

INTEGRABILITY AND LYAPUNOV EXPONENTS

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ABSTRACT. A smooth distribution, invariant under a dynamical system, integrates to give an invariant foliation, unless certain resonance conditions are present.

1. INTRODUCTION

The analysis of smooth dynamical systems is aided greatly by the existence of foliations invariant under the action of the system. In this paper, we show that smooth, invariant distributions on the tangent space typically integrate to give invariant foliations, and that special resonance conditions must be present to obstruct this.

THEOREM 1.1. *Let M be a compact Riemannian manifold, $f : M \rightarrow M$ a C^1 diffeomorphism which preserves a measure equivalent to Lebesgue, and $TM = E \oplus F$ a C^1 , Tf -invariant splitting. Suppose E is not integrable. Then at some point on M , there are a Lyapunov exponent μ of order two of E and a Lyapunov exponent λ of order one of F such that $\mu = \lambda$.*

We say that a vector $v \in TM$ is a *Lyapunov vector* if the limit

$$(1) \quad \lim_{|n| \rightarrow \infty} \frac{1}{n} \log \|Tf^n v\|$$

exists; this limit is the *Lyapunov exponent* of order one corresponding to the vector. A point $x \in M$ is *Lyapunov regular* if (among other properties) its tangent space $T_x M$ has a basis of Lyapunov vectors. The theorem of Oseledets says that the set of Lyapunov regular points is of full probability; that is, with respect to any f -invariant measure, almost every point is Lyapunov regular. A light introduction to the theory is given in [13] and proofs of Oseledets' theorem are given in [12] and [11].

At a Lyapunov regular point, a Lyapunov exponent of order k is the sum of k exponents of order one corresponding to k linearly independent vectors. If TM has a Tf -invariant splitting, then at regular points, a basis of Lyapunov vectors can be chosen that respects this splitting. Therefore, we may speak of Lyapunov exponents associated to the bundles E and F above.

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Theorem 1.1 states that if an invariant, C^1 splitting $TM = E \oplus F$ fails to be integrable, this must be due to a special type of “Lyapunov resonance” between E and F .

If X and Y are C^1 vector fields and g is a C^∞ function, then $X(Y(g))$, the derivative of the C^1 function $Y(g)$ taken in the direction of the C^1 vector field X , is well-defined and continuous. As is $Y(X(g))$. Further, there is a unique continuous vector field $[X, Y]$ such that

$$[X, Y](g) = X(Y(g)) - Y(X(g))$$

for every C^∞ function g . A C^1 distribution $E \subset TM$ is *involutive* if for any C^1 vector fields X, Y taking values in E , the bracket $[X, Y]$ also lies in E . By Frobenius’ theorem, a C^1 distribution is integrable if and only if it is involutive. Theorem 1.1 then follows from a local result about involutivity at a Lyapunov regular point:

THEOREM 1.2. *Let M be a compact Riemannian manifold, $f : M \rightarrow M$ a C^1 diffeomorphism, and $TM = E \oplus F$ a C^1 , Tf -invariant splitting. Suppose E is not involutive at a Lyapunov regular point $p \in M$. Then at p there are a Lyapunov exponent μ of order two of E and a Lyapunov exponent λ of order one of F such that $\mu = \lambda$.*

To prove the main theorem from Theorem 1.2, note that involutivity is a closed condition. If E is not integrable, it fails to be involutive on an open subset of M . Then apply Oseledets theorem to find a Lyapunov regular point in that subset.

Remark. As can be observed from the last paragraph, the condition in Theorem 1.1 that the diffeomorphism preserve a smooth measure could be replaced with the weaker assumption that the Lyapunov regular points are dense.

Remark. For the splitting $TM = E \oplus F$ we will only use that E is C^1 and that F is continuous. Frobenius’ theorem and the notion of involutivity can even be extended to the case where the distribution is Lipschitz [15, 14]. Using these techniques, we could prove Theorem 1.1 in the case where E is merely Lipschitz. This would, however, introduce annoying technicalities into the proof, so for simplicity, we assume C^1 smoothness throughout.

Section 2 discusses how Theorem 1.1 relates to other integrability results in the study of dynamical systems and gives examples of its application. Section 3 details the proof of Theorem 1.2. Section 4 gives the statement and proof of a slightly more technical result that establishes integrability without relying on a smooth invariant measure or Lyapunov regularity. Finally, Section 5 looks at the regularity of non-integrable subbundles, showing that these integrability theorems cannot be extended to the Hölder case.

2. DISCUSSION AND EXAMPLES

To understand Theorem 1.1, we apply it to examples of hyperbolic and partially hyperbolic systems.

A diffeomorphism f of a compact Riemannian manifold M is *partially hyperbolic* if there are constants $\sigma < \hat{\gamma} < 1 < \gamma < \mu$ and a Tf -invariant splitting of TM such that for every $x \in M$, the splitting $T_x M = E^u(x) \oplus E^c(x) \oplus E^s(x)$ satisfies

$$\begin{aligned} \mu \|v\| &\leq \|Tf v\| && \text{for } v \in E^u(x), \\ \hat{\gamma} \|v\| &\leq \|Tf v\| \leq \gamma \|v\| && \text{for } v \in E^c(x), \\ \|Tf v\| &\leq \sigma \|v\| && \text{for } v \in E^s(x). \end{aligned}$$

There are slightly more general definitions of partial hyperbolicity, but we use this formulation for simplicity. See [3] or [10] for a more detailed discussion of these systems.

