

Diff Geom 2017

Prop ~~If~~ Let $v_p \in T_p M$.

If f is zero in a nbhd U of p , then $v_p(f) = 0$.

pf Take a bump function $h \in C^\infty(\mathbb{R}^n)$
s.t. $h(x) = 0$ if $x \notin U$.

Then $(f \cdot h)(x) = 0$ for all x .

$$\begin{aligned} 0 &= v_p(0) = v_p(f \cdot h) \\ &= v_p(f) \cdot \underbrace{h(p)}_1 + \underbrace{f(p)}_0 \cdot v_p(h) \\ &= v_p(f). \end{aligned}$$
□

Cor If f and g agree on a nbhd of p , then $v_p(f) = v_p(g)$.

Proof

$$0 = v_p(f - g) = v_p(f) - v_p(g).$$
□

Proposition If U is an open submanifold of M , then and $p \in U$, then

$$T_p U \text{ and } T_p M$$

are isomorphic.

proof We define linear maps

$$\begin{array}{ccc} & A & \\ T_p U & \xrightarrow{\quad} & T_p M \\ B & \leftarrow & \end{array}$$

as follows:

Let h be a bump function at p (so $h=1$ in a neighbourhood of p) with $\text{supp}(h) \subset U$.

For any $\Phi \in C^\infty(U)$,

the function $M \rightarrow \mathbb{R}$, $x \mapsto \begin{cases} h(x)\Phi(x), & x \in U \\ 0 & x \notin U \end{cases}$

is smooth.

Call this function $h \cdot \Phi \in C^\infty(M)$

For $u_p \in T_p U$, define

$A(u_p) \in T_p M$ by

$$A(u_p)(\Phi) = u_p(h \cdot \Phi).$$

For $v_p \in T_p M$

define $B(v_p)$ by

$$B(v_p)(\Phi) = v_p(\Phi|_U).$$

for $\Phi \in C^\infty(U)$

$$\text{Now } AB(v_p)(\Phi) = v_p(h \cdot \Phi|_U)$$

~~$$\text{and } BA(u_p)(\Phi) = u_p((h \cdot \Phi)|_U)$$~~

since Φ agrees with $h \cdot \Phi|_U$
in a neighbourhood of p

$$AB(v_p)(\Phi) = v_p(\Phi)$$

That is, $AB(v_p) = v_p$.

Similarly $BA(u_p) = u_p$

so A and B are inverses. \square

Remark: If $i: U \hookrightarrow M$

is the inclusion map, one can
check that the isomorphism
is actually given by

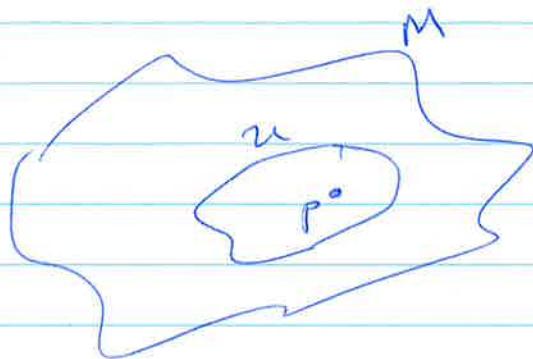
$$T_p i : T_p U \rightarrow T_p M.$$

Prop If M is an m -dimensional

manifold and $p \in M$, then

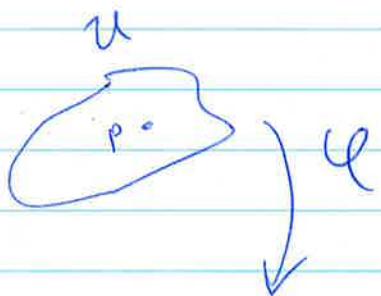
$T_p M$ is an m -dimensional
vector space.

