MTH 41041 (Semester 1, 2017) Assignment 2

Important

(a) The assignment is due at 11:30 am on Friday, 17 March, 2017. You can either hand your assignment in at the start of class, hand it to me in person beforehand, or slip it under my office door.

(b) You can talk with the other students and with me about the problems, but you must write up and hand in your own work.

(c) We expect that you will put enough thought and effort into the presentation of your solutions so that they are neat, clear, and concise. Poorly presented assignments will be penalized.

Problem 1 Let $\mathbb{RP}^n$ be the quotient space of[1] $\mathbb{R}^{n+1}_\times$ by the following equivalence relation: for $x, y \in \mathbb{R}^{n+1}_\times$, $x \approx y \Leftrightarrow x = \lambda y$ for some $\lambda \neq 0$.

(i) Let $\pi : \mathbb{R}^{n+1}_\times \to \mathbb{RP}^n$ be the canonical projection map, that is

$$\pi(x^0, x^1, \ldots, x^n) := [x^0, x^1, \ldots, x^n],$$

where $[x^0, x^1, \ldots, x^n]$ denotes the equivalence class of the point

$$(x^0, x^1, \ldots, x^n) \in \mathbb{R}^{n+1}_\times.$$

For $i = 0, \ldots, n$, let $U_i = \{[x^0, x^1, \ldots, x^n] \in \mathbb{RP}^n \,|\, x^i \neq 0\}$ and define $\phi_i : U_i \to \mathbb{R}^n$ by

$$\phi_i([x^0, x^1, \ldots, x^n]) = \left(\frac{x^0}{x^i}, \frac{x^1}{x^i}, \ldots, \widehat{\frac{x^i}{x^i}}, \ldots, \frac{x^n}{x^i}\right),$$

where $\widehat{\frac{x^i}{x^i}}$ means it is omitted. Show that ϕ_i is a well-defined bijection.

(ii) For $i, j \in \{0, \ldots, n\}$ $i \neq j$, compute the transition maps

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$

and show that they are smooth (in the usual sense as maps between open sets of $\mathbb{R}^n$).

(iii) Show that $\mathcal{A} = \cup_{i=0}^n \{(U_i, \phi_i)\}$ is an atlas for $\mathbb{RP}^n$, and hence, that $\mathbb{RP}^n$ is an n-dimensional, smooth manifold, known as *real projective n-space* .

(iv) Show that the projection $\pi : \mathbb{R}^{n+1}_\times \to \mathbb{RP}^n$ is smooth.

Problem 2 Let $F : \mathbb{R}^{n+1}_\times \longrightarrow \mathbb{R}^{m+1}_\times$ be a smooth function, and suppose that for some $k \in \mathbb{Z}$, $F(\lambda x) = \lambda^k F(x)$ for all $x \in \mathbb{R}^{n+1}_\times$ and $\lambda \in \mathbb{R}_\times$. Show that the map $f : \mathbb{RP}^n \longrightarrow \mathbb{RP}^m$ defined by $f([x]) := [F(x)]$ is well-defined and smooth.

[1]Here, we are using the notation $V_\times := V \setminus \{0\}$, where V is a vector space.