

# CONSTRUCTING CENTER-STABLE TORI

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ABSTRACT. We show that any weakly partially hyperbolic diffeomorphism on the 2-torus may be realized as the dynamics on a center-stable or center-unstable torus of a 3-dimensional strongly partially hyperbolic system. We also construct examples of center-stable and center-unstable tori in higher dimensions.

## 1. INTRODUCTION

Partially hyperbolic dynamical systems have received a large amount of attention in recent years. These systems display a wide variety of highly chaotic behaviour [Bon11], but they have enough structure to allow, in some cases, for the dynamics to be understood and classified [CRHRHU15, HP15b].

A diffeomorphism  $f$  is *strongly partially hyperbolic* if there is a splitting of the tangent bundle into three invariant subbundles  $TM = E^u \oplus E^c \oplus E^s$  such that the derivative  $Df$  expands vectors in the unstable bundle  $E^u$ , contracts vectors in stable bundle  $E^s$ , and these dominate any expansion or contraction in the center direction  $E^c$ . (See section 2 for a precise, if slightly unorthodox, definition.) The global properties of these systems are often determined by analysing invariant foliations tangent to the subbundles of the splitting.

The bundles  $E^u$  and  $E^s$  are uniquely integrable [HPS77]. That is, there are foliations  $W^u$  and  $W^s$  such that any curve tangent to  $E^u$  or to  $E^s$  lies in a single leaf of the respective foliation. For the center bundle  $E^c$ , however, the situation is more complicated. There may not be a foliation tangent to  $E^c$ , and even if such a foliation exists, the bundle may not be uniquely integrable. The first discovered examples of partially hyperbolic systems without center foliations were algebraic in nature. In these examples, both  $f$  and the splitting can be taken as smooth, and the center bundle is not integrable because it does not satisfy Frobenius' condition of involutivity [Wil98]. Such non-involutive examples are only possible if the dimension of the center bundle is at least two, and for a long time it was an open question if a one-dimensional center bundle was necessarily integrable.

Rodriguez Hertz, Rodriguez Hertz, and Ures recently answered this question by constructing a counterexample [RHRHU16]. They defined a partially hyperbolic system on the 3-torus with a center bundle which is uniquely integrable everywhere except for an invariant embedded 2-torus tangent to  $E^c \oplus E^u$ . This discovery has shifted our view on the possible dynamics a partially hyperbolic

system can possess, and leads to questions of how commonly invariant submanifolds of this type occur in general. In this paper, we build further examples of partially hyperbolic systems having compact submanifolds tangent either to  $E^c \oplus E^u$  or  $E^c \oplus E^s$ , both in dimension 3 and in higher dimensions.

In the construction in [RHRHU16], the dynamics on the 2-torus tangent to  $E^c \oplus E^u$  is Anosov. In fact, it is given by a hyperbolic linear map on  $\mathbb{T}^2$ , the cat map. It has long been known that a *weakly partially hyperbolic* system, that is, a diffeomorphism  $g : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  with a splitting of the form  $E^c \oplus E^u$  or  $E^c \oplus E^s$ , need not be Anosov. Therefore, one can ask exactly which weakly partially hyperbolic systems may be realized as the dynamics on an invariant 2-torus sitting inside a 3-dimensional strongly partially hyperbolic system. We show, in fact, that there are no obstructions on the choice of dynamics.

**Theorem 1.1.** *For any weakly partially hyperbolic diffeomorphism  $g_0 : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ , there is an embedding  $i : \mathbb{T}^2 \rightarrow \mathbb{T}^3$  and a strongly partially hyperbolic diffeomorphism  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  such that  $i(\mathbb{T}^2)$  is either a center-stable or center-unstable torus (depending on the splitting of  $g_0$ ) and  $i^{-1} \circ f \circ i = g_0$ .*

To be precise, a *center-stable torus* is an embedded copy of  $\mathbb{T}^D$  with  $D \geq 2$  which is tangent to  $E_f^{cs} := E_f^c \oplus E_f^s$ . Similarly, a *center-unstable torus* is tangent to  $E_f^{cu} := E_f^c \oplus E_f^u$ . We also use the terms *cs-torus* and *cu-torus* as shorthand.

In the case where the derivative of  $g_0$  preserves the orientation of the center bundle,  $E_{g_0}^c$ , we may be more specific about the construction.

**Theorem 1.2.** *Let  $g_0 : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  be a weakly partially hyperbolic diffeomorphism which preserves the orientation of its center bundle and is homotopic to a linear Anosov diffeomorphism  $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  and let  $0 < \epsilon < \frac{1}{2}$ . Then there is a strongly partially hyperbolic diffeomorphism  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  such that*

- (1)  $f(x, t) = (A(x), t)$  for all  $(x, t) \in \mathbb{T}^3$  with  $|t| > \epsilon$ ,
- (2)  $f(x, t) = (g_0(x), t)$  for all  $(x, t) \in \mathbb{T}^3$  with  $|t| < \frac{\epsilon}{2}$ , and
- (3)  $\mathbb{T}^2 \times 0$  is either a center-stable or center-unstable torus, depending on the splitting for  $g_0$ .

Since the construction is local in nature, different weakly partially hyperbolic diffeomorphisms may be inserted into the system at different places.

**Corollary 1.3.** *Suppose  $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  is a hyperbolic linear automorphism and  $g_i : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  for each  $i \in \{1, \dots, n\}$  is a weakly partially hyperbolic diffeomorphism which is homotopic to  $A$  and preserves the orientation of its center bundle. Let  $\{t_1, \dots, t_n\}$  be a finite subset of the circle,  $S^1$ . Then there is a strongly partially hyperbolic diffeomorphism  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  such that*

- (1)  $f(x, t_i) = (g_i(x), t_i)$  for each  $t_i$  and all  $x \in \mathbb{T}^2$ , and
- (2) each  $\mathbb{T}^2 \times t_i$  is either a center-stable or center-unstable torus, depending on the splitting for  $g_i$ .