Pf



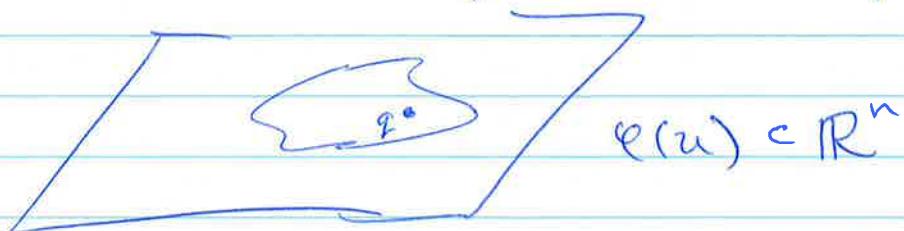
let (φ, \mathcal{U}) be a chart

$$T_p M \cong T_p \mathcal{U}$$



φ is a diffco

$$\Rightarrow T_p \mathcal{U} \cong T_{\varphi(p)} \varphi(\mathcal{U})$$



$$q = \varphi(p)$$

$\varphi(\mathcal{U})$ is an open
subset of \mathbb{R}^m

$$\Rightarrow T_q \varphi(\mathcal{U}) \text{ is } m\text{-dimil.}$$



The Tangent Bundle.

Def

The set

$$TM = \bigcup_{p \in M} T_p M$$

is called the tangent bundle,

and maps

$$\begin{array}{ccc} TM & \xrightarrow{\pi} & M \\ \downarrow & & \downarrow \\ v_p & \in & p \in M \end{array}$$

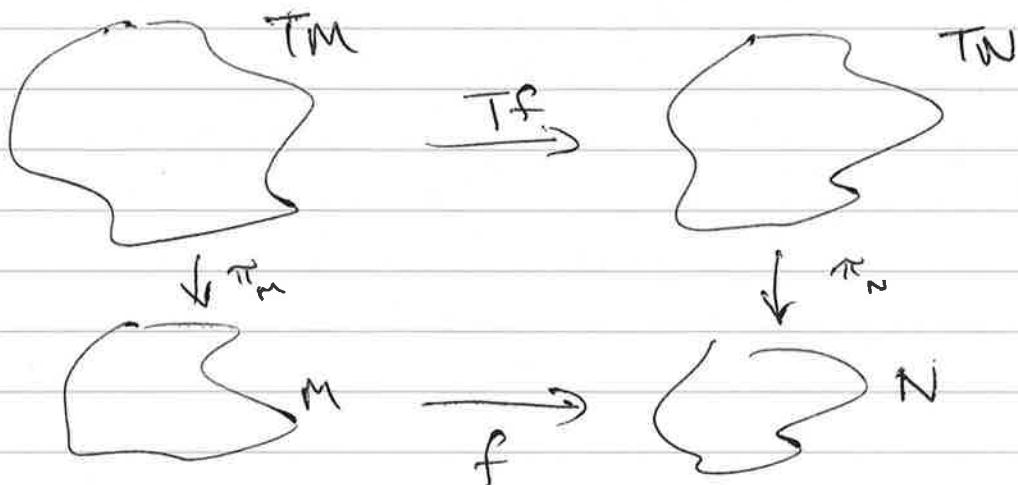
is known as the projection map.

