

Vector Fields

A vector field is a smooth map

$$X : M \rightarrow TM$$

such that  $\pi \circ X = \text{id}_M$

i.e.,

$$\begin{array}{ccc} & & TM \\ & \nearrow X & \downarrow \pi \\ M & \xrightarrow{\text{id}} & M \end{array}$$

commutes.

The space of all vector fields on  $M$  is denoted  $\mathfrak{X}(M)$ .

Each vector field defines a derivation of  $C^\infty(M)$ :

$$X : C^\infty(M) \rightarrow C^\infty(M)$$

$$\text{by } (X(f))_p = X_p(f).$$

This satisfies the following

(i)  $X$  is  $\mathbb{R}$ -linear:  $X(af + bg) = aX(f) + bX(g)$

(ii) Leibniz rule:  $X(fg) = f \cdot X(g) + g \cdot X(f)$ .

for all  $a, b \in \mathbb{R}$ ,  $f, g \in C^\infty(M)$ .

derivations on  $C^\infty(M) \iff$  vector fields.

Exercise: Given  $X, Y \in \mathfrak{X}(M)$ ,

show that

$$[X, Y](f) := X(Y(f)) - Y(X(f))$$

defines a derivation, and hence a vector field.

This map

$$[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$$

is called the Lie bracket.

Ex in  $\mathbb{R}^3$

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}$$

Properties of the Lie bracket

Bilinear:

(i)  $[X+Y, Z] = [X, Z] + [Y, Z]$

$[X, Y+Z] = [X, Y] + [X, Z]$

if  $c \in \mathbb{R}$ , then

$[cX, Y] = c[X, Y] = [X, cY]$

(ii) anti-symmetric:

$[X, Y] = -[Y, X]$

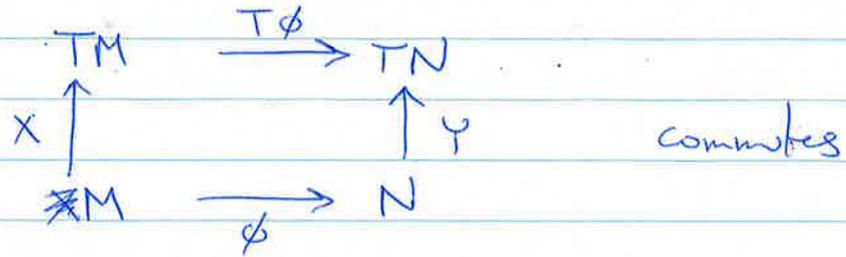
(iii) the Jacobi identity

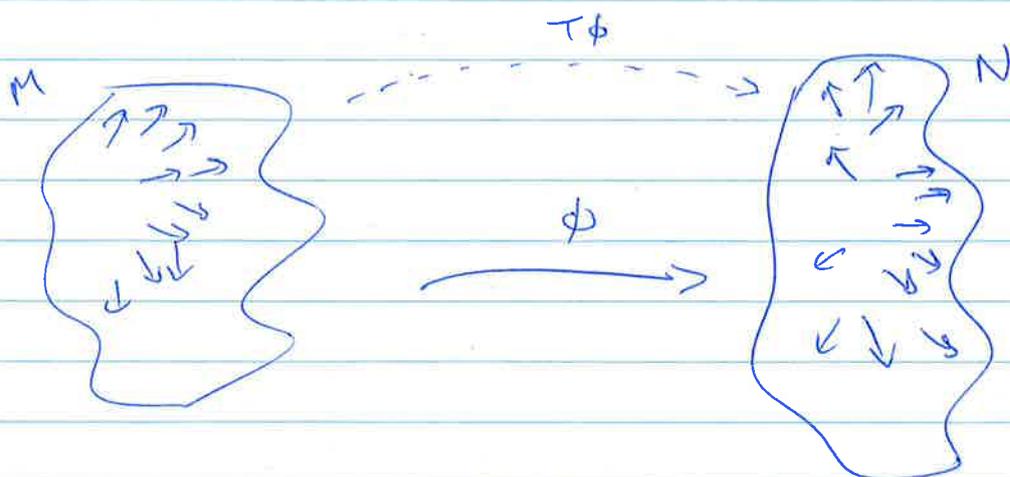
$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Def Let  $\phi: M \rightarrow N$  be smooth.

