MTH 41041 (Semester 1, 2017) Assignment 1

Important

(a) The assignment is due at 11:30 am on Friday, 10 March, Week 2. You can either hand them in at the start of class, hand them to me in person beforehand, or slip it under my office door.

(b) You can talk with the other students and with me about the problems, but you must write up and hand in your own work.

(c) We expect that you will put enough thought and effort into the presentation of your solutions so that they are neat, clear, and concise. Poorly presented assignments will be penalized.

Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a function. In this assignment, we say f is *linearly bounded* if there is $C > 0$ and a neighbourhood $U \subset \mathbb{R}^m$ of 0 such that $\|f(x)\| \le C\|x\|$ for all $x \in U$. We say f is *sublinearly bounded* if $f(0) = 0$ and $\lim_{x\to 0} \frac{\|f(x)\|}{\|x\|} = 0$.

(1) Show that a linear map $A : \mathbb{R}^m \to \mathbb{R}^n$ is linearly bounded.

(2) Show that if f and g are linearly bounded functions and r and q are sublinearly bounded functions then:

  (a) $f + g$ is linearly bounded,

  (b) $r + q$ is sublinearly bounded,

  (c) $f \circ r$ is sublinearly bounded, and

  (d) $r \circ f$ is sublinearly bounded.

  (In each case, assume the sum or composition is well-defined.)

(3) Recall that a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is (Fréchet) differentiable at $a \in \mathbb{R}^m$ if there is linear map $A : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{x\to 0} \frac{1}{\|x\|} \|f(a + x) - f(a) - A(x)\| = 0.$$

Show that a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at a if and only if there is a linear map $A : \mathbb{R}^m \to \mathbb{R}^n$ and a sublinearly bounded map $r : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$f(a + x) = f(a) + A(x) + r(x).$$

(4) Using the formulation given in (3), prove the chain rule for differentiable functions $f : \mathbb{R}^k \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^n$.