For $f: M \rightarrow N$ smooth,

the tangent map $Tf: TM \rightarrow TN$

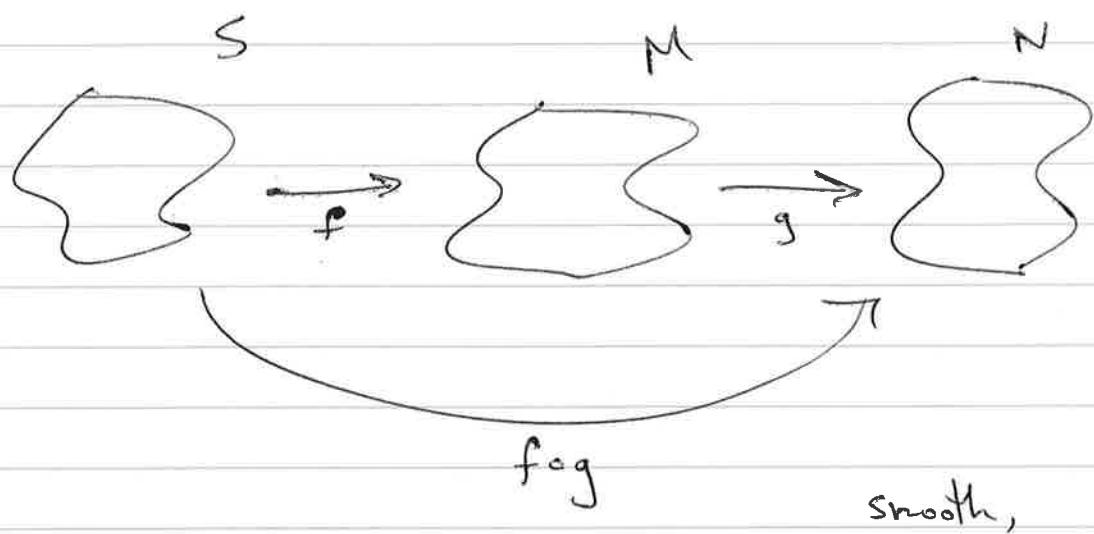
is defined by

$$Tf(v_p) = T_p f(v_p).$$

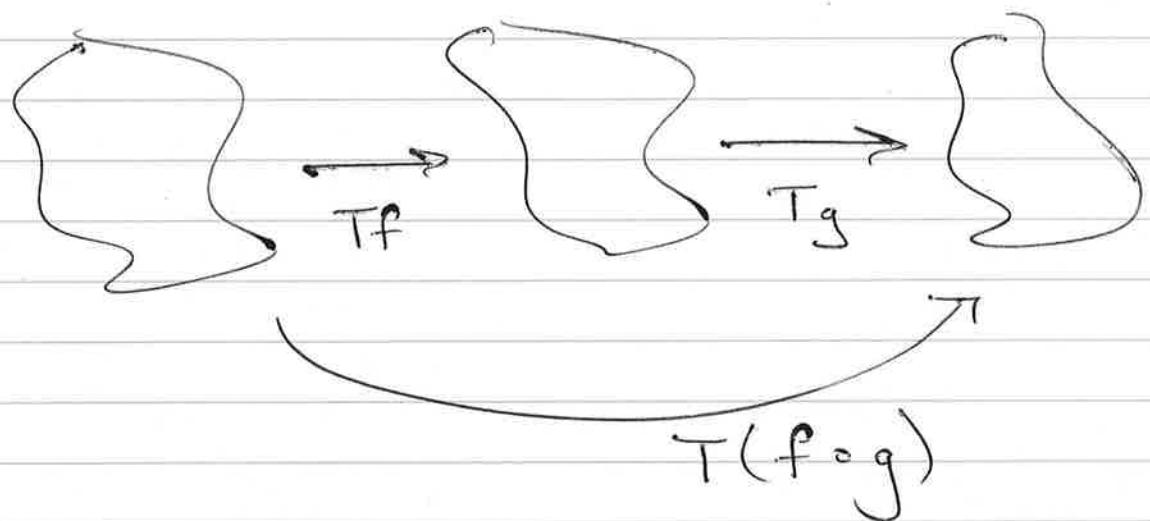


The Chain Rule

If



then



$$T(f \circ g) = Tf \circ Tg$$

In fancy words,

T is a covariant functor.

Exercises

- $T(id_M) = Id_{TM}$

- If $f: M \rightarrow N$ is a diff.,
then

$Tf: TM \rightarrow TN$
is bijective and

$$(Tf)^{-1} = T(f^{-1})$$

We want to show that TM is a smooth manifold and that $TM \xrightarrow{Tf} TN$ is a smooth map.

To do this, we first have to go back to \mathbb{R}^m .

Let $U \subset \mathbb{R}^m$ be open.
Then there is a map

$$U \times \mathbb{R}^m \rightarrow TU$$

$$(p, u) \mapsto \left. \frac{d}{dt} \right|_{t=0} (f \mapsto f(p+tu))$$

going vector
 in \mathbb{R}^m a vector as
 a directional
 derivative

$$\text{Each } \{p\} \times \mathbb{R}^m \rightarrow T_p U$$

is a linear isomorphism,

$$\text{so } U \times \mathbb{R}^m \rightarrow TU \text{ is a}$$

bijection.