It is often required that each of the subbundles be non-zero for the system to be truly called partially hyperbolic. If the *center* bundle, E^c , is zero, that is, $TM = E^u \oplus E^s$, then the system is called *hyperbolic* or *Anosov* [1]. It is possible, by different choices of splitting, for a system to be both hyperbolic and partially hyperbolic. These definitions are robust in that a C^1 -small perturbation of a hyperbolic (or partially hyperbolic) system is also hyperbolic (or partially hyperbolic).

In a hyperbolic or partially hyperbolic system, the *stable*, E^s , and *unstable*, E^u , subbundles always integrate to give foliations, and the study of these foliations is critical for understanding the system. The center subbundle is in some cases integrable and in other cases not. The direct sums $E^{cs} = E^c \oplus E^s$ and $E^{cu} = E^c \oplus E^u$ are similarly fickle. If the three subbundles, E^c , E^{cs} , and E^{cu} , are uniquely integrable, the system is called *dynamically coherent*. For a C^1 distribution, integrability and unique integrability are equivalent, but for less regular distributions, the distinction is important.

In [5], K. Burns and A. Wilkinson discuss dynamical coherence and present a condition under which it holds.

THEOREM 2.1 (Burns-Wilkinson [5]). *Suppose f is partially hyperbolic and has a C^2 splitting with constants as in the definition. If $\sigma < \hat{\gamma}^2$, then E^{cu} is uniquely integrable. If $\gamma^2 < \mu$, then E^{cs} is uniquely integrable. If both these inequalities hold, the intersection $E^c = E^{cu} \cap E^{cs}$ is integrable as well, and the system is dynamically coherent.*

There, C^2 smoothness was chosen for convenience, and the proof readily generalizes to the C^1 case. For a measure-preserving system, Theorem 1.1 is a generalization of Theorem 2.1, since the inequalities guarantee that a Lyapunov resonance as described in Theorem 1.1 cannot occur. In fact taking $\hat{\gamma}$ and γ to the power two in Theorem 2.1 relates directly to the use of a Lyapunov exponent of order two. In essence, the result of Burns and Wilkinson shows that integrability is achieved under certain inequalities; the results in this paper show that for integrability to fail, certain *equalities* must hold.

Examples of non-dynamically coherent systems with smooth splittings can be constructed from Lie group automorphisms [16, 17]. These satisfy resonance

conditions as described in Theorem 1.1, where Lyapunov exponents associated to Lie algebra elements X and Y sum to give the Lyapunov exponent associated to $[X, Y]$.

In general, partially hyperbolic splittings are Hölder continuous, instead of C^1 or C^2 , and it is an open question whether Theorem 2.1 is true for a general splitting. Since, as we will soon show, Theorem 1.1 is false for Hölder splittings, it is doubtful that any such generalization exists.

Dynamical coherence deals with the integrability of $E^c \oplus E^u$ and $E^c \oplus E^s$, but we can also investigate the integrability of $E^u \oplus E^s$. This is important when trying to establish the ergodicity of a partially hyperbolic system. A partially hyperbolic system is *accessible* if any two points of the manifold are joined by a continuous path that is the concatenation of a finite number of subpaths, each tangent either to E^u or E^s . The system is *center bunched* if

$$\sigma < \hat{\gamma}\gamma^{-1} \quad \text{and} \quad \gamma\hat{\gamma}^{-1} < \mu.$$

THEOREM 2.2 (Burns-Wilkinson, [6]). *If a measure-preserving, partially hyperbolic C^2 diffeomorphism is accessible and center bunched, it is ergodic.*

Many believe the technical assumption of center bunching to be extraneous here (one of the so-called Pugh-Shub conjectures). Because the inequalities for center bunching look similar to those in Theorem 2.1, one is tempted to think there is a connection between the two concepts. There are, however, examples of non-dynamically coherent, center bunched systems. These counterexamples were originally discovered by the author, and first published in [5].

In contrast to center bunching, the theorem clearly relies on the condition of accessibility. There are many examples of non-accessible systems which are also non-ergodic. These include diffeomorphisms of the form $f \times id : M \times N \rightarrow M \times N$ where f is hyperbolic. For a general system, if $E^u \oplus E^s$ is integrable, the system cannot be accessible, since any path piecewise tangent to E^u or E^s must be confined to one leaf of the foliation.

E. Rodriguez Hertz, M. A. Rodriguez Hertz, and R. Ures have shown that in the space of conservative, partially hyperbolic systems with one-dimensional center, accessibility holds on an open and dense set, and consequently that ergodicity is a generic condition [9]. In fact, for any center dimension, the subset of systems where $E^u \oplus E^s$ fails to be integrable is C^1 -open and C^r -dense ($r \geq 1$). This subset is also open and dense when we take the surrounding space to include all partially hyperbolic systems instead of just the conservative ones [4].

This abundance of non-integrability differs sharply from the special case of partially hyperbolic systems with C^1 splittings. For instance, consider a linear toral automorphism $g : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ with eigenvalues $\lambda_s < 1 < \lambda_c < \lambda_u$. The automorphism given by the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

is such an example. The partially hyperbolic splitting $T\mathbb{T}^3 = E^u \oplus E^c \oplus E^s$ is found by using the eigenspaces of the linear map. $E^u \oplus E^s$ is integrable as can be seen directly, but integrability can also be proven from Theorem 1.1, since the Lyapunov exponents are the logarithms of the eigenvalues and

$$\log(\lambda_s) + \log(\lambda_u) = -\log(\lambda_c) \neq \log(\lambda_c).$$

Further, any small perturbation of g will not change the Lyapunov exponents by much, so if the splitting of the perturbation is still C^1 , its stable and unstable subbundles will also be jointly integrable. This behaviour is markedly different from a generic perturbation of the linear system, as generically the splitting is not C^1 and the stable and unstable directions are not jointly integrable.

