MTH 1035 – Handout 1 – Lipschitz continuity

Exercise. For the following claims about real numbers a, b, c, d, which always hold? Which hold under extra assumptions?

Note: for this question (and only this question), it is enough just to say which are true without full justification or proof. In fact, some of these are *axioms* of the real numbers, meaning they cannot be proved and instead must just be assumed.

(1) If $a < b$, then $a + c < b + c$.
(2) If $a \le b$, then $a + c \le b + c$.
(3) If $a \le b$, then $a \cdot c \le b \cdot c$.
(4) If $a \le b$, then $a^2 \le b^2$.
(5) If $a^2 \le b^2$, then $a \le b$.
(6) $\sqrt{a^2} = |a|$.
(7) $(\sqrt{a})^2 = |a|$.
(8) $|a + b| = |a| + |b|$.
(9) $|a + b| \le |a| + |b|$.
(10) $|a - b + c - d| \le |a - b| + |c - d|$.
(11) $|ab| = |a|\,|b|$
(12) If $a \le b$, then $\frac{1}{b} \le \frac{1}{a}$.

Which number lies halfway between 1 and 9? In an arithmetic sense, $5 = \frac{1}{2}(1+9)$. In a geometric sense, $3 = 3^1$ is halfway between $1 = 3^0$ and $9 = 3^2$. For positive numbers x and y, define $\frac{1}{2}(x + y)$ as the *arithmetic mean* and $\sqrt{xy}$ as the *geometric mean*.

Exercise. Show the AM-GM inequality: $\sqrt{xy} \le \frac{1}{2}(x + y)$.

Exercise. Show that if $1 \le x, y \le 2$, then $|x^2 - y^2| \le 4|x - y|$.

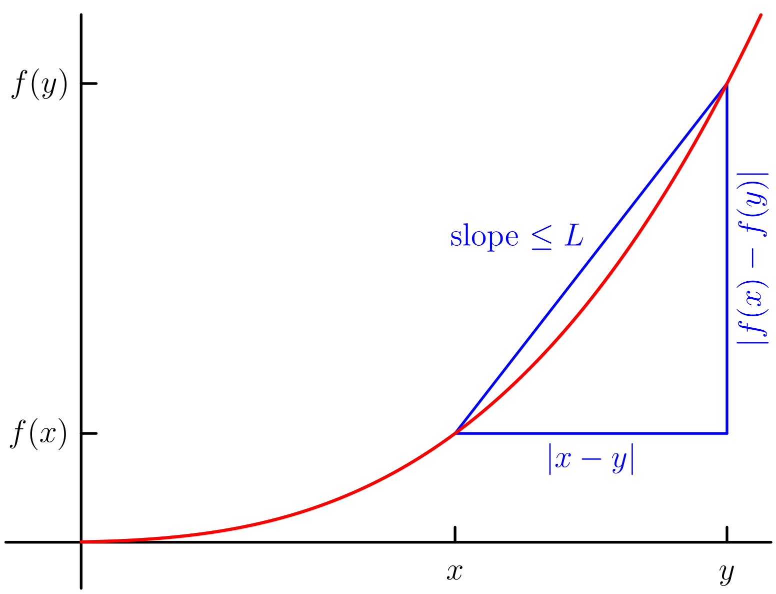
Note: the notation $1 \le x, y \le 2$ means that *both* x and y are between 1 and 2.

Let X be a subset of $\mathbb{R}$. (For example, $X = [1,2]$ or $X = [0,\infty)$ or $X = \mathbb{R}$.) A function $f : X \to \mathbb{R}$ is called *Lipschitz* if there is a constant $L > 0$ such that

$$|f(x) - f(y)| \le L|x - y|$$

for any two numbers $x, y \in X$.

For instance, the previous exercise shows that the function $f : [1,2] \to \mathbb{R}$ defined by $f(x) = x^2$ is Lipschitz with constant $L = 4$.



Exercises.

(1) Show that the function $g : [1,2] \to \mathbb{R}$ defined by $g(x) = x^3$ is Lipschitz.

(2) Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is **NOT** Lipschitz. That is, for any $L > 0$, find $x, y \in \mathbb{R}$ such that

$$|f(x) - f(y)| > L|x - y|.$$

(3) Prove the following: if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions, then the composition $f \circ g : \mathbb{R} \to \mathbb{R}$ is Lipschitz.

(4) Prove the following: if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions, then the sum $f + g : \mathbb{R} \to \mathbb{R}$ defined by $(f + g)(x) = f(x) + g(x)$ is Lipschitz.

(5) A function $f : X \to \mathbb{R}$ is *bounded* if there is a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in X$.
For instance, sin is bounded, since $|\sin(x)| \leq 1 = M$ for all $x \in \mathbb{R}$.
Give an example of a function which is Lipschitz, but not bounded.

(6) Show that any Lipschitz function $f : [a, b] \to \mathbb{R}$ defined on an interval of the form $[a, b]$ is a bounded function

(7) Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both Lipschitz functions and bounded functions. Show that the product $f \cdot g : \mathbb{R} \to \mathbb{R}$ defined by $(f \cdot g)(x) = f(x)g(x)$ is Lipschitz.

(8) Show that $h : [0, 1] \to \mathbb{R}$ given by $h(x) = \sqrt{x}$, is bounded, but not Lipschitz.

(9) Show that the absolute value function $f : \mathbb{R} \to [0, \infty)$ given by $f(x) = |x|$ is Lipschitz, but is not differentiable at zero.