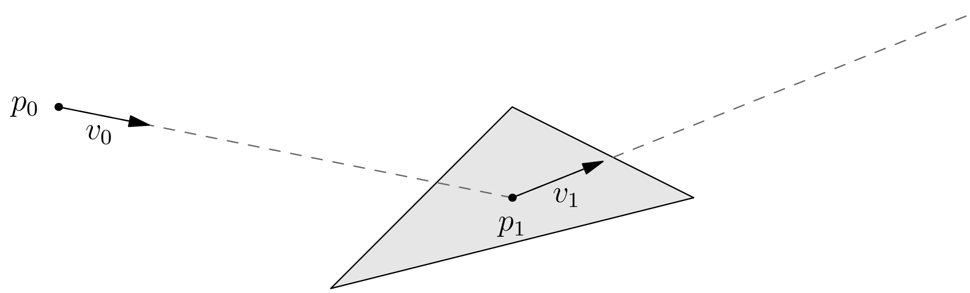
As we know the direction v_0 and normal vector n, we may compute the reflected direction exactly as in the previous question.

Finally, if the roots satisfy $t_1 < 0 < t_2$, this corresponds to the situation where the ray of light starts inside the sphere and hits the inward facing side of the sphere. Here, we may use $p_1 = p_0 + t_2 v_0$ to find the point of intersection. The normal is then

$$n = -\frac{1}{p_1} p_1$$

in this case.

(3) Consider a triangle with vertices $q_1, q_2, q_3 \in \mathbb{R}^3$. As before, consider a ray of light starting from a point $p_0 \in \mathbb{R}^3$ and pointing in the direction of $v_0 \in \mathbb{R}^3$. Describe an algorithm which determines whether or not the ray of light hits the triangle. If the ray hits the triangle, also determine the position p_1 and direction v_1 of the ray of light when it is reflected off of the triangle.



Answer. One approach is to solve for the intersection by finding real numbers t_1, t_2, and t_3 such that

$$p_0 + t_1 v_0 = q_1 + t_2(q_2 - q_1) + t_3(q_3 - q_1).$$

Writing out such a vector equation in terms of its x, y, and z coordinates gives a linear system of three equations in three unknowns that can be solved using linear algebra. (If you are curious to know more, there is an efficient method called the Möller-Trombore algorithm for solving this specific system of equations.)

As before, we require $t_1 > 0$ so that the point of intersection is in the ray pointing in the direction of v_0 instead of $-v_0$. The triangle is given by

$$\{q_1 + t_2(q_2 - q_1) + t_3(q_3 - q_1) \,:\, 0 \le t_2 \le 1,\ 0 \le t_3 \le 1,\ 0 \le t_2 + t_3 \le 1\}$$

and we may test t_2 and t_3 to see that they satisfy these inequalities. Once an intersection between the ray and triangle is known to exist, we can solve for the direction reflection exactly as in the first question. Here, the normal vector is given by the cross product

$$n = \frac{(q_2 - q_1) \times (q_3 - q_1)}{\|(q_2 - q_1) \times (q_3 - q_1)\|}.$$

rescaled to have norm equal to one.

(4) For a specific example, consider two triangles, one with vertices

$$(1,0,0) \qquad (5,2,3) \qquad (3,5,3)$$

and the other with vertices

$$(2,1,0) \qquad (7,1,2) \qquad (5,5,3).$$

Also consider a ray of light starting at the point $(1,0,-2)$ and having a direction given by the vector $(1,1,5)$. How many times does this ray of light bounce off of each triangle?

(You will likely need a computer to fully solve this question.)

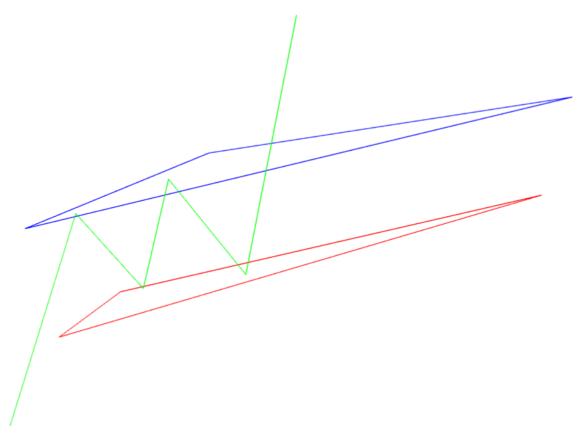
Answer. To solve this, we can apply the technique in the last question multiple times. The result is that the ray of light bounces off each triangle twice starting with the first triangle, so four reflections total. The points of reflection are, in order:

$$(1.4923, 0.4923, 0.4615)$$
$$(2.5123, 1.2545, 0.3194)$$
$$(3.1136, 1.4635, 1.7377)$$
$$(4.9030, 2.4066, 1.7942)$$

After each reflection, the new direction vector is:

$$(0.7961, 0.5949, -0.1109)$$
$$(0.3868, 0.1344, 0.9123)$$
$$(0.8843, 0.4661, 0.0279)$$
$$(0.5698, 0.1123, 0.8140)$$

We give a picture of the triangles and the ray of light.



Bonus Exercises.

(5) Suppose a sphere is made of glass with a given index of refraction. Determine the outgoing direction of a *refracted* ray of light entering the sphere.

(6) In your favorite math-capable programming language, write code for some of the algorithms above. For instance, set up two or more spheres close to each other and aim a ray towards the spheres in such a way that it reflects off each of the spheres multiple times.