The toral automorphism g can be viewed as hyperbolic by grouping the E^c and E^u directions together into a two-dimensional unstable direction. Then, the diffeomorphism $g \times id : \mathbb{T}^3 \times N \rightarrow \mathbb{T}^3 \times N$ is partially hyperbolic for any choice of manifold N . This product has the Lyapunov exponents of g as well as a Lyapunov exponent of zero arising from the identity map. No two of these exponents add to any of the others, so again for a small perturbation with C^1 splitting, the stable and unstable directions remain jointly integrable, and again this contrasts the behaviour of a generic system.

As a final example, consider the toral automorphism $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ given by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

This is Arnold’s so-called “cat map.” Take the direct product $h = f^3 \times f^4 \times f^5$ defined on the six-dimensional torus. If the cat map has eigenvalues, say, $\lambda > 1$ and $\lambda^{-1} < 1$, then h has eigenvalues

$$\lambda^{-5} < \lambda^{-4} < \lambda^{-3} < \lambda^3 < \lambda^4 < \lambda^5$$

and the tangent space has a corresponding Th -invariant splitting

$$T\mathbb{T}^6 = E_{-5} \oplus E_{-4} \oplus E_{-3} \oplus E_3 \oplus E_4 \oplus E_5$$

into one-dimensional subbundles. This splitting is robust under perturbations of the system [10]. Note that no two eigenvalues have a product which is another eigenvalue; equivalently, no two Lyapunov exponents sum to give another. Therefore, in any perturbation where the splitting is still C^1 , any direct sum of any of the subbundles will be integrable. For instance, $E_{-4} \oplus E_3 \oplus E_5$ will remain integrable for a perturbation with C^1 splitting.

In this section, the author has been sloppy about restricting discussion to measure-preserving diffeomorphisms, as required by the conditions of Theorem 1.1. However, the results in Section 4 will show that this assumption is not necessary for establishing integrability in the above examples. Also, perturbations of linear systems were considered for convenience, but any system, so long as it has similar Lyapunov exponents, will enjoy the same integrability results.

We give two examples of general conditions that imply integrability.

COROLLARY 2.3. *Suppose $f : M \rightarrow M$ is a conservative, partially hyperbolic diffeomorphism with C^1 splitting. If all of the central Lyapunov exponents are zero then f is dynamically coherent.*

To prove this, note that the stable and unstable Lyapunov exponents are all non-zero.

The next corollary was suggested by F. Rodriguez Hertz.

COROLLARY 2.4. *Suppose $f : M \rightarrow M$ is a partially hyperbolic, four-dimensional symplectomorphism with C^1 splitting and two-dimensional center. Then f is dynamically coherent. Moreover, if the central Lyapunov exponents are non-zero, then E^u and E^s are jointly integrable.*

Proof. As a symplectomorphism, f preserves a smooth volume form. As it is a partially hyperbolic symplectomorphism, at a Lyapunov regular point, the Lyapunov exponents have the form $-\mu < -\lambda \leq 0 \leq \lambda < \mu$. Then $-\lambda + \lambda = 0 \neq \pm\mu$, so the center is integrable. If $\lambda \neq 0$ then $-\mu + \mu = 0 \neq \pm\lambda$ and so $E^u \oplus E^s$ is integrable. \square

All of the examples in this section serve to show that in the case of C^1 subbundles, it is entirely reasonable to expect integrability. This highlights how differently smooth subbundles behave compared to the merely Hölder continuous subbundles that generically occur.

In dynamical systems that arise in the study of mathematics or the sciences, we may be able to establish the existence of a smooth splitting. Theorem 1.1 then gives us a way to automatically deduce the existence of invariant manifolds when it may be intractable to do so directly. Finally, as the theorem deals only with Lyapunov exponents, it applies just as readily to the non-uniform generalizations of the notions of hyperbolicity and partial hyperbolicity.

In related results, X. Cabré, E. Fontich, and R. de la Llave have shown the existence of certain invariant manifolds through a fixed point on a Banach manifold in the absence of special resonance conditions somewhat similar to those in Theorem 1.1 [7]. L. Barreira and C. Valls have shown center integrability for a non-uniformly partially hyperbolic system with Lipschitz splitting under certain inequalities for both the splitting and the Lipschitz constant [2].

3. PROOF OF THEOREM 1.2

As noted in the introduction, one can define the bracket $[X, Y]$ of C^1 vector fields X and Y . Many of the identities for smooth vector fields also hold in the C^1 case. The bracket is bilinear and anti-symmetric, and for C^1 functions $a, b : M \rightarrow \mathbb{R}$, the equation

$$(2) \quad [aX, bY] = aX(b)Y - bY(a)X + ab[X, Y]$$

is satisfied. If η is a C^1 differential 1-form, we have the so-called invariant formula

$$(3) \quad d\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta([X, Y]).$$

Finally, if f is a C^1 diffeomorphism on M , then its derivative Tf acts on vector fields in a pointwise manner, and

$$(4) \quad Tf[X, Y] = [TfX, TfY].$$

In a coordinate chart, the bracket of $X = \sum_i a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_i b_i \frac{\partial}{\partial x_i}$ is given by

$$(5) \quad [X, Y] = \sum_{i,j} \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j},$$

and the identities above can be proved, quite tediously, by writing everything out in terms of coordinates.

