

Let  $V$  be a vector space. 11

Define  $V^*$  as the space of all linear maps from  $V$  to  $\mathbb{R}$ .

Exercise  $V^*$  is itself a vector space

Call  $V^*$  the dual space of  $V$ .

If  $\{e_1, \dots, e_d\}$  is a basis for  $V$ ,

we can define a dual basis for  $V^*$

$$\{\theta^1, \theta^2, \dots, \theta^d\}$$

$$\text{by } \theta^i: V \rightarrow \mathbb{R}$$

$$\theta^i(e_j) = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Using Dirac delta  $\theta^i(e_j) = \delta_j^i$ .

If  $V = \mathbb{R}^d$  then the standard inner product

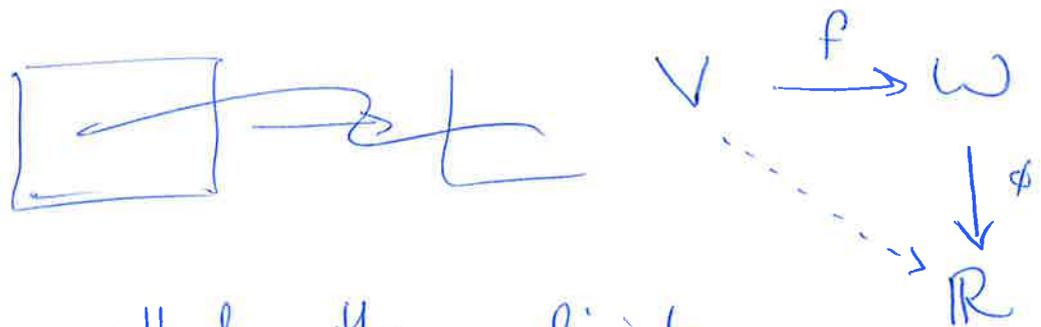
gives us a (non-canonical) way to identify  $V$  with  $V^*$ .

Every linear map  $\phi: \mathbb{R}^d \rightarrow \mathbb{R}$  is  
~~may be represented is~~ of the form

$$\phi(x) = \langle u, x \rangle$$

for some  $u \in \mathbb{R}^d$ .

Suppose  $L: V \rightarrow W$  is a linear  
 map. Then it induces a linear  
 map  $L^*: W^* \rightarrow V^*$



$L^*$  is called the adjoint  
 or transpose of  $L$ .

Example Suppose  $L: \mathbb{R}^m \rightarrow \mathbb{R}^n$  is linear.

Then it is given by an  $m \times n$  matrix  $A \in \mathbb{R}^{m \times n}$   $\left[ \begin{matrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{matrix} \right]_m \left[ \begin{matrix} \phantom{0} & \phantom{0} & \phantom{0} \\ \phantom{0} & \phantom{0} & \phantom{0} \\ \phantom{0} & \phantom{0} & \phantom{0} \end{matrix} \right]_n$

Suppose  $\phi: \mathbb{R}^m \rightarrow \mathbb{R}$  is linear

$$\text{then } (L^* \phi)(x) = \phi(L(x)) = \phi(Ax).$$

$$\text{If } \phi(x) = \langle u, x \rangle = u^T x \quad \underline{\text{L3}}$$

$$\begin{aligned} \text{Then } \phi(Ax) &= \langle u, Ax \rangle \\ &= u^T Ax \\ &= (A^T u)^T x \\ &= \langle A^T u, x \rangle \\ &= (L^* \phi)(x). \end{aligned}$$

Let  $M$  be a smooth manifold and  $p \in M$ .

Then the dual space to  $T_p M$   
is called the cotangent space at  $p$   
and denoted  $T_p^* M$ .

An element  $\theta_p \in T_p^* M$  (i.e.,  $\theta_p: T_p M \rightarrow \mathbb{R}$  <sup>a linear map</sup>)  
is called a cotangent vector. <sub>is linear</sub>

The set  $T^* M = \bigsqcup_{p \in M} T_p^* M$   
is called the cotangent bundle.

Prop If  $M$  is an  $m$ -dim manifold  
then  $T^* M$  is a  $2m$ -dim manifold

proof Exercise. Use that the map  $L_4$

$$\mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{n \times m}$$

$$A \mapsto A^T$$

is smooth.  $\square$

Suppose  $f: M \rightarrow N$  is a diffeo.