We also note that the presence of a  $cs$  or  $cu$ -torus affects the dynamics only in a neighbourhood of that torus and does not place global restrictions on the dynamics on  $\mathbb{T}^3$ . For instance, one could easily construct a system which has a  $cs$  or  $cu$ -torus  $\mathbb{T}^2 \times 0$  and has a robustly transitive blender elsewhere on  $\mathbb{T}^3$  [BD96].

The results as stated above rely on work announced by Gourmelon and Potrie which shows that in the  $C^1$ -open set of diffeomorphisms of  $\mathbb{T}^2$  with dominated splittings, the subset of diffeomorphisms isotopic to a given hyperbolic toral automorphism is connected. See section 4 for further details about this property.

The original construction of Rodriguez Hertz, Rodriguez Hertz, and Ures on the 3-torus may be viewed as a skew product with Anosov dynamics in the fibers. In fact, the example can be given as a map of the form

$$F(x, v) = (f(x), Av + h(x))$$

where  $f$  is a Morse-Smale diffeomorphism of the circle,  $A$  is the cat map on  $\mathbb{T}^2$ , and  $h : S^1 \rightarrow \mathbb{T}^2$  is smooth. The diffeomorphism  $f$  has a sink at a point  $x_0$  and the fiber  $x_0 \times \mathbb{T}^2$  over this sink gives the embedded 2-torus tangent to  $E_F^{cu}$ .

This description of  $F$  naturally suggests a way to construct higher-dimensional examples of the same form. We will show that, starting from any diffeomorphism  $f$  of any closed manifold  $M$ , one may construct a strongly partially hyperbolic diffeomorphism  $F$  of  $M \times \mathbb{T}^D$  using sinks of  $f$  to construct center-unstable tori for  $F$  and sources to construct center-stable tori.

**Theorem 1.4.** *Let  $f_0 : M \rightarrow M$  be a diffeomorphism and  $X \subset M$  a finite invariant set such that every  $x \in X$  is either a periodic source or sink. Then there is a strongly partially hyperbolic diffeomorphism  $F : M \times \mathbb{T}^D \rightarrow M \times \mathbb{T}^D$  of the form*

$$F(x, v) = (f(x), Av + h(x))$$

*such that  $f$  is isotopic to  $f_0$  and, for each  $x \in X$ , the submanifold  $x \times \mathbb{T}^D$  is tangent either to  $E^{cs}$  or  $E^{cu}$ .*

In dimension 3, the presence of a compact submanifold tangent to  $E^{cs}$  or  $E^{cu}$  has strong consequences on the global topology of the manifold. In fact, Rodriguez Hertz, Rodriguez Hertz, and Ures showed that the 3-manifold can only be one of a few possibilities [RHRHU11]. The proof of theorem 1.4 uses a local argument and the global topology of  $M$  has no impact on the construction. This suggests that in higher dimensions, compact submanifolds tangent to  $E^{cs}$  and  $E^{cu}$  may arise naturally in many examples of partially hyperbolic systems.

Theorem 1.4 is stated for the trivial fiber bundle  $M \times \mathbb{T}^D$  only for the sake of simplicity. As the proof is entirely local in nature, the same technique may be used to introduce center-stable and center-unstable tori in a system defined on a non-trivial fiber bundle, so long as the dynamics in the fibers is given by a linear Anosov map. By adapting the examples in [GORH15], it might be possible to define a system with a center-stable torus so that the total space is simply

connected. See also [Gog16] for further constructions, and [FG16] for conditions which imply that the fiber bundle must be trivial. We suspect that, just as in the case of dimension 3, the future study of compact center-stable submanifolds in higher dimensions will be full of surprises.

In order to prove the theorems listed above, dominated splittings must be constructed in a variety of settings. Before constructing specific examples, sections 2 and 3 first introduce a number of helpful tools in a general setting which give sufficient and easy-to-verify conditions for dominated splittings to exist. Section 4 establishes properties for dominated splittings in dimension 2 specifically. Section 5 gives the proof of theorem 1.2. Section 6 generalizes this construction and proves theorem 1.1. Finally, section 7 handles higher-dimensional examples and proves theorem 1.4.

## 2. SPLITTINGS AND INEQUALITIES

Many concepts in dynamical systems are defined by an invariant splitting with one or more inequalities related to the splitting. This section shows that, in many cases, the inequalities need only be verified on the non-wandering set of the system. The results in this section are similar in nature to those established in [Cao03] and earlier work referenced therein. As the proofs are short, we give them here for completeness.

Throughout this section assume  $f$  is a homeomorphism of a compact metric space  $M$ . Let  $NW(f)$  denote its non-wandering set.

**Proposition 2.1.** *If  $U$  is a neighborhood of  $NW(f)$ , there is a uniform bound  $N$  such that any orbit  $\{f^n(x) : n \in \mathbb{Z}\}$  has at most  $N$  points lying outside of  $U$ .*

*Proof.* Suppose no such  $N$  exists. As  $M \setminus U$  is totally bounded, for any  $k \in \mathbb{N}$ , there is a point  $x_k \in M \setminus U$  and an iterate  $n_k \geq 1$  such that  $d(x_k, f^{n_k}(x_k)) < \frac{1}{k}$ . The sequence  $\{x_k\}$  accumulates on a non-wandering point outside of  $U$ , which gives a contradiction.  $\square$

A *cochain* for  $f$  (in the context of this section) is a collection of continuous functions  $\alpha_n : X \rightarrow \mathbb{R}$  for  $n \in \mathbb{Z}$ . The cochain is *additive* if

$$\alpha_{n+m}(x) = \alpha_n(f^m(x)) + \alpha_m(x)$$

for all  $n, m \in \mathbb{Z}$  and  $x \in X$ . It is *superadditive* if

$$\alpha_{n+m}(x) \geq \alpha_n(f^m(x)) + \alpha_m(x)$$

for all  $n, m \in \mathbb{Z}$  and  $x \in X$ . It is *eventually positive* if there is  $n_0$  such that  $\alpha_n$  is positive for all  $n > n_0$ . Note that any positive linear combination of superadditive cochains is again superadditive.