To prove Theorem 1.2, we first establish an elementary lemma on brackets of vector fields independent of any dynamical system acting on M .

LEMMA 3.1. *Let M be a smooth manifold with C^1 , k -dimensional distribution $E \subset TM$ and suppose E is not involutive at $p \in M$. Choose a basis $\{e_1, \dots, e_k\}$ of $E_p \subset T_pM$, and a complementary subspace $F_p \subset T_pM$ where $T_pM = E_p \oplus F_p$. Then, there exist indices i and j with $1 \leq i < j \leq k$, and vector fields X_i, X_j lying in E such that $X_i(p) = e_i, X_j(p) = e_j$, and $0 \neq [X_i, X_j](p) \in F_p$.*

Remark. In this lemma and its proof, we use $X(p)$ to denote the value of a vector field at a point on the manifold. This is due to the subscripts on X_i and X_j . Later in the paper, we switch to the more compact notation X_p .

Proof. Since this lemma only involves vector fields in a neighbourhood of p , we restrict ourselves to a coordinate chart, and assume, without loss of generality, that $M = \mathbb{R}^n$ and that at the point p , the subspaces E_p and F_p align with the axes. That is, $E_p = \text{span}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\}$, where $e_i = \frac{\partial}{\partial x_i}$ and $F_p = \text{span}\{\frac{\partial}{\partial x_{k+1}}, \dots, \frac{\partial}{\partial x_n}\}$.

In a neighbourhood U of p , define $Z_i : U \rightarrow F_p$ and $X_i(x) = \frac{\partial}{\partial x_i} + Z_i(x)$ for $i = 1, \dots, k$ such that

$$E_x = \text{span}\{X_1(x), \dots, X_k(x)\}$$

for $x \in U$. To clarify, $Z_i(x)$ is in F_p independent of x and so the vector fields could be written as

$$Z_i(x) = z_{i,k+1}(x) \frac{\partial}{\partial x_{k+1}} + \dots + z_{i,n}(x) \frac{\partial}{\partial x_n}$$

where the $z_{i,\ell}$ are C^1 functions $U \rightarrow \mathbb{R}$.

For $1 \leq i, j \leq k$,

$$\begin{aligned} [X_i, X_j] &= \left[\frac{\partial}{\partial x_i} + Z_i, \frac{\partial}{\partial x_j} + Z_j \right] \\ &= \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] + \left[\frac{\partial}{\partial x_i}, Z_j \right] + \left[Z_i, \frac{\partial}{\partial x_j} \right] + [Z_i, Z_j]. \end{aligned}$$

Here the first bracket is zero, and since Z_i and Z_j are in F_p for all points in U , it can be verified from (5) that the remaining three brackets are also in F_p . Therefore, $[X_i, X_j](p) \in F_p$.

If there are i and j such that $[X_i, X_j](p)$ is non-zero, the conclusion of the lemma is satisfied. Therefore, we assume that $[X_i, X_j](p) = 0$ for all indices and show that E is involutive at that point.

If X and Y are arbitrary C^1 vector fields defined on U and tangent to E , then

$$X = \sum_{i=1}^k a_i X_i \quad \text{and} \quad Y = \sum_{j=1}^k b_j X_j$$

for C^1 functions $a_i, b_j : U \rightarrow \mathbb{R}$, and by (2),

$$\begin{aligned} [X, Y] &= \sum_{i,j} [a_i X_i, b_j X_j] \\ &= \sum_{i,j} a_i X_i (b_j) X_j - \sum_{i,j} b_j X_j (a_i) X_i + \sum_{i,j} a_i b_j [X_i, X_j]. \end{aligned}$$

The first two summations are linear combinations of the X_i and so lie in E , while in the third summation, every term is zero at p by assumption. Consequently, $[X, Y](p)$ is in E_p showing that, at the point p , the distribution is involutive. \square

We now prove Theorem 1.2.

Proof. Assume that E is not involutive at $p \in M$ and that p is Lyapunov regular. Choose a basis of E_p consisting of Lyapunov vectors and apply the lemma to get vector fields X and Y tangent to E such that X_p and Y_p are Lyapunov vectors and $0 \neq [X, Y]_p \in F_p$.

Let

$$\mu_1 = \lim_{|n| \rightarrow \infty} \frac{1}{n} \log \|Tf^n X_p\|$$

and

$$\mu_2 = \lim_{|n| \rightarrow \infty} \frac{1}{n} \log \|Tf^n Y_p\|.$$

If V is an inner product space with orthonormal basis $\{e_1, \dots, e_k\}$ then the second exterior power, $\Lambda^2(V)$, is an inner product space with orthonormal basis $\{e_1 \wedge e_2 : 1 \leq i < j \leq k\}$.

In our case, $X_p \wedge Y_p \in \Lambda^2(T_p M)$ and it is a consequence of Lyapunov regularity that the limit

$$\mu = \lim_{|n| \rightarrow \infty} \frac{1}{n} \log \|Tf^n X_p \wedge Tf^n Y_p\|$$

exists and is equal to $\mu_1 + \mu_2$. (This follows from Theorem 4 as stated in [13], which handles both the case where μ_1 and μ_2 are distinct and the case where they are equal.)

