MTH 41041 (Semester 1, 2017) Assignment 4

Important

(a) The assignment is due at 11:30 am on Friday, 31 March, 2017. You can either hand them in at the start of class, hand them to me in person beforehand, or slip it under my office door.

(b) You can talk with the other students and with me about the problems, but you must write up and hand in your own work.

(c) We expect that you will put enough thought and effort into the presentation of your solutions so that they are neat, clear, and concise. Poorly presented assignments will be penalized.

Let $U \subset \mathbb{R}$ be open and let $f : U \to \mathbb{R}$ be a smooth function. We say that f is *flat* at a point $p \in U$ if $f^{(k)}(p) = 0$ for all derivatives $f^{(k)}$ of all orders $k > 0$.

(1) For smooth functions $f, g : \mathbb{R} \to \mathbb{R}$, show that
   (a) if f and g are flat at p, then $f + g$ is flat at p;
   (b) if f is flat at p, then the product $f \cdot g$ is flat at p;
   (c) if f is flat at p and g is positive, then $\frac{f}{g}$ is flat at p.

(2) Suppose $U \subset \mathbb{C}$ is open and connected and $f : U \to \mathbb{C}$ is an analytic function such that the restriction of f to $U \cap \mathbb{R}$ is a real-valued function which is flat at a point $p \in U \cap \mathbb{R}$. Show that f is constant.

(3) Show that there is a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho(0) = 0$, $\rho(x) > 0$ for all $x \neq 0$, and ρ is flat at 0.

(4) For real numbers $a < b$, show that there is a smooth function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 0$ for all $x \leq a$ and $g(x) = 1$ for all $x \geq b$.

(5) Suppose $U \subset \mathbb{R}^d$ is open and $p \in U$. Show that there is a smooth function $h : \mathbb{R}^d \to [0, 1]$ such that the support of h is a compact subset of U and there is a neighbourhood V of p such that $h(x) = 1$ for all $x \in V$.