**Proposition 2.2.** *If  $\alpha$  is a superadditive cochain, the following are equivalent:*

- (1)  $\alpha$  is eventually positive;
- (2) there is  $n \geq 1$  such that  $\alpha_n(x) > 0$  for all  $x \in M$ ;

(3) *there is  $n \geq 1$  such that  $\alpha_n(x) > 0$  for all  $x \in NW(f)$ ;*

*Proof.* Clearly (1) implies (2) and (2) implies (3).

Proof of (2) implies (1): Suppose (2) holds for some  $n$ . As  $\alpha_n$  and  $\alpha_1$  are continuous, there are  $\delta > 0$  and  $C > 0$  such that  $\alpha_n(x) > \delta$  and  $\alpha_1(x) > -C$  for all  $x \in M$ . Write  $m \in \mathbb{Z}$  as  $m = qn + r$  with  $q \in \mathbb{Z}$  and  $0 \leq r < n$ . Then  $\alpha_m(x) \geq q\delta - Cn$ . If  $m$  is sufficiently large and positive, then so is  $q\delta - Cn$ .

Proof of (3) implies (2): First, note that if  $\alpha$  is a superadditive cochain for  $f$  and  $k \geq 1$ , then  $\beta_n = \alpha_{nk}$  defines a superadditive cochain for  $f^k$ . Therefore, we may assume  $\alpha_1(x) > 0$  for all  $x \in NW(f)$ . Next, if  $\gamma$  is the unique additive cochain with  $\gamma_1 = \alpha_1$ , then  $\alpha_n \geq \gamma_n$  for all  $n \geq 1$ . Therefore, we may assume  $\alpha$  is additive. Let  $\epsilon > 0$  be small enough that  $U := \{x \in M : \alpha_1(x) > \epsilon\}$  is a neighborhood of  $NW(f)$ . Let  $N$  be the bound in proposition 2.1, and let  $C$  be such that  $\alpha_1(x) > -C$  for all  $x \in M$ . Then  $\alpha_m(x) > \epsilon(m - N) - CN$  for all  $m$  and  $x$ . Thus, for large  $m$ ,  $\alpha_m$  is positive.  $\square$

For a linear operator,  $A$ , between normed vector spaces, the norm  $\|A\|$  and conorm  $m(A)$  are defined by

$$\|A\| = \sup\{\|Av\| : \|v\| = 1\} \quad \text{and} \quad m(A) = \inf\{\|Av\| : \|v\| = 1\}.$$

If  $f$  is a diffeomorphism and  $E \subset TM$  is a continuous invariant subbundle, then each of  $\log m(Df^n|_{E(x)})$  and  $-\log \|Df^n|_{E(x)}\|$  defines a superadditive cochain. We formulate a number of dynamical concepts in terms of linear combinations of such cochains. Here, all bundles considered are non-zero.

(1) An invariant subbundle  $E$  is *expanding* if

$$\log m(Df^n|_{E(x)})$$

is eventually positive.

(2) An invariant subbundle  $E$  is *contracting* if

$$-\log \|Df^n|_{E(x)}\|$$

is eventually positive.

(3) An invariant splitting  $E^u \oplus E^s$  is *dominated* if

$$\log m(Df^n|_{E^u(x)}) - \log \|Df^n|_{E^s(x)}\|$$

is eventually positive. Write  $E^u \oplus_{>} E^s$  to indicate the direction of the domination.

(4) An invariant splitting  $E^u \oplus E^s$  is *absolutely dominated* if there is a constant  $c \in \mathbb{R}$  such that both

$$\log m(Df^n|_{E^u(x)}) - cn \quad \text{and} \quad cn - \log \|Df^n|_{E^s(x)}\|$$

are eventually positive.

(5) A dominated splitting  $E^u \oplus_{>} E^s$  is *hyperbolic* if  $E^s$  is contracting and  $E^u$  is expanding.

- (6) A dominated splitting is *weakly partially hyperbolic* if it is either of the form  $E^c \oplus_{>} E^s$  with  $E^s$  contracting or  $E^u \oplus_{>} E^c$  with  $E^u$  expanding.
- (7) An invariant splitting  $E^u \oplus E^c \oplus E^s$  is *strongly partially hyperbolic* if both  $(E^u \oplus E^c) \oplus_{>} E^s$  and  $E^u \oplus_{>} (E^c \oplus E^s)$  are dominated splittings,  $E^s$  is contracting, and  $E^u$  is expanding.
- (8) For  $r \geq 1$ , a strongly partially hyperbolic splitting is *r-partially hyperbolic* if both

$$\log m(Df^n|_{E^u(x)}) - r \log \|Df^n|_{E^c(x)}\|$$

and

$$r \log m(Df^n|_{E^c(x)}) - \log \|Df^n|_{E^s(x)}\|$$

are eventually positive.

Sometimes, one also requires that  $f$  is a  $C^r$  diffeomorphism [HPS77].