Vector fields  $X \in \mathcal{X}(M)$  and  $Y \in \mathcal{X}(N)$

are  $\phi$ -related if





Prop  $X$  and  $Y$  are  $\phi$ -related

if and only if

$$X(g \circ \phi) = Y(g) \circ \phi \quad \text{for all } g \in C^\infty(M)$$

(Note: Write  $X_p$  for  $X(p) \in T_p M$ .)

proof

$$X(g \circ \phi) = Y(g) \circ \phi \quad \forall g$$

$$\Leftrightarrow \cancel{X_p(g \circ \phi) = Y(g) \circ \phi}$$

$$T\phi(X)(g) = Y(g) \circ \phi \quad \forall g \in C^\infty(M)$$

$$\Leftrightarrow \cancel{T\phi(X_p)(g) = Y(g) \circ \phi}$$

$$T\phi(X_p)(g) = Y_{\phi(p)}(g) \quad \forall g \in C^\infty(M), p \in M$$

$$\Leftrightarrow T\phi(X_p) = Y_{\phi(p)} \quad \forall p \in M$$

$$\Leftrightarrow T\phi \circ X = Y \circ \phi \quad \square$$

Corollary

If  $X_1$  is  $\phi$ -related to  $Y_1$

and  $X_2$  is  $\phi$ -related to  $Y_2$ ,

then  $[X_1, X_2]$  is  $\phi$ -related to  $[Y_1, Y_2]$ .

proof

$$\begin{aligned} X_1(X_2(g \circ \phi)) &= X_1(Y_2(g) \circ \phi) \\ &= Y_1(Y_2(g)) \circ \phi \end{aligned}$$

Similarly  $X_2(X_1(g \circ \phi)) = Y_2(Y_1(g)) \circ \phi$

so  $[X_1, X_2](g \circ \phi) = ([Y_1, Y_2](g)) \circ \phi$

for all  $g \in C^\infty(M)$ . □

# Integral Curves

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Def ~~A~~ Let  $I \subset \mathbb{R}$  be an open interval ( $I = (a, b)$  or  $(-\infty, b)$  or  $(a, +\infty)$ , or  $\mathbb{R}$ ).

A smooth map from  $\gamma: I \rightarrow M$  is called a smooth curve.

Note that for  $s \in I$ ,

$$\gamma \# \mapsto \left. \frac{d}{dt} \right|_{t=s} \gamma(t)$$

is a derivation.

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$$\text{That is } \left. \frac{d}{dt} \right|_{t=s} \in T_s I.$$

Define a curve  $\gamma': I \rightarrow TM$  by  $\gamma'(s) = \# T\gamma \left( \left. \frac{d}{dt} \right|_{t=s} \right)$ .

Note that for  $f \in C^\infty(M)$

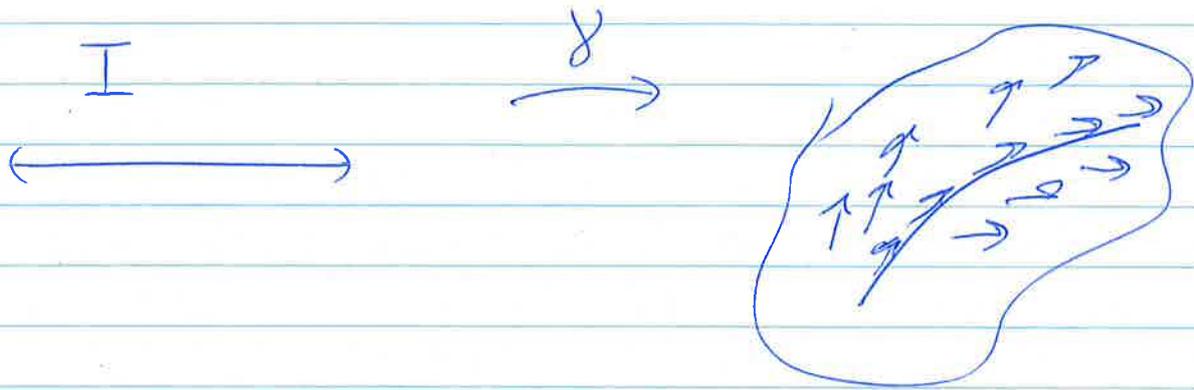
$$\gamma'(\#s)(f) = \left. \frac{d}{dt} \right|_{t=s} f(\gamma(t)).$$