We have established that μ is a Lyapunov exponent of order two. The proof proceeds by showing that $[X, Y]_p \in F_p$ is a Lyapunov vector and that its Lyapunov exponent is equal to μ . Using that p is Lyapunov regular and that F is a Tf -invariant distribution, write the vector $[X, Y]_p \in F_p$ as

$$[X, Y]_p = u_1 + \dots + u_\ell$$

where the u_i are (non-zero) Lyapunov vectors associated to distinct Lyapunov exponents, λ_i . Due to (1), for each i , $\|Tf^n u_i\|$ is proportional to $e^{\lambda_i n}$ modulo a factor which grows sub-exponentially with $|n|$. Therefore,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|Tf^n [X, Y]_p\| = \lambda^+$$

where λ^+ is the greatest of the exponents, and

$$\lim_{n \rightarrow -\infty} \frac{1}{n} \log \|Tf^n[X, Y]_p\| = \lambda^-$$

where λ^- is the least. We will show that these two exponents are equal, and hence $[X, Y]_p$ is a Lyapunov vector. For now, the most we can say is that $\lambda^- \leq \lambda^+$.

To compare μ to λ^+ and λ^- , we adapt the proof of Theorem 2.1 as given in [8]. Let q be in $\omega(p)$, the omega limit set of p , and take $n_j \rightarrow +\infty$ such that $f^{n_j}(p) \rightarrow q$. Define

$$v_j = \frac{Tf^{n_j}[X, Y]_p}{\|Tf^{n_j}[X, Y]_p\|}$$

and assume $v_j \rightarrow v$ for some $v \in T_qM$ by replacing n_j with another subsequence. Because F is Tf -invariant and closed, this limit vector v also lies in F .

In particular, v is a non-zero vector lying outside of E . Let η be a 1-form defined in a neighbourhood of q such that $E \subset \ker \eta$ when restricted to this neighbourhood and $\eta(v) \neq 0$.

For large j , by identities (3) and (4),

$$\begin{aligned} d\eta(Tf^{n_j}X, Tf^{n_j}Y) &= (Tf^{n_j}X)\eta(Tf^{n_j}Y) - (Tf^{n_j}Y)\eta(Tf^{n_j}X) \\ &\quad - \eta(Tf^{n_j}[X, Y]). \end{aligned}$$

Since $Tf^{n_j}X$ and $Tf^{n_j}Y$ are in $E \subset \ker \eta$, the first two terms on the right hand side are zero, so

$$(6) \quad |d\eta(Tf^{n_j}X, Tf^{n_j}Y)| = |\eta(Tf^{n_j}[X, Y])|.$$

Now,

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{1}{n_j} \log |\eta(Tf^{n_j}[X, Y]_p)| &= \lim_{j \rightarrow \infty} \frac{1}{n_j} \log (\|Tf^{n_j}[X, Y]_p\| \cdot |\eta(v_j)|) \\ &= \lim_{j \rightarrow \infty} \frac{1}{n_j} \log \|Tf^{n_j}[X, Y]_p\| + \lim_{j \rightarrow \infty} \frac{1}{n_j} \log |\eta(v_j)| \\ &= \lambda^+ \end{aligned}$$

since $\eta(v_j) \rightarrow \eta(v) \neq 0$.

The 2-form $d\eta_x : T_xM \times T_xM \rightarrow \mathbb{R}$ descends to a linear map $d\eta_x : \Lambda^2(T_xM) \rightarrow \mathbb{R}$ where

$$d\eta(u_1 \wedge u_2) = d\eta(u_1, u_2).$$

Define

$$w_j = \frac{Tf^{n_j}X_p \wedge Tf^{n_j}Y_p}{\|Tf^{n_j}X_p \wedge Tf^{n_j}Y_p\|}.$$

Then,

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{1}{n_j} \log |d\eta(Tf^{n_j}X_p \wedge Tf^{n_j}Y_p)| &= \lim_{j \rightarrow \infty} \frac{1}{n_j} \log (\|Tf^{n_j}X_p \wedge Tf^{n_j}Y_p\| \cdot |d\eta(w_j)|) \\ &= \mu + \lim_{j \rightarrow \infty} \frac{1}{n_j} \log |d\eta(w_j)|. \end{aligned}$$

We cannot deduce directly from this that $\lambda^+ = \mu$, because it is possible that $d\eta(w_j) \rightarrow 0$ so that $\log|d\eta(w_j)| \rightarrow -\infty$. Since $d\eta$ is a continuous 2-form, we do know that $\log|d\eta(w_j)|$ is bounded above, so

$$\lim_{j \rightarrow \infty} \frac{1}{n_j} \log|d\eta(w_j)| \leq 0$$

and it follows from (6) that $\lambda^+ \leq \mu$.

In a similar manner, we can construct a subsequence $n_j \rightarrow -\infty$ and deduce that $\mu \leq \lambda^-$. Then

$$\mu \leq \lambda^- \leq \lambda^+ \leq \mu,$$

so the three exponents are equal and the theorem is proved. \square

4. INTEGRABILITY WITHOUT LYAPUNOV REGULARITY

If the system does not satisfy Lyapunov regularity on a dense set, a similar result still holds, but the formulation is more technical.