- (9) A strongly partially hyperbolic splitting is *center bunched* if both

$$\log m(Df^n|_{E^u(x)}) - \log \|Df^n|_{E^c(x)}\| + \log m(Df^n|_{E^c(x)})$$

and

$$-\log \|Df^n|_{E^c(x)}\| + \log m(Df^n|_{E^c(x)}) - \log \|Df^n|_{E^s(x)}\|$$

are eventually positive.

**Corollary 2.3.** *Let  $f$  be a diffeomorphism on a compact manifold. For an invariant splitting, any of the properties listed above holds on all of  $M$  if and only if the property holds on the non-wandering set.*

Since the log of the Jacobian of  $Df^n|_{E(x)}$  defines an additive cochain, one could also establish similar results for volume partial hyperbolicity as studied in [BDP03]. Further, the techniques in [Cao03] show that all of these properties hold uniformly if and only if they hold in a non-uniform sense on all invariant measures.

### 3. SPLITTINGS FROM SEQUENCES

Here we present what are hopefully “user-friendly” techniques to prove the existence of a dominated splitting. The techniques here are similar in form to results developed by Mañé to study quasi-Anosov systems [Mañ77, Lemma 1.9], by Hirsch, Pugh, Shub in regards to normally hyperbolicity [HPS77, Theorem 2.17], and by Franks and Williams in constructing non-transitive Anosov flows [FW80, Theorem 1.2].

This section uses  $E^u$  and  $E^s$  to denote the bundles of a dominated splitting, even though the splitting may not necessarily be uniformly hyperbolic. It is far easier, at least for the author, to remember that  $E^u$  dominates  $E^s$  than to remember which of, say,  $E^1$  and  $E^2$  dominates the other.

**Notation.** For a non-zero vector  $v \in TM$  and  $n \in \mathbb{Z}$ , let  $v^n$  denote the unit vector

$$v^n = \frac{Df^n v}{\|Df^n v\|}.$$

Of course,  $v^n$  depends on the diffeomorphism  $f : M \rightarrow M$  being studied, so this notation is used only when the  $f$  under study is clear.

**Theorem 3.1.** *Suppose  $f$  is a diffeomorphism of a closed manifold  $M$  and  $Z$  is an invariant subset which contains all chain-recurrent points and has a dominated splitting*

$$T_Z M = E^u \oplus E^s$$

with  $d = \dim E^u$ . Suppose that for every  $x \in M \setminus Z$ , there is a point  $y$  in the orbit of  $x$  and a subspace  $V_y$  of dimension  $d$  such that for any non-zero  $v \in V_y$ , each of the sequences  $v^n$  and  $v^{-n}$  accumulates on  $E^u$  as  $n \rightarrow \infty$ . Then, the dominated splitting on  $Z$  extends to a dominated splitting on all of  $M$ .

A key step in proving the theorem is the following

**Proposition 3.2.** *Let  $f : M \rightarrow M$  be a diffeomorphism,  $\Lambda$  a compact invariant subset, and let  $U \subset \Lambda$  be open in the topology of  $\Lambda$  such that*

- (1)  $f(U)$  is compactly contained in  $U$ ,
- (2) each of

$$\bigcap_{n>0} f^n(U) \quad \text{and} \quad \bigcap_{n>0} \Lambda \setminus f^{-n}(U)$$

has a dominated splitting with  $d = \dim E^u$ , and

- (3) for each  $x \in \overline{U} \setminus f(U)$  there is a  $d$ -dimensional subspace  $V_x$  such that for all  $0 \neq v \in V_x$ , both  $v^n$  and  $v^{-n}$  accumulate on  $E^u$  as  $n \rightarrow \infty$ .

Then, there is a dominated splitting on all of  $\Lambda$ .

From the proof, it will be evident that if  $x \in \overline{U} \setminus f(U)$ , then  $E^u(x) = V_x$  in the resulting dominated splitting on  $M$ . Therefore, it is not immediately clear how applying theorem 3.1 or proposition 3.2 would compare favorably to constructing a dominated splitting directly. Still, there are a number of advantages. First, only  $E^u$  needs to be known, not  $E^s$ , and only on a single fundamental domain where, depending on  $f$ , it may be easy to define. Next, to verify the hypotheses, one need only consider individual convergent subsequences rather than an entire cone field or splitting. Finally, as long as one already knows that the original splitting on  $Z$  is dominated, there are no further inequalities to verify.

While cone fields do not appear in the statement of proposition 3.2, they are needed for its proof. We follow the conventions given in [CP15, Section 2]. If  $\Lambda \subset M$  and  $\mathcal{C} \subset T_\Lambda M$  is a cone field, then for each  $x \in \Lambda$ , the cone  $\mathcal{C}(x)$  at  $x$  is of the form

$$\mathcal{C}(x) = \{v \in T_x M : Q_x(v) \geq 0\}.$$

where  $Q_x$  is a non-positive, non-zero quadratic form which depends continuously on  $x \in \Lambda$ . The *interior* of  $\mathcal{C}(x)$  is

$$\text{int } \mathcal{C}(x) := \{0\} \cup \{v \in T_x M : Q_x(v) > 0\}$$

and the *dual cone* is

$$\mathcal{C}^*(x) := \{v \in T_x M : -Q_x(v) \geq 0\}.$$

**Lemma 3.3.** *Let  $\Lambda \subset M$  be an invariant set with a dominated splitting  $T_\Lambda M = E^u \oplus E^s$ . Then there is a neighborhood  $U$  of  $\Lambda$  and a cone field  $\mathcal{C}$  defined on  $U$  such that*