Now suppose $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ and $f: U \rightarrow V$.

Then there is a derivative

$$Tf: TU \rightarrow TV$$

and a derivative

$$Df: U \times \mathbb{R}^m \rightarrow U \times \mathbb{R}^n$$

Here $Df(p, u) = Df_p(u)$

from the first class.

Are these the same? Yes.

$$\begin{array}{ccccc}
 & & f & & \\
 U & \xrightarrow{\quad} & V & \xrightarrow{\quad g \quad} & \mathbb{R} \\
 h \mapsto \frac{d}{dt} h(p+tu) & \cdots \cdots \cdots \rightarrow & g \mapsto \frac{d}{dt} g(f(p+tu)) \\
 \text{PAK} \rightsquigarrow & & & & g \mapsto \frac{d}{dt} g(q+tv) \\
 Tp \rightsquigarrow & & Tf & \longrightarrow & TV \\
 TU & \xrightarrow{\quad Tf \quad} & TV & & \\
 \uparrow & & \uparrow & & \\
 U \times \mathbb{R}^m & \xrightarrow{\quad Df \quad} & U \times \mathbb{R}^n & & \\
 (p, u) & & & & (q, v) = (f(p), Df_p(u))
 \end{array}$$

Does $\frac{d}{dt} g(f(p+tu)) = \frac{d}{dt} (q+tv)$?

$$\begin{aligned}
 & g(f(p+tu)) \\
 &= g(f(p) + t Df_p(u) + \text{small}) \\
 &= g(\cancel{f(p)} + t q + tv + \text{small})
 \end{aligned}$$

$$\left. \frac{d}{dt} \right|_{t=0} g(q + tv + \text{small}) = \frac{d}{dt} g(q + tv)$$

since g is smooth.

$\frac{r(t)}{t} \rightarrow 0$ small.

Note that we can write Df as a big ugly matrix

$$Df_p(u) = \begin{bmatrix} & & & & u^1 \\ & & & \vdots & \vdots \\ & \frac{\partial f^i}{\partial x^j}|_p & - & - & u^m \\ & \vdots & - & - & \vdots \end{bmatrix}$$

and f is smooth, \Rightarrow

and its partial derivatives
are smooth in p ,

It follows that

$$Df : U \times \mathbb{R}^m \rightarrow V \times \mathbb{R}^n$$

If $f: U \rightarrow V$
is smooth. If $f: U \rightarrow V$
has inverse \exists a diffeo, then so is Df and $D(f^{-1}) = (Df)^{-1}$
Now things are easy.

Proposition If M is an n -diml manifold

then TM is a $2n$ -diml manifold.