Then each linear map  $T_p f: T_p M \rightarrow T_p N$

~~defines an~~ has an adjoint map  
from  $T_p^* N$  to  $T_p^* M$ .

This defines a map  $T^* f: T^* N \rightarrow T^* M$ .

Exercise:  $T^* f$  is a diffeo.

$$T^* \text{id}_M = \text{Id}_{T^* M}$$

Chain Rule ~~is~~

$$\begin{array}{ccccc} S & \xrightarrow{f} & M & \xrightarrow{g} & N \\ & & & \searrow & \\ & & & & \text{f} \circ \text{g} \end{array}$$

induces

$$\begin{array}{ccccc} T^* S & \xleftarrow{T^* f} & T^* M & \xleftarrow{T^* g} & T^* N \\ & & & \searrow & \\ & & & & T^*(g \circ f) \end{array}$$

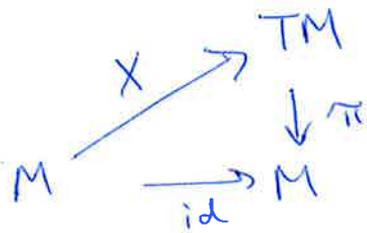
and  $T^*(g \circ f) = T^*f \circ T^*g$ . LS

In fancy words,

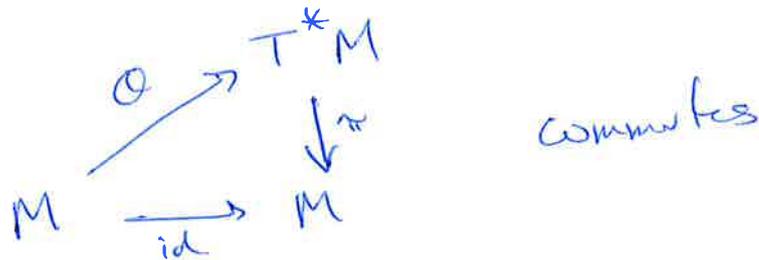
$T^*$  is a contravariant functor.

More generally, any vector bundle  $E \downarrow M$  has a dual bundle  $E^* \downarrow M$ .

Recall a vector field is a map



A map  $\omega : M \rightarrow T^*M$  such that



is called a 1-form. Let  $\mathcal{X}^*(M)$  denote the space of all 1-forms on  $M$ .

Q How can we combine  $X \in \mathcal{X}(M)$  and  $\Theta \in \mathcal{X}^*(M)$ ? (6)

A For each  $p \in M$ ,  $X(p) = X_p \in T_p M$   
and  $\Theta(p) = \Theta_p \in T_p^* M$ , so

$\Theta_p$  is a linear map  $\Theta_p : T_p M \rightarrow \mathbb{R}$

Then  $\Theta_p(X_p) \in \mathbb{R}$ .

This defines a smooth function on  $M$ .

$$\Theta(X) : M \rightarrow \mathbb{R}$$

Thus,  $\Theta \in \mathcal{X}^*(M)$  induces a linear map

$$\Theta : \mathcal{X}(M) \rightarrow C^\infty(M).$$

Consider  $f : M \rightarrow \mathbb{R}$  smooth and its

the tangent map  $Tf : TM \rightarrow T\mathbb{R}$ .

At each point  $p \in M$ ,  $T_p M \rightarrow T_{f(p)} \mathbb{R}$

and we can identify  $T_{f(p)} \mathbb{R}$  with  $\mathbb{R}$

Recall:  $f \mapsto \frac{d}{dt} f \Big|_{t=s} \in T_p$

$$\frac{d}{dt} \Big|_{t=s} \in T_s \mathbb{R}$$

identify  $\frac{d}{dt} \Big|_{t=s}$  with  $1 \in \mathbb{R}$

Inverse:

$$T_s \mathbb{R} \rightarrow \mathbb{R}$$

$$X_s \mapsto X_s(\text{id}_{\mathbb{R}})$$

Hence each  $T_p f: T_p M \rightarrow T_{f(p)} \mathbb{R}$  □ 7

~~it~~ gives a map  $df_p: T_p M \rightarrow \mathbb{R}$ .