THEOREM 4.1. *Suppose $f : M \rightarrow M$ is a C^1 diffeomorphism on a compact, Riemannian manifold, and the tangent bundle has a C^1 , Tf -invariant splitting*

$$TM = \underbrace{E_1 \oplus \cdots \oplus E_r}_E \oplus \underbrace{F_1 \oplus \cdots \oplus F_s}_F$$

with positive constants a_i, b_i for $i \in \{1, \dots, r\}$ and c_ℓ, d_ℓ for $\ell \in \{1, \dots, s\}$ such that

$$a_i \|v\| \leq \|Tf v\| \leq b_i \|v\| \quad \text{for } v \in E_i$$

and

$$c_\ell \|v\| \leq \|Tf v\| \leq d_\ell \|v\| \quad \text{for } v \in F_\ell.$$

Suppose E is not involutive. Then, there are $i, j \in \{1, \dots, r\}$ and $\ell \in \{1, \dots, s\}$ such that the intervals $[a_i a_j, b_i b_j]$ and $[c_\ell, d_\ell]$ intersect.

Moreover, if E_i is one-dimensional, then $i \neq j$.

Remark. We can obtain Theorem 2.1 as a special case of this result. To show the center is integrable, set

$$E = E_1 = E^c, \quad F = F_1 \oplus F_2 = E^u \oplus E^s$$

and

$$[a_1, b_1] = [\hat{\gamma}, \gamma], \quad [c_1, d_1] = [\mu, N], \quad \text{and} \quad [c_2, d_2] = [N^{-1}, \lambda],$$

where N is sufficiently large. The inequalities of Theorem 2.1 guarantee that $[a_1 a_1, b_1 b_1] = [\hat{\gamma}^2, \gamma^2]$ does not intersect the other intervals. Thus, $E = E^c$ is integrable.

Proof of Theorem 4.1. Without loss of generality, assume the intervals $[c_\ell, d_\ell]$ are disjoint. If, say $[c_1, d_1]$ and $[c_2, d_2]$ intersect, replace F_1 and F_2 by $F_{12} = F_1 \oplus F_2$ and associate to it the interval

$$[c_{12}, d_{12}] = [\min(c_1, c_2), \max(d_1, d_2)].$$

If an interval $[a_i a_j, b_i b_j]$ intersects this new interval, it must do so by intersecting one of the original intervals. By repeating this joining of subbundles, reduce to the case where all of the $[c_\ell, d_\ell]$ are disjoint.

Suppose E is not involutive at a point $p \in M$. Then, by Lemma 3.1, there are X and Y defined in a neighbourhood of p such that X and Y are contained in E , $X_p \in E_i(p)$ and $Y_p \in E_j(p)$ for indices i, j , and $0 \neq [X, Y]_p \in F_p$. These will be the indices used in the conclusion of the theorem. Moreover, as the lemma yields vectors X_p and Y_p which are linearly independent, if E_i is one-dimensional, then $i \neq j$.

Since X_p and Y_p are not necessarily Lyapunov vectors, we need to restrict to a subsequence in order to continue the proof. Let $u = [X, Y]_p \in T_p M$.

LEMMA 4.2. *There is a bi-infinite subsequence $\{n_k\}_{k \in \mathbb{Z}}$ such that*

$$\lim_{k \rightarrow -\infty} n_k = -\infty \quad \text{and} \quad \lim_{k \rightarrow +\infty} n_k = +\infty,$$

the limits

$$\lambda^+ = \lim_{k \rightarrow +\infty} \frac{1}{n_k} \log \|Tf^{n_k} u\|, \quad \text{and} \quad \lambda^- = \lim_{k \rightarrow -\infty} \frac{1}{n_k} \log \|Tf^{n_k} u\|$$

exist, and either there is $\ell \in \{1, \dots, s\}$ such that

$$(7) \quad e^{\lambda^-}, e^{\lambda^+} \in [c_\ell, d_\ell],$$

or there are $\ell_1, \ell_2 \in \{1, \dots, s\}$ such that

$$(8) \quad c_{\ell_1} \leq e^{\lambda^-} \leq d_{\ell_1} < c_{\ell_2} \leq e^{\lambda^+} \leq d_{\ell_2}.$$

Proof. Decompose $u \in F(p)$ into components $u = \sum_{i=1}^s u_i$ where $u_i \in F_i(p)$. Then, there are indices i^- and i^+ , possibly equal, such that the components u_{i^-} and u_{i^+} are both non-zero, and for all other indices i , either $u_i = 0$ or

$$d_{i^-} < c_i < d_i < c_{i^+}.$$

As $n \rightarrow +\infty$, $Tf^n u_{i^+}$ is the fastest growing component of $Tf^n u$ (or the slowest shrinking component if $\|Tf^n u\| \rightarrow 0$). $Tf^n u_{i^-}$ is the slowest growing (or fastest shrinking) component of the decomposition.

By looking at the growth rates, one sees that

$$\lim_{n \rightarrow -\infty} \frac{\|Tf^n u_{i^-}\|}{\|Tf^n u\|} = 1 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \frac{\|Tf^n u_{i^+}\|}{\|Tf^n u\|} = 1.$$

Take a bi-infinite subsequence $\{n_k\}$ of the integers such that the limits

$$\lim_{k \rightarrow -\infty} \frac{1}{n_k} \log \|Tf^{n_k} u\| = \lim_{k \rightarrow -\infty} \frac{1}{n_k} \log \|Tf^{n_k} u_{i^-}\|$$

and

$$\lim_{k \rightarrow +\infty} \frac{1}{n_k} \log \|Tf^{n_k} u\| = \lim_{k \rightarrow +\infty} \frac{1}{n_k} \log \|Tf^{n_k} u_{i^+}\|$$

exist. Since $Tf^{n_k} u_{i^\pm} \in F_{i^\pm}$ for all k , the two limits are in the corresponding intervals $[\log c_{i^\pm}, \log d_{i^\pm}]$. If the indices are equal, then (7) holds with $\ell = i^- = i^+$. Otherwise, (8) holds with $\ell_1 = i^-$ and $\ell_2 = i^+$. This finishes the proof of the lemma. \square