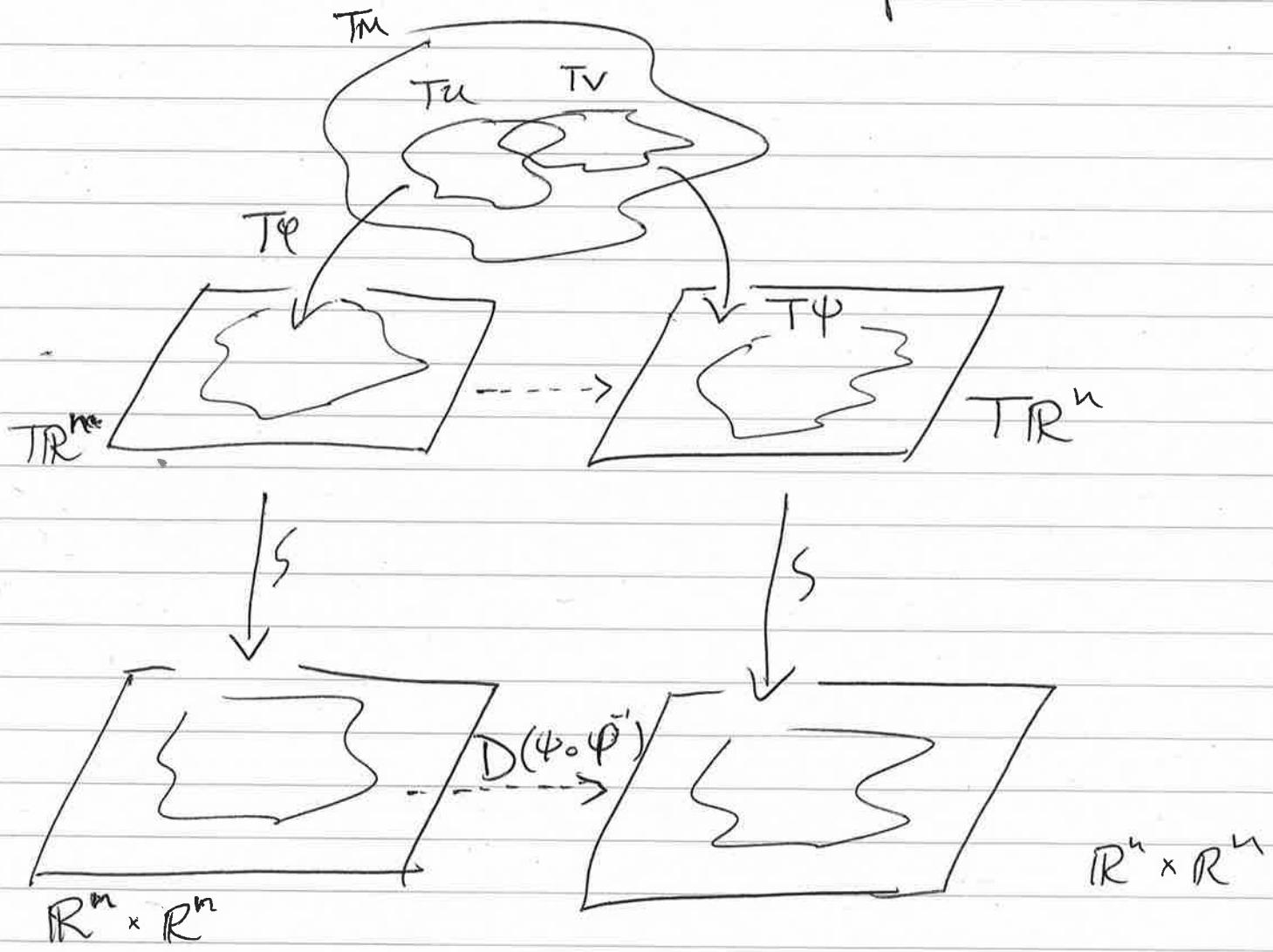
proof

Let A be an atlas
for each chart (U, φ)
Define a chart $(TU, \tilde{\varphi} \circ \tilde{\psi})$

by $TU \xrightarrow{T\varphi} TR^n \xrightarrow{\sim} \mathbb{R}^n \times \mathbb{R}^n$