- (1) *if a sequence  $\{v_k\}$  of unit vectors in  $TM$  converges to  $v \in E^u$ , then  $v_k \in \mathcal{C}$  for all large positive  $k$ ;*
- (2) *if  $x, f(x) \in U$ , then  $Df(\mathcal{C}(x)) \subset \text{int } \mathcal{C}(f(x))$ ;*
- (3) *if  $x \in M$  and  $N \in \mathbb{Z}$  are such that  $f^{-n}(x) \in U$  for all  $n > N$ , then*

$$\bigcap_{n>N} Df^n(\mathcal{C}(f^{-n}(x)))$$

*is a subspace of  $T_x M$  with the same dimension as  $E^u$ ;*

- (4) *if  $x \in M$  and  $N \in \mathbb{Z}$  are such that  $f^n(x) \in U$  for all  $n > N$ , then*

$$\bigcap_{n>N} Df^{-n}(\mathcal{C}^*(f^n(x)))$$

*is a subspace of  $T_x M$  with the same dimension as  $E^s$ .*

- (5) *the subspaces given by (3) and (4) define an extension of the dominated splitting to all of  $\bigcap_{n \in \mathbb{Z}} f^n(U)$ .*

The proof of lemma 3.3 uses the same techniques as in [CP15, Section 2] and is left to the reader.

**Lemma 3.4.** *In the setting of proposition 3.2, if there are cone fields  $\mathcal{B}$  defined on  $\Lambda \setminus f(U)$  and  $\mathcal{C}$  defined on  $\overline{U}$  such that  $d = \dim \mathcal{B} = \dim \mathcal{C}$  and*

$$\begin{aligned} Df(\mathcal{B}(x)) &\subset \text{int } \mathcal{B}(f(x)) && \text{if } x \in M \setminus U, \\ Df(\mathcal{C}(x)) &\subset \text{int } \mathcal{C}(f(x)) && \text{if } x \in \overline{f(U)}, \\ \mathcal{B}(x) &\subset \mathcal{C}(x) && \text{if } x \in \overline{U} \setminus f(U), \end{aligned}$$

*then there is a dominated splitting of dimension  $d$  defined on all of  $\Lambda$ .*

*Proof.* Let  $\alpha : \Lambda \rightarrow [0, 1]$  be a continuous function such that  $\alpha(M \setminus U) = \{0\}$  and  $\alpha(f(U)) = \{1\}$ . If  $P_x$  is the continuous family of quadratic forms defining  $\mathcal{B}$  and  $Q_x$  is the family defining  $\mathcal{C}$ , then

$$(1 - \alpha(x))P_x + \alpha(x)Q_x$$

defines a cone field  $\mathcal{A}$  on  $\Lambda$  such that  $Df(\mathcal{A}(x)) \subset \text{int } \mathcal{A}(f(x))$  for all  $x \in \Lambda$ . This inclusion implies the existence of a dominated splitting.  $\square$

*Proof of proposition 3.2.* Let  $\Lambda_C$  and  $\Lambda_B$  denote the two intersections respectively in item (2) of the proposition. By lemma 3.3, there is a cone field  $\mathcal{C}_0$  defined on a neighborhood  $U_C$  of  $\Lambda_C$ . For  $n \in \mathbb{Z}$ , define a cone field  $\mathcal{C}_n$  on  $f^n(U_C)$  by  $\mathcal{C}_n(x) = Df^n(\mathcal{C}_0(f^{-n}(x)))$ . Similarly, define a cone field  $\mathcal{B}_0$  on a neighborhood  $U_B$  of  $\Lambda_B$  and for each  $n \in \mathbb{Z}$  define the cone field  $\mathcal{B}_n(x) = Df^n(\mathcal{B}_0(f^{-n}(x)))$ .

We claim here that  $\bigcap_m \mathcal{B}_m(x) = V_x$  for all  $x \in \overline{U} \setminus f(U)$  where the intersection is over all  $m \in \mathbb{Z}$  for which  $\mathcal{B}_m(x)$  is defined and  $V_x$  is the subspace given in



the statement of the proposition. Indeed, if  $v \in V_x$  is non-zero, then there is a sequence  $n_j \rightarrow \infty$  such that  $v^{-n_j}$  converges to a vector in  $E^u$ . Hence,  $v^{-n_j} \in \mathcal{B}_0$  for all large  $j$ . Equivalently,  $v \in \mathcal{B}_{n_j}$  for all large  $j$ . Since the sequence  $\mathcal{B}_n$  is nested,

$$\bigcap_j \mathcal{B}_{n_j}(x) = \bigcap_n \mathcal{B}_n(x).$$

This shows that  $V_x \subset \bigcap_n \mathcal{B}_n(x)$ . Since both sets are  $d$ -dimensional subspaces of  $T_x M$ , they must be equal. This proves the claim

If, for some  $m, n \in \mathbb{Z}$ , the cone fields  $\mathcal{B}_m$  and  $\mathcal{C}_n$  satisfied the conditions of lemma 3.4, then the desired dominated splitting would exist. Hence, we may assume that for every  $m, n \in \mathbb{Z}$ , the open set

$$\{x : \mathcal{B}_m(x) \subset \text{int} \mathcal{C}_n(x)\}$$

does not cover all of  $\bar{U} \setminus f(U)$ . By compactness, there is  $y \in \bar{U} \setminus f(U)$  such that

$$\mathcal{B}_m(y) \setminus \text{int} \mathcal{C}_n(y)$$

is non-empty for all  $m, n \in \mathbb{Z}$ . By compactness of the unit sphere in  $T_y M$ , the intersection

$$\bigcap_{m,n} \mathcal{B}_m(y) \setminus \text{int} \mathcal{C}_n(y)$$

is non-empty. Let  $u$  be a unit vector in this intersection. Since  $u \in \bigcap_m \mathcal{B}_m(y)$ , the above claim shows that  $u \in V_y$ . Therefore, there is  $n_j \rightarrow \infty$  such that  $u^{n_j}$  converges to a vector in  $E^u$ . Then  $u^{n_j} \in \mathcal{C}_0$  for all large  $j$ , and therefore  $u \in \mathcal{C}_{-n_j} \subset \text{int} \mathcal{C}_{-n_j-1}$  for all large  $j$  as well. This gives a contradiction.  $\square$

*Proof of theorem 3.1.* By the so-called ‘‘Fundamental Theorem of Dynamical Systems’’ due to Conley [Nor95], there is a continuous function  $\ell : M \rightarrow \mathbb{R}$  such that  $\ell(f(x)) \leq \ell(x)$  with equality if and only if  $x$  is in the set  $R(f)$  of chain-recurrent points. Further,  $\ell(R(f))$  is a compact, nowhere dense subset of  $\mathbb{R}$ .