Let  $X$  be a v. field on  $M$  (7)

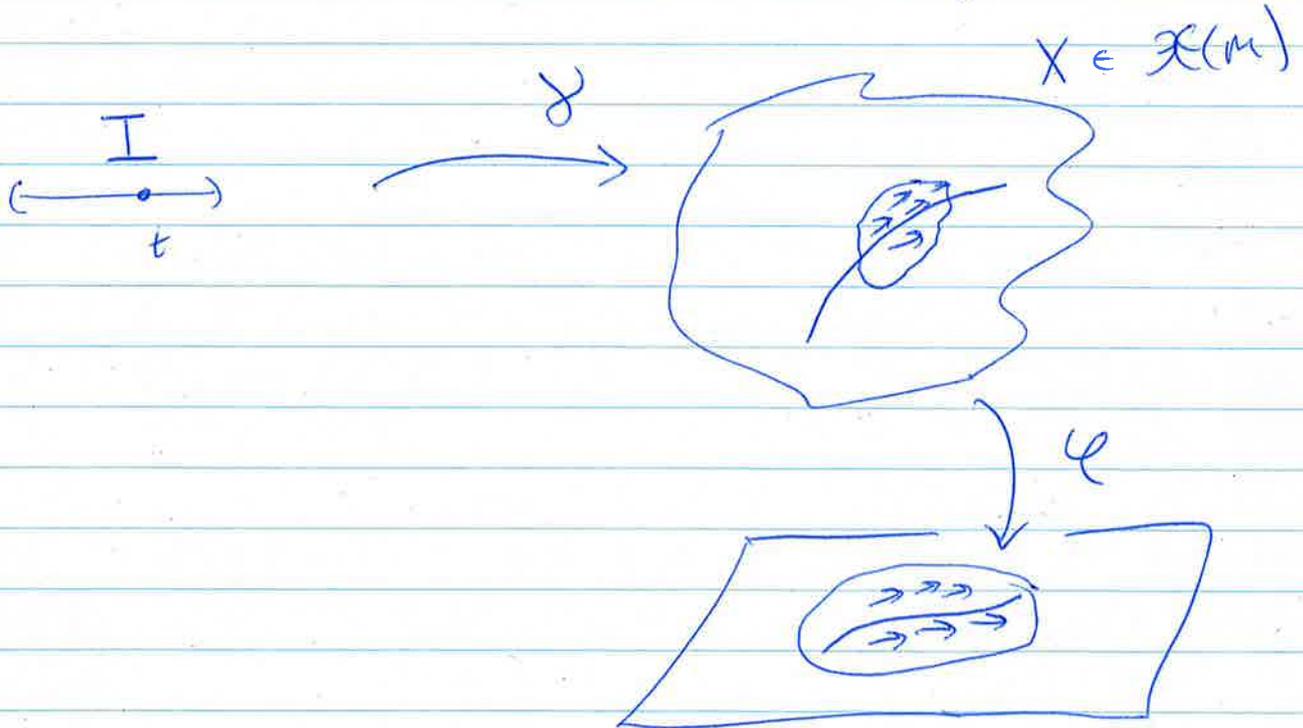
Def A curve  $\gamma: I \rightarrow M$

is an integral curve of  $X$

if  $\gamma'(t) = X(\gamma(t))$  for all  $t \in I$ .