For now, just consider the tail of the bi-infinite sequence $\{n_k\}$ which tends to positive infinity. By further restricting the subsequence, assume that $f^{n_k}(p)$ converges to a point $q \in M$, that

$$\frac{Tf^{n_k}u}{\|Tf^{n_k}u\|} = \frac{Tf^{n_k}[X, Y]_p}{\|Tf^{n_k}[X, Y]_p\|}$$

converges to a vector $v \in F_q$, and that the limits

$$\begin{aligned}\mu_1 &= \lim_{k \rightarrow +\infty} \frac{1}{n_k} \log \|Tf^{n_k}X_p\|, \\ \mu_2 &= \lim_{k \rightarrow +\infty} \frac{1}{n_k} \log \|Tf^{n_k}Y_p\|, \quad \text{and} \\ \mu &= \lim_{k \rightarrow +\infty} \frac{1}{n_k} \log \|Tf^{n_k}X_p \wedge Tf^{n_k}Y_p\|\end{aligned}$$

exist. Note that $\mu \leq \mu_1 + \mu_2$ and, a priori, this inequality may be strict if the angle between $Tf^{n_k}X_p$ and $Tf^{n_k}Y_p$ tends to zero.

Define a 1-form η in a neighbourhood of q such that $E \subset \ker \eta$ and $\eta(v) \neq 0$. Then, using equation (6) and the same analysis as in the proof of Theorem 1.2, $\lambda^+ \leq \mu$ and therefore

$$\lambda^+ \leq \mu \leq \mu_1 + \mu_2 \leq \log b_i + \log b_j \Rightarrow e^{\lambda^+} \leq b_i b_j$$

Looking at n_k as $k \rightarrow -\infty$ and restricting to an appropriate subsequence, one also sees that $a_i a_j \leq e^{\lambda^-}$.

Lemma 4.2 shows that $[a_i a_j, b_i b_j]$ intersects one of the intervals $[c_\ell, d_\ell]$, as follows. If (7) holds, either $e^{\lambda^-} \leq e^{\lambda^+}$, in which case

$$\emptyset \neq [e^{\lambda^-}, e^{\lambda^+}] \subset [a_i a_j, b_i b_j] \cap [c_\ell, d_\ell]$$

or $e^{\lambda^+} < e^{\lambda^-}$, in which case

$$[a_i a_j, b_i b_j] \cap [e^{\lambda^+}, e^{\lambda^-}] \subset [a_i a_j, b_i b_j] \cap [c_\ell, d_\ell]$$

where the left hand side is non-empty as both $a_i a_j$ and e^{λ^+} are bounded above by $b_i b_j$ and by e^{λ^-} . If (8) holds, then $[a_i a_j, b_i b_j]$ contains $[e^{\lambda^-}, e^{\lambda^+}]$ and therefore intersects both $[c_{\ell_1}, d_{\ell_1}]$ and $[c_{\ell_2}, d_{\ell_2}]$. \square

5. HÖLDER COUNTEREXAMPLES

Let M be a smooth manifold. A subbundle $E \subset TM$ is *Hölder with exponent θ* if there are $H, \delta > 0$ such that

$$(9) \quad d(p, q) < \delta \Rightarrow \|E_p - E_q\| \leq H d(p, q)^\theta$$

for $p, q \in M$. Here, the distance between the subspaces E_p and E_q is defined by restricting to a coordinate chart so that $T_p M$ and $T_q M$ may be identified. While the constant H depends on the choice of coordinate chart, the exponent θ does not.

It is a consequence of the Hölder Section Theorem in [10] that the subbundles of a dominated splitting are Hölder continuous:

THEOREM 5.1 (Hirsch-Pugh-Shub [10]). *Suppose $f : M \rightarrow M$ is a diffeomorphism with invariant splitting $TM = E \oplus F$ and there are $0 < a < b < c$ such that*

$$\begin{aligned} c\|v\| &\leq \|Tf v\| && \text{for } v \in E, \\ a\|v\| &\leq \|Tf v\| \leq b\|v\| && \text{for } v \in F. \end{aligned}$$

If $0 < \theta \leq 1$ satisfies $b < a^\theta c$, then $E \subset TM$ is Hölder with exponent θ .

We use this to establish the Hölder regularity of the partially hyperbolic splittings of perturbations of linear toral automorphisms:

COROLLARY 5.2. *Let $g : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a hyperbolic toral automorphism given by a 3×3 matrix with eigenvalues $\lambda_s < \lambda_c < \lambda_u$. Suppose $0 < \theta < 1$ satisfies*

$$(10) \quad \lambda_u^\theta \lambda_s < \lambda_c < \lambda_s^\theta \lambda_u.$$

Then, there is a C^1 neighbourhood U of partially hyperbolic diffeomorphisms near g such that the stable and unstable subbundles of any $g' \in U$ are Hölder with exponent θ .