Let  $\mathcal{C}$  be a cone field defined on a neighborhood  $U$  of  $R(f)$  as in lemma 3.3. Then, there is  $\delta > 0$  such that  $\ell(x) - \ell(f(x)) > \delta$  for all  $x \notin U$ . Define

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_q < b_q$$

such that  $b_i - a_i < \delta$  for all  $i$  and the union of closed intervals  $[a_i, b_i]$  covers  $\ell(R(f))$ . For  $a, b \in \mathbb{R}$  define

$$\Lambda[a, b] := \{x \in M : \ell(f^n(x)) \in [a, b] \text{ for all } n \in \mathbb{Z}\}$$

If  $x \in \Lambda[a_i, b_i]$ , then  $b_i - a_i < \delta$  implies that  $f^n(x) \in U$  for all  $n$ . Therefore, the dominated splitting may be extended to each  $\Lambda[a_i, b_i]$ . By the inductive hypothesis, assume the dominated splitting has been extended to all of  $\Lambda[a_1, b_k]$ . Choose  $t_k \in (b_k, a_{k+1})$  and use

$$\Lambda = \Lambda[a_i, b_{k+1}] \quad \text{and} \quad U = \{x \in \Lambda : \ell(x) < t_k\}$$

in proposition 3.2 to extend the dominated splitting to all of  $\Lambda$ . By induction, the dominated splitting extends to all of  $\Lambda[a_1, b_q] = M$ .  $\square$

When applying theorem 3.1, it may be a hassle to show directly that  $v^n$  accumulates on  $E^u$ . Suppose instead we know that there is a sequence  $\{n_j\}$  with  $\lim_j n_j = +\infty$  such that  $v^{n_j}$  converges to a unit vector  $w \in T_Z M$  which does not lie in  $E^s$ . As with  $v$ , we use the notation

$$w^m = \frac{Df^m(w)}{\|Df^m(w)\|}.$$

The properties of the dominated splitting on  $Z$  imply that there is a sequence  $\{m_j\}$  tending to  $+\infty$  such that  $\lim_j w^{m_j}$  exists and lies in  $E^u$ . By replacing  $\{n_j\}$  with a further subsequence, one may establish that  $\lim_j v^{n_j+m_j} = \lim_j w^{m_j}$ . This reasoning shows that if  $v^n$  accumulates on a vector in  $T_Z M \setminus E^s$ , it also accumulates on a vector in  $E^u$ .

Iterating in the opposite direction, suppose there is a sequence  $\{n_j\}$  tending to  $+\infty$  such that  $\{v^{-n_j}\}$  converges to  $w \in T_Z M \setminus E^s$ . Then there is a sequence  $\{m_j\}$  tending to  $+\infty$  such that  $\lim_j w^{m_j}$  exists and lies in  $E^u$ . By replacing  $\{n_j\}$  with a subsequence, one may establish both that  $\lim_j (-n_j + m_j) = -\infty$  and  $\lim_j v^{-n_j+m_j} = \lim_j w^{m_j}$ . Hence, if  $v^{-n}$  accumulates on a vector in  $T_Z M \setminus E^s$ , it also accumulates on a vector in  $E^u$ .

With these observations in mind, we now state a slightly generalized version of theorem 3.1. The proof is highly similar and is left to the reader.

**Theorem 3.5.** *Suppose  $f$  is a diffeomorphism of a manifold  $M$ , and  $Y$  and  $Z$  are compact invariant subsets such that*

- (1) *all chain recurrent points of  $f|_Y$  lie in  $Z$ ,*
- (2)  *$Z$  has a dominated splitting  $T_Z M = E^u \oplus E^s$  with  $d = \dim E^u$ , and*
- (3) *for every  $x \in Y \setminus Z$ , there is a point  $y$  in the orbit of  $x$  and a subspace  $V_y$  of dimension  $d$  such that for any non-zero  $v \in V_y$ , each of the sequences  $v^n$  and  $v^{-n}$  accumulates on a vector in  $T_Z M \setminus E^s$  as  $n \rightarrow +\infty$ .*

*Then the dominated splitting on  $Z$  extends to a dominated splitting on  $Y \cup Z$ .*

#### 4. SPLITTINGS ON THE 2-TORUS

This section introduces a number of properties of dominated splittings in dimension 2 that will be used in the next section to prove theorem 1.2. First, we state the announced result of Gourmelon and Potrie mentioned in the introduction.

**Proposition 4.1.** *If  $g_0 : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  has a global dominated splitting and  $g_0$  is isotopic to a linear Anosov diffeomorphism  $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ , then there is a continuous parameterized family of diffeomorphisms  $g : \mathbb{T}^2 \times [0, 1] \rightarrow \mathbb{T}^2$  such that  $g(\cdot, 0) = g_0$ ,  $g(\cdot, 1) = A$ , and each  $g(\cdot, t)$  for  $t \in [0, 1]$  has a dominated splitting.*

If, out of caution, one wants to avoid using this announced but not yet published result, then a condition must be added to the  $g_0$  and  $g_i$  in theorems 1.1 and 1.2 that they lie in the connected component of  $A$ . It is already well established that this connected component contains weakly partially hyperbolic diffeomorphisms which are not Anosov. For completeness, proposition 4.5 below gives a specific example of a weakly partially hyperbolic system which can be reached from a linear Anosov system by a path in the space of systems with dominated splittings.