Consider a chart  $(U, \varphi)$



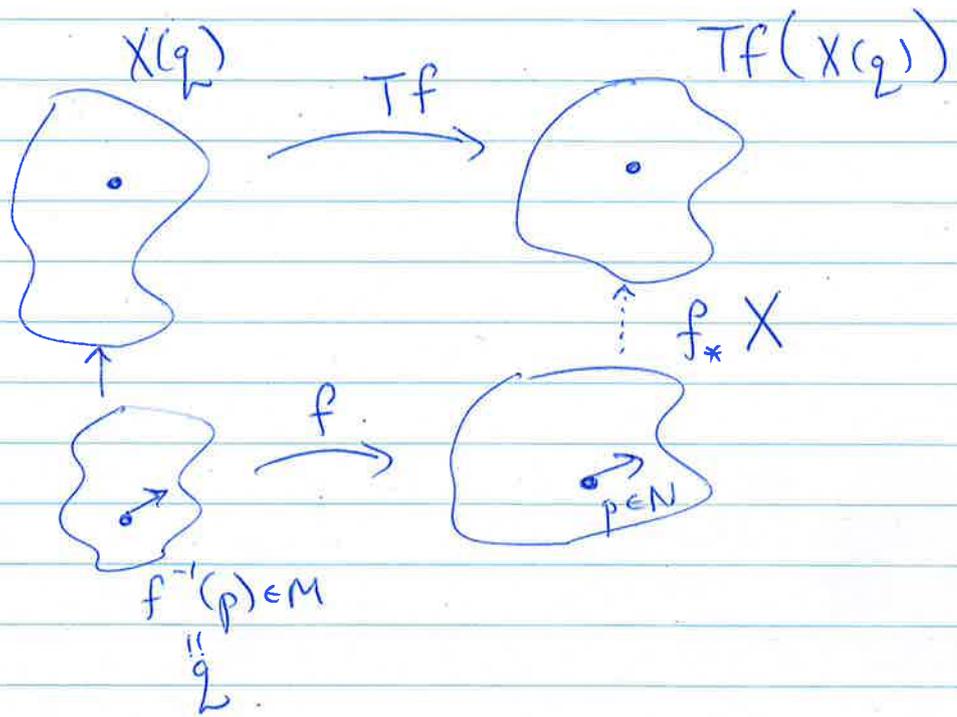
$\varphi_* X$   
v. field on a  
subset of  $\mathbb{R}^m$ .

Suppose  $f: M \rightarrow N$  is a diffeo.

Then  $Tf: TM \rightarrow TN$  is also a diffeo

Let  $X$  be a v. field on  $M$ .

How do we define a v. field on  $N$ ?



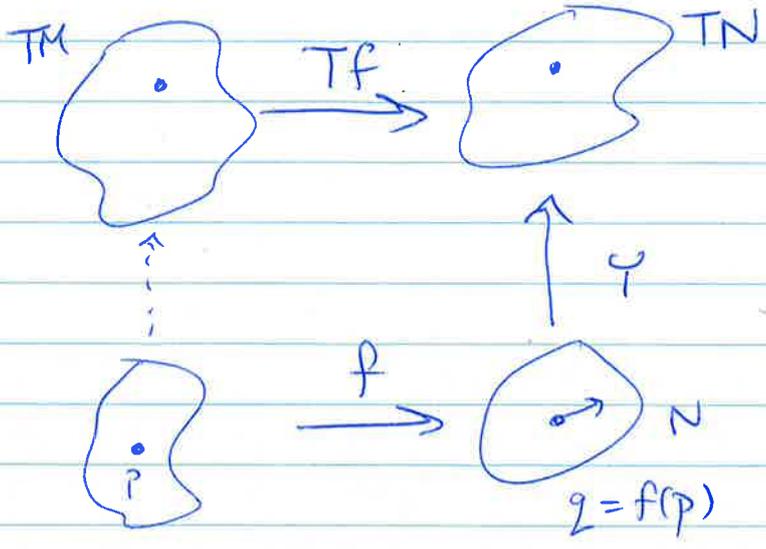
Define the push forward of  $X$

by

$$f_* X = Tf \circ X \circ f^{-1} \in \mathcal{X}(N)$$

↑ Note subscript.

If instead we have a vector field  $Y$  on  $N \neq M$

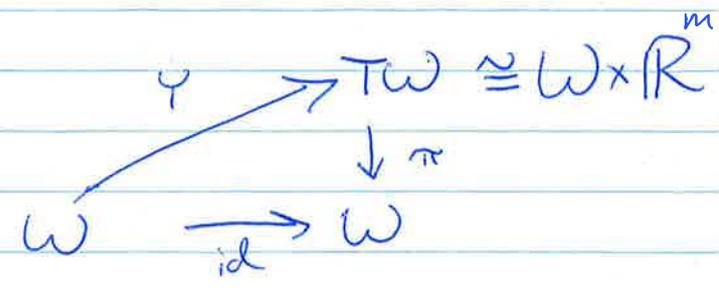


Define the pull back of  $Y$  by

$$f^*Y = Tf^{-1} \circ Y \circ f \in \mathcal{X}(M).$$

A v. field  $\Upsilon$  on a subset  $W \subset \mathbb{R}^m$

is a section

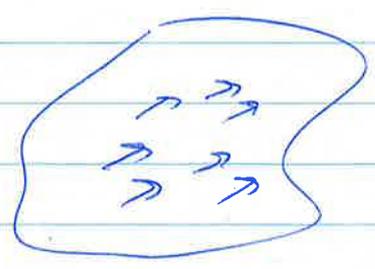


Each <sup>tangent</sup> vector  $\Upsilon(p) \in T_p W$

may be identified with a Euclidean <sup>tangent</sup> vector in  $\mathbb{R}^m$ ,

so  $\Upsilon$  may be described by

functions  $\Upsilon^1, \Upsilon^2, \dots, \Upsilon^m : W \rightarrow \mathbb{R}$