That is  $df_p \in T_p^* M$ .

Since this is defined for ~~each~~ <sup>each</sup>  $p \in M$   
it yields a 1-form  $df: M \rightarrow T_p^* M$

That is  $df \in \mathcal{X}^*(M)$ .

If  $X \in \mathcal{X}(M)$ , what is  $df(X)$ ?

$$(df_p)(X_p) = X(p).$$

proof

$$\begin{array}{ccc} T_p M & \xrightarrow{T_p f} & T_{f(p)} \mathbb{R} & \longrightarrow & \mathbb{R} \\ \downarrow \omega & & \downarrow X_p(\cdot \circ f) & & \downarrow X_p(\text{id}_{\mathbb{R}} \circ f) \\ X_p(\cdot) & & X_p(\mathbb{E} \circ f) & & X_p(\text{id}_{\mathbb{R}} \circ f) \\ & & & & \square \\ & & & & \mathbb{E} = \text{id}_{\mathbb{R}} \end{array}$$

Given  $f \in C^\infty(M)$ ,  $df \in \mathcal{E}^*(M)$  [8]

so we may regard  $d$  as a function

$$d: C^\infty(M) \rightarrow \mathcal{E}^*(M)$$

called the exterior derivative.

Properties (i)  $d$  is  $\mathbb{R}$ -linear

$$(ii) \quad d(fg) = f \cdot dg + g \cdot df$$

$$(iii) \quad d(h \circ f) = (h' \circ f) \cdot df$$

for  $f \in C^\infty(M)$  and  $h \in C^\infty(\mathbb{R})$

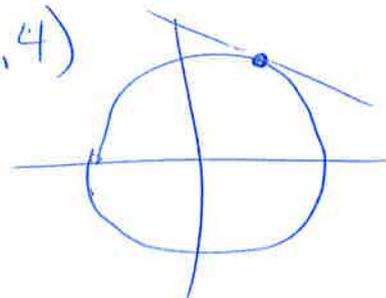
$$(or \ just \ dy = \frac{dy}{dx} \cdot dx)$$

Aside:

What is the slope of

$$the \ circle \ x^2 + y^2 = 5^2$$

at the point  $(3, 4)$



UGLY METHOD:

$$y^2 = 5^2 - x^2$$

$$y = \pm \sqrt{5^2 - x^2} \quad (\text{take } +)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{5^2 - x^2}} (-2x)$$

WRONG  
DERIV.  $\rightarrow$

$$\frac{dy}{dx} \text{ at } (x=3) \text{ is } \frac{1}{\sqrt{5^2 - 3^2}} (-2 \cdot 3) = \frac{-6}{4}$$

PRETTY METHOD :

$$x^2 + y^2 = 5^2$$

$$\frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} (5^2)$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$x + y \frac{dy}{dx} = 0$$

~~$$2 \cdot 3 + 2 \cdot 4 \frac{dy}{dx} = 0$$~~

$$y \frac{dy}{dx} = -x$$

~~$$4 \frac{dy}{dx} = -3$$~~

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\rightarrow \frac{dy}{dx} = -\frac{3}{4}$$

BEAUTIFUL METHOD :

$$x^2 + y^2 = 5^2$$

$$d(x^2 + y^2) = d(5^2)$$

$$2x dx + 2y dy = 0$$

$$x dx + y dy = 0$$

~~$$2 \cdot 3 dx + 2 \cdot 4 dy = 0$$~~

$$y dy = -x dx$$

~~$$3 dx + 4 dy = 0$$~~

$$\frac{dy}{dx} = -\frac{x}{y}$$

~~$$4 dy = -3 dx$$~~

$$\frac{dy}{dx} = -\frac{3}{4}$$

Suppose  $X \in \mathcal{X}(M)$  and  $f \in C^\infty(M)$ . 10

Then  $X(f) \in C^\infty(M)$

Note that  $df \in \mathcal{X}^*(M)$

so  $df(X) \in C^\infty(M)$

Are these the same? Yes.

$$X(f) = df(X)$$

Intuitively, this function at a point  $p \in M$  tells us how  $f$  is changing in the direction of  $X_p \in T_p M$ .