**Addendum 4.2.** *We may assume  $g$  in proposition 4.1 is a  $C^1$  function both in  $\mathbb{T}^2$  and  $[0, 1]$ .*

*Proof of addendum.* Suppose that  $g$  is  $C^0$  in the parameter  $[0, 1]$ . As diffeomorphisms with a dominated splitting comprise an open subset of all  $C^1$  diffeomorphisms, there is a partition  $0 = t_0 < t_1 < \dots < t_m = 1$  such that if  $t \in [t_i, t_{i+1}]$  then the linear interpolation defined by

$$x \mapsto g(x, t_i) + \frac{t - t_i}{t_{i+1} - t_i} [g(x, t_{i+1}) - g(x, t_i)]$$

has a dominated splitting. Hence, we may assume without loss of generality that  $g$  is piecewise linear. Define a smooth monotonic function  $\alpha : [0, 1] \rightarrow [0, 1]$  such that  $\alpha(t_i) = t_i$  and  $\frac{d\alpha}{dt}|_{t=t_i} = 0$  for all  $i$ . Then  $(x, t) \mapsto g(x, \alpha(t))$  is a  $C^1$  function.  $\square$

To keep consistent notation with section 5, the next proposition uses  $E^c$  and  $E^u$  to denote the bundles of the dominated splitting. In this context, the  $E^u$  bundle may not necessarily be uniformly expanding for all  $t \in [0, 1]$ .

**Proposition 4.3.** *If  $g : \mathbb{T}^2 \times [0, 1] \rightarrow \mathbb{T}^2$  is a  $C^1$  function such that each  $g(\cdot, t)$  has a dominated splitting, then there are cone fields for each  $g(\cdot, t)$  which vary continuously in  $t$ .*

*In particular, there is  $\eta > 0$  such that if, at a point  $(x, t) \in \mathbb{T}^2 \times [0, 1]$ , the dominated splitting is given by*

$$T_x \mathbb{T}^2 = E^c(x, t) \oplus E^u(x, t),$$

*then the cone  $\mathcal{C}(x, t) \subset T_x \mathbb{T}^2$  satisfies the properties*

- (1)  $E^u(x, t) \subset \mathcal{C}(x, t)$ ,
- (2)  $E^c(x, t) \subset \mathcal{C}^*(x, t)$ , and
- (3)  $Dg(\mathcal{C}(x, s))$  lies in the interior of  $\mathcal{C}(g(x, t), t)$  for all  $s \in [0, 1]$  with  $|s - t| < \eta$ .

*Proof.* If a diffeomorphism  $f : M \rightarrow M$  has a dominated splitting of the form  $TM = E_1 \oplus_{<} E_2$ , then it is possible to define  $\lambda < 1$  and a Riemannian metric  $\|\cdot\|$  which depends smoothly on  $x$  and such that

$$\frac{\|Df v_1\|}{\|v_1\|} < \lambda \frac{\|Df v_2\|}{\|v_2\|}$$

for all  $x \in M$  and non-zero  $v_1 \in E_1(x)$  and  $v_2 \in E_2(x)$ . With respect to this metric, the domination is seen under one application of  $f$  instead of an iterate  $f^N$ . See, for instance, [CP15, Section 2.4] for a proof. By adapting the proof to our current setting, one can show that there is  $\lambda < 1$  and a smooth choice of metric  $\|\cdot\|$  such that

$$\frac{\|Df v^c\|}{\|v^c\|} < \lambda \frac{\|Df v^u\|}{\|v^u\|}$$

for all  $(x, t) \in \mathbb{T}^2 \times [0, 1]$  and non-zero  $v^c \in E^c(x, t)$  and  $v^u \in E^u(x, t)$ . Define

$$\mathcal{C}(x, t) = \{v^c + v^u : v^c \in E^c(x, t), v^u \in E^u(x, t), \|v^c\| \leq \|v^u\|\}$$

and note that  $Dg(\mathcal{C}(x, t))$  is in the interior of  $\mathcal{C}(g(x, t), t)$  for all  $x$  and  $t$ . As  $E^c$ ,  $E^u$ , and  $\|\cdot\|$  are all continuous, the function  $(x, t) \mapsto \mathcal{C}(x, t)$  is continuous. Therefore, at each  $(x, t)$ , there is  $\eta > 0$  such that  $Dg(\mathcal{C}(x, s))$  lies in the interior of  $\mathcal{C}(g(x, t), t)$  for all  $s \in [0, 1]$  with  $|s - t| < \eta$ . Since the domain is compact, this  $\eta$  may be chosen uniformly.  $\square$

The next lemma is used to determine the effect of shearing in the proof of theorem 1.2.

**Lemma 4.4.** *Suppose  $A: \mathbb{T}^2 \rightarrow \mathbb{T}^2$  is a hyperbolic toral automorphism which preserves the orientation of its stable bundle. Lift  $A$  to a linear map on  $\mathbb{R}^2$  and let  $E_A^u(0)$  denote the lifted unstable manifold through the origin. For any  $C > 1$ , there is  $z \in \mathbb{Z}^2$  such that  $\text{dist}(\zeta \cdot A(z) + \xi \cdot z, E_A^u(0)) \geq C(\zeta + \xi)$  for all  $\zeta, \xi \geq 0$ .*

The proof is left to the reader. Note how the condition on orientation is necessary.