The curve  $\varphi \circ \gamma : I \rightarrow W$  is satisfies an integral curve for

$\Upsilon$ . That is,  $(\varphi \circ \gamma)'(t) = \Upsilon(\varphi \circ \gamma(t))$

Say  $\alpha = \varphi \circ \gamma$ . Then

$$\alpha : I \rightarrow W \subset \mathbb{R}^m$$

satisfies  $\alpha'(t) = Y(\alpha(t))$ . (11)

Write the coords of  $\alpha$  as

$$\alpha = (\alpha^1, \alpha^2, \dots, \alpha^m).$$

Then this become

$$\frac{d\alpha^i}{dt} = Y^i(\alpha(t)).$$

This is an ODE on a subset of  $\mathbb{R}^m$ .

Since  $Y$  is smooth, a local solution exists and is unique (Picard - Lindelöf).

Prop For a v. field  $X \in \mathcal{X}(M)$  and point  $p \in M$ , there exists an open interval  $I$  about 0 and an integral curve  $\gamma: I \rightarrow M$  such that  $\gamma(0) = p$ .

Further, if  $\beta: J \rightarrow M$  is also  
 an integral curve with  $\beta(0) = p$ ,  
 then  $\gamma(t) = \beta(t)$  for all  $t \in I \cap J$ .

proof Charts + Picard-Lindelöf.

Remark We may take the  
 union of all domains of  
 all integral curves through  
 $p$ , then we get the  
maximal integral curve  
 through  $p$ .

Moreover, if maximal integral  
 curves  $\alpha, \beta$  for points  $p$  and  $q$   
 have a common point of  
 intersection  $\alpha(t_1) = \beta(t_2) \in M$   
 then  $\beta \circ \alpha^{-1}(t) = \alpha(t + c)$   
 where  $c = t_2 - t_1$ .

Exercise ~~Sho~~

Def A vector field is complete if every maximal integral curve is defined on  $\mathbb{R}$ .

Exercises (i) Show that the v. field

$$X = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$

on  $\mathbb{R}^2$  is complete.

(ii) Show that the v. field

$$X = x^2 \frac{\partial}{\partial x}$$

on  $\mathbb{R}$  is NOT complete.

A v. field generates a flow.

First consider the case where

$X \in \mathcal{X}(M)$  is complete.

Def The flow generated by  $X$  is the map

$$F: \mathbb{R} \times M \rightarrow M, \quad (t, p) \mapsto \gamma_p(t)$$

where  $\gamma_p$  is the max integral curve of  $X$  with  $\gamma_p(0) = p$ .

Use the notation  $F_t(p) = F(t, p)$

Properties of the flow

(i)  $F$  is smooth

(ii)  $F_0 = \text{id}_M$ ,

(iii)  $F_s \circ F_t = F_{s+t}$  for  $s, t \in \mathbb{R}$

(iv)  $F_{-t} = (F_t)^{-1}$  for  $t \in \mathbb{R}$

(In particular, each  $F_t: M \rightarrow M$  is a diffeo.) and

$$(v) \quad \frac{d}{dt} F_t(p) = X(F_t(p))$$

for all  $(t, p) \in \mathbb{R} \times M$ .

### Remark

If  $X \in \mathcal{X}(M)$  is not complete,  
the flow is defined on

$$\bigcup_{p \in M} I_p \times \{p\} \subset \mathbb{R} \times M$$

where  $I_p$  is the domain of  
the maximal integral curve through

$p$ .

Thm If  $M$  is a compact manifold (without boundary), then any vector field on  $M$  is complete.

proof

Lemma ~~In the setting of~~  
If  $M$  is compact,  $X \in \mathcal{X}(M)$ , then there is  $\epsilon > 0$  such that each maximal integral curve contains  $(-\epsilon, \epsilon)$  in its domain.

proof of lemma Suppose not.