To see how the corollary follows from the theorem, suppose $\lambda_c < \lambda_s^\theta \lambda_u$. Then, there is $\epsilon > 0$ such that $b < a^\theta c$ where $a = \lambda_s - \epsilon$, $b = \lambda_c + \epsilon$, and $c = \lambda_u + \epsilon$. In a C^1 neighbourhood of g , any perturbation will satisfy the conditions of Theorem 5.1 with $E = E^u$, $F = E^s \oplus E^c$ and a, b, c as above. Thus, E^u is Hölder with exponent θ . To see that E^s is Hölder with the same exponent, consider f^{-1} .

As stated in Section 2, $E^u \oplus E^s$ is non-integrable for an open and dense set of partially hyperbolic systems. For any given $\theta < 1$, one can find a 3×3 matrix whose eigenvalues satisfy (10). Thus, for any $\theta < 1$, there is a partially hyperbolic system $g' : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ having a non-integrable subbundle $E^u \oplus E^s$ which is Hölder with exponent θ . For small enough perturbations, the derivative of the new system closely approximates that of the linear one, and since the linear system does not satisfy the resonance conditions given in the conclusions of Theorem 1.1 and Theorem 4.1 neither does g' . This shows that these two theorems do not hold in the case where the subbundle is Hölder continuous, even for Hölder exponents arbitrarily close to one.

Given the divide between Hölder and Lipschitz regularity, one could ask about the integrability of subbundles which are C^{Lip^-} , that is, Hölder for every exponent $\theta < 1$ (but not necessarily with the same constant H for each θ). For example, if a three-dimensional, partially hyperbolic system acts as an isometry in the center direction ($\|Tf v\| = \|v\|$ for all $v \in E^c$), the results in [10] guarantee that the unstable and stable subbundles are C^{Lip^-} , and may or may not be jointly integrable. In a measure-preserving system, the condition of isometry on the center implies

$$\|Tf^n v^s\| = \frac{1}{\|Tf^n v^u\|}$$

for unit vectors $v^s \in E_p^s$ and $v^u \in E_p^u$. Therefore, at any Lyapunov regular point, the stable Lyapunov exponent is the negative of the unstable one. As the central Lyapunov exponent is zero, a Lyapunov resonance as described in Theorem 1.1

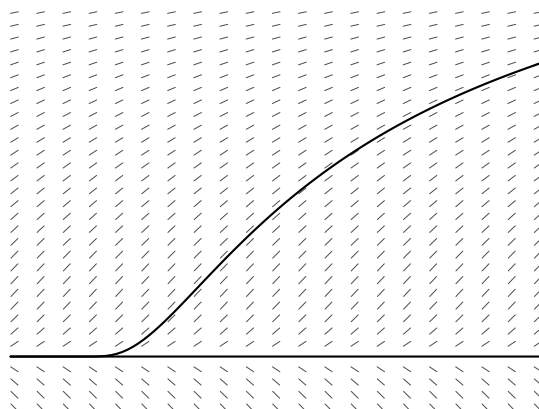


FIGURE 1. The line field of slope $\phi(y)$ at (x, y) gives a C^{Lip^-} vector field on the plane which is not uniquely integrable. Tangent curves are given by horizontal lines where $\phi(y) = 0$ and by the graphs of functions $x \mapsto e^{\frac{-1}{x-c}}$ for $c \in \mathbb{R}$.

is unavoidable. In general, the conditions for which the Hölder Section Theorem gives C^{Lip^-} splittings of the tangent bundle seem too restrictive to use this technique to construct a resonance-free counterexample to integrability.

A one-dimensional, invariant subbundle has no Lyapunov exponents of order two. As such, we may interpret as a consequence of Theorem 1.1 the known fact that any C^1 line field (automatically invariant under the identity map) is uniquely integrable. There are C^{Lip^-} line fields which are not uniquely integrable. For example, define $\phi : \mathbb{R} \rightarrow \mathbb{R}$ by $\phi(0) = 0$ and $\phi(y) = y [\log|y|]^2$ elsewhere. Then the line field given by a line of slope $\phi(y)$ at each point $(x, y) \in \mathbb{R}^2$ is C^{Lip^-} and, as illustrated in Figure 5, there is no foliation tangent to it. Denote this line field by E^h , and the vertical line field by E^v . Then, $E^h \oplus E^v$ gives a splitting of the tangent bundle of \mathbb{R}^2 . Restrict to $[-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ and identify the edges to give a splitting defined over the compact manifold \mathbb{T}^2 . This example, with f as the identity, shows that Theorem 1.1 is false if the C^1 condition on the splitting is replaced by C^{Lip^-} .

Using the identity map as an illustrative example of a dynamical system is, of course, highly questionable. For a slightly less dubious example, take $g : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ to be a toral automorphism with hyperbolic splitting $T\mathbb{T}^3 = E^u \oplus E^s$. Then $g \times id$ is a partially hyperbolic system defined on $\mathbb{T}^5 = \mathbb{T}^3 \times \mathbb{T}^2$, and the C^{Lip^-} splitting $T\mathbb{T}^5 = E \oplus F$ with $E = E^h \oplus E^u$ and $F = E^v \oplus E^s$ is free of the resonance conditions in Theorem 1.1. Still, E is not uniquely integrable.

One might ask if there is a C^{Lip^-} counterexample where the spectrum of $Tf|_E$ is disjoint from the spectrum of $Tf|_F$, or even a Hölder counterexample where the splitting $TM = E \oplus F$ is dominated. The cases presented in this section, however, suggest that Theorem 2.1 and its generalizations in Theorems 1.1 and 4.1

rely critically on Frobenius' Theorem, and cannot be meaningfully extended beyond the Lipschitz case.

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