std form

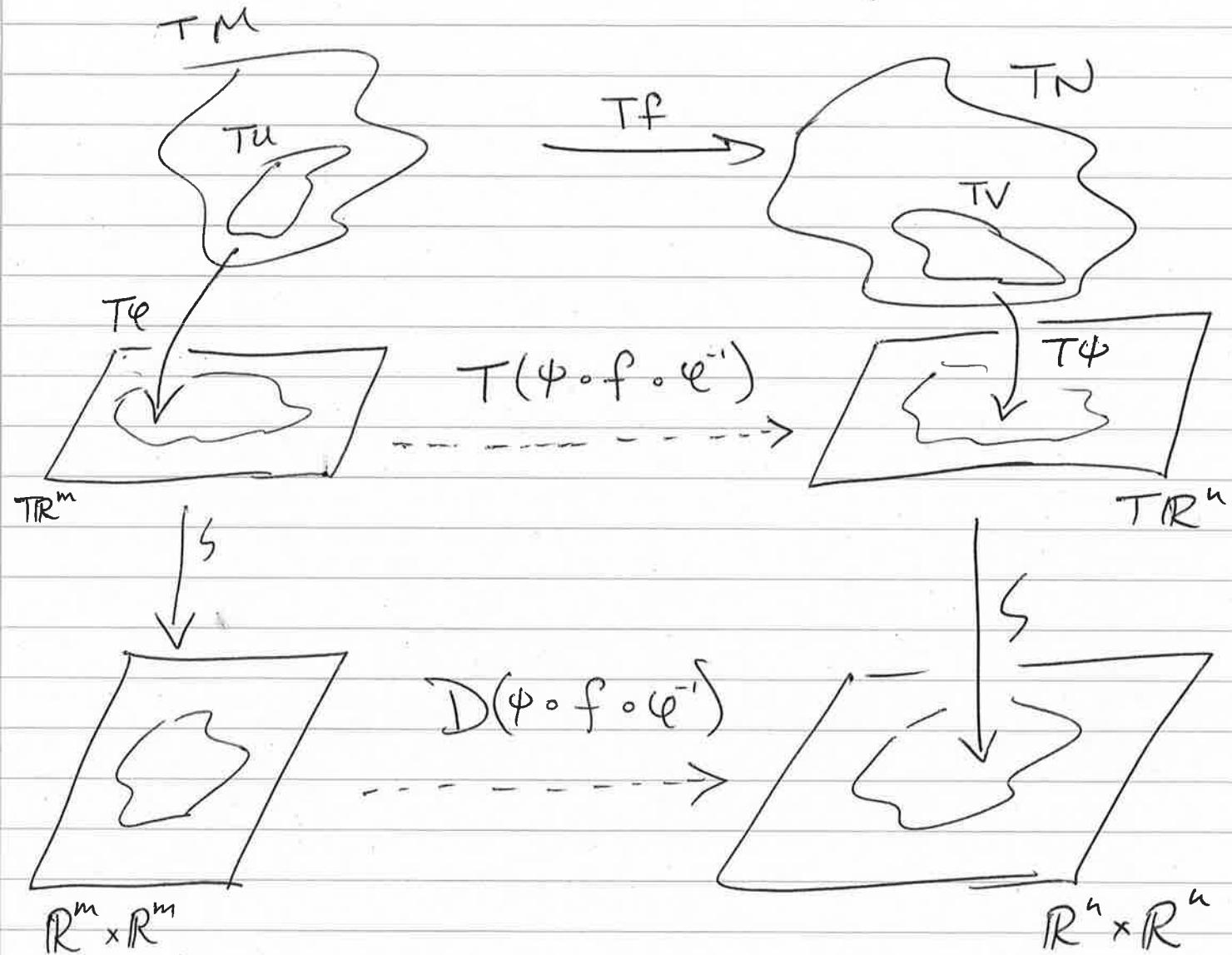
These charts are compatible



Prop If $f: M \rightarrow N$ is smooth,

then $Tf: TM \rightarrow TN$ is smooth.

proof Almost the same proof:



□

Def A vector bundle of rank k
 over a manifold M is a pair
 (E, π) where

- 1) E is a manifold
- 2) $\pi: M \rightarrow E$ is surjective map
- 3) each fiber $E_p = \pi^{-1}(p)$
 is a k -dim'l vector space

(This structure is part of the definition of the bundle.)

4) For each $p \in M$, there is \exists
 a nbhd $U \subset M$ and a diffeo
 $\pi^{-1}(U) \xrightarrow{\varphi} U \times \mathbb{R}^k$

such that each

$$\varphi|_{E_x}: E_x \rightarrow \{x\} \times \mathbb{R}^k$$

is a linear isomorphism.

φ is called a local trivialization of E .

More on vector bundles later

Ex S^2 2-sphere

TS^2 is a v. bundle over S^2 of rank 2

$S^2 \times \mathbb{R}^2$ is also a v. bundle over S^2 of rank 2.

These are not the same bundle.

Why?

The hairy ball theorem.