Then there are points  $p_n$  such that the maximal integral curve  $I_n$  for  $p_n$  does not contain  $(-\frac{1}{n}, \frac{1}{n})$ .

$p$ . As  $M$  is compact,

$\{p_n\}$  has a convergent subsequence.

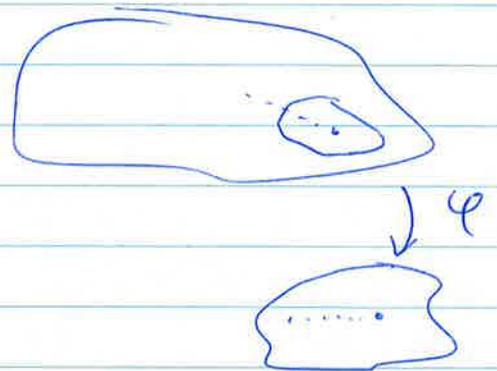
~~$\{p_n\}$~~  Replace  $\{p_n\}$  by this subsequence.

Then  $p_n \rightarrow p$ . Take a chart  $(U, \varphi)$  containing  $p$

and define

$$q_n = \varphi(x_n) \text{ for } p_n \in U$$

$$\text{and } Y = \varphi^* X.$$



Now  $Y$  is a smooth vector field on an open subset  $W$

of  $\mathbb{R}^m$  and  $q_n$  is a convergent

sequence in  $W$ .

By standard ODE theory on  $\mathbb{R}^m$ ,

there is  $\delta > 0$  and a nbhd

of  $q$  such that every point

has an integral curve defined on  $(-\delta, +\delta)$ .  $\square$

proof of thm

Let  $p \in M$  and let  $\gamma_p: I_p \rightarrow M$  be the domain of the maximal integral curve through  $p$ .

Iff  $t \in I_p$ , then the maximal integral curve  $\gamma_q$  through  $q = \gamma_p(t)$  is defined on  $(-\epsilon, \epsilon)$  and so  $\gamma_p$  is defined on  $(t-\epsilon, t+\epsilon)$ .

That is  $I_p \subset \mathbb{R}$  satisfies the property if  $t \in I_p$  then  $(t-\epsilon, t+\epsilon) \subset I_p$ .

This is only possible if  $I_p = \mathbb{R}$ .

□

\* Say  $X$  generates a flow

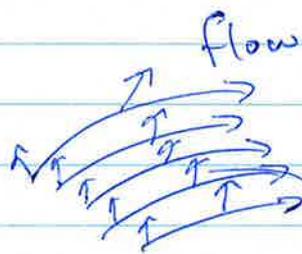
$$F_t : \mathbb{R} \times M \rightarrow M$$

and  $\psi$  is another v. field.

As  $F_t$  is a diffeo,

we may ~~push~~ pull  $\psi$  ~~forward~~ back by  $F_t$ .

\*



Prop

$$\frac{d}{dt} (F_t^* \psi) = F_t^* ([X, \psi])$$

In particular

$$[X, \psi] = \left. \frac{d}{dt} \right|_{t=0} F_t^* \psi.$$

proof ~~Exercise.~~

See Conlon Thm 2.8.16.

One has to use coordinates.

Idea: let  $Z(t) = \frac{F_t^* Y - Y}{t}$

then  $Z_p(t) = \frac{TF_{\epsilon}^{-1} \circ Y_{F_{\epsilon}(p)} - Y_p}{t}$

"="  $TF_{\epsilon}^{-1} \circ Y$

$$Z_p(t)(\Phi) = \frac{Y_{F_{\epsilon}(p)}(\Phi \circ F_{-t}) - Y_p(\Phi)}{t}$$

$$= \frac{Y_{F_{\epsilon}(p)}(\Phi) - Y_p(\Phi)}{t}$$

$$\text{"="} \frac{Y_{F_{\epsilon}(p)}(\Phi \circ F_{\epsilon}) - Y_{F_{\epsilon}(p)}(\Phi)}{t} + \frac{Y_{F_{\epsilon}(p)}(\Phi) - Y_p(\Phi)}{t}$$

$\xrightarrow{\text{as } t \rightarrow 0}$   $X(Y(\Phi)) - Y(X(\Phi))$