To conclude the section, we give a simple, concrete example of how a linear Anosov map on  $\mathbb{T}^2$  may be isotoped into a derived-from-Anosov system. This example has the nice additional property that the cone field is independent of both  $x$  and  $t$ . For this example, assume  $\mathbb{T}^2 = \mathbb{R}^2 / 2\pi\mathbb{Z}^2$ .

**Proposition 4.5.** *For  $t \in [0, 1]$ , let  $g_t: \mathbb{T}^2 \rightarrow \mathbb{T}^2$  be defined by*

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x - \frac{9(1-t)}{10} \sin(x) & 0 \\ 0 & y - \frac{1-t}{2} \sin(y) \end{pmatrix}.$$

*Then  $g_t$  is weakly partially hyperbolic with a splitting of the form  $E^c \oplus E^s$ . Moreover, if  $\mathcal{C}$  is the cone field defined in each  $T_x\mathbb{T}^2 \cong \mathbb{R}^2$  by  $\{(u, v) \in \mathbb{R}^2 : uv \geq 0\}$ , then  $Dg_t(\mathcal{C})$  lies in the interior of  $\mathcal{C}$ .*

*If  $t = 1$ , then  $g_t = g_1$  is a hyperbolic toral automorphism.*

*If  $t = 0$ , then  $g_t = g_0$  has a sink at  $(x, y) = (0, 0)$  and is not Anosov.*

*Proof.* Most of this is routine multivariable calculus. The only minor difficulty is proving that  $E^s$  is uniformly contracting. Suppose  $(u_0, v_0) \in \mathbb{R}^2$  is a non-zero vector in the  $E^s$  subbundle and write  $(u_n, v_n) = Dg_t^n(u_0, v_0)$ . Then

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_x & 0 \\ 0 & c_y \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

for some  $c_x, c_y \in \mathbb{R}$  with  $|c_x - 1| < \frac{9}{10}$  and  $|c_y - 1| < \frac{1}{2}$ . Inverting gives

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} c_x^{-1} & 0 \\ 0 & c_y^{-1} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}.$$

Since  $E^s \cap \mathcal{C} = 0$ , the product of  $u_1$  and  $v_1$  is negative. Assume without loss of generality that  $u_1 < 0 < v_1$ . Then  $v_0 = c_y^{-1}(5v_1 - 2u_1) > 3v_1$ . This shows that  $v_n$  shrinks exponentially fast to zero as  $n \rightarrow \infty$ . For any point  $p \in \mathbb{T}^2$  and non-zero vector  $(u, v)$  in  $E^s(p)$ , the ratio  $\frac{u}{v}$  is well defined and depends continuously on  $x$ . Therefore the ratio is uniformly bounded and so  $u_n$  also converges exponentially quickly to zero.  $\square$

## 5. PROOF OF THEOREM 1.2

We now construct the diffeomorphism in the conclusion of theorem 1.2 and show that it is strongly partially hyperbolic. Let  $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ ,  $g_0 : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ , and  $\epsilon > 0$  be as in the theorem. Choose constants  $\epsilon/2 < a < b < c < d < e < \epsilon$ .

We briefly give an intuitive description of the construction before diving into the details. The diffeomorphism  $f$  will contract the region  $\mathbb{T}^2 \times (-e, e)$  down towards  $\mathbb{T}^2 \times 0$ . In the region  $\mathbb{T}^2 \times [c, d]$ , a strong shear pushes the vertical center direction to be almost horizontal. Then in the region  $\mathbb{T}^2 \times [a, b]$ , the dynamics in the horizontal direction is changed from  $A$  to  $g_0$ . Finally in  $\mathbb{T}^2 \times [0, a]$ , the vertical contraction is dialled up so that  $\mathbb{T}^2 \times 0$  is a normally attracting submanifold. The effect of the dynamics on the  $E^{cu}$  and  $E^u$  subbundles is shown in figure 1.

Let  $g : \mathbb{T}^2 \times [0, 1] \rightarrow \mathbb{T}^2$  be a  $C^1$  function as in proposition 4.1. By a reparameterization of the  $[0, 1]$  coordinate, assume without loss of generality that  $g(\cdot, t) = g_0$  for all  $t \leq a$  and  $g(\cdot, t) = A$  for all  $t \geq b$ . With  $g$  now determined, let  $\eta > 0$  be as in proposition 4.3.

Fix a value  $\lambda \in (0, 1)$  such that  $\|Dg_0 v\| > 2\lambda$  for all unit vectors  $v$  in the tangent bundle of  $\mathbb{T}^2$ . Define a smooth diffeomorphism  $h : [0, \epsilon] \rightarrow [0, \epsilon]$  with the following properties.

- (1)  $h(s) = \lambda s$  for all  $s \in [0, \epsilon/2]$ ,
- (2)  $h(s) < s$  for all  $s \in (0, e)$ ,
- (3)  $|h(s) - s| < \eta$  for all  $s \in [a, b]$ ,
- (4)  $h^2(d) < c < h(d)$ , and
- (5)  $h(s) = s$  for all  $s \in [e, \epsilon]$ .

Define a smooth bump function  $\rho : [0, \epsilon] \rightarrow [0, 1]$  such that

- (1)  $\rho(s) = 0$  for all  $s \in [0, c]$ ,
- (2)  $\rho'(s) > 0$  for all  $s \in (c, d)$ , and
- (3)  $\rho(s) = 1$  for all  $s \in [d, \epsilon]$ .

Let  $z$  be a non-zero element of  $\mathbb{Z}^2$ . The precise conditions for choosing  $z$  will be given later in this section.

With these objects in place, define  $f$  for  $(x, s) \in \mathbb{T}^2 \times [0, \epsilon]$  by

$$f(x, s) = (g(x, s) + \rho(s) \cdot z, h(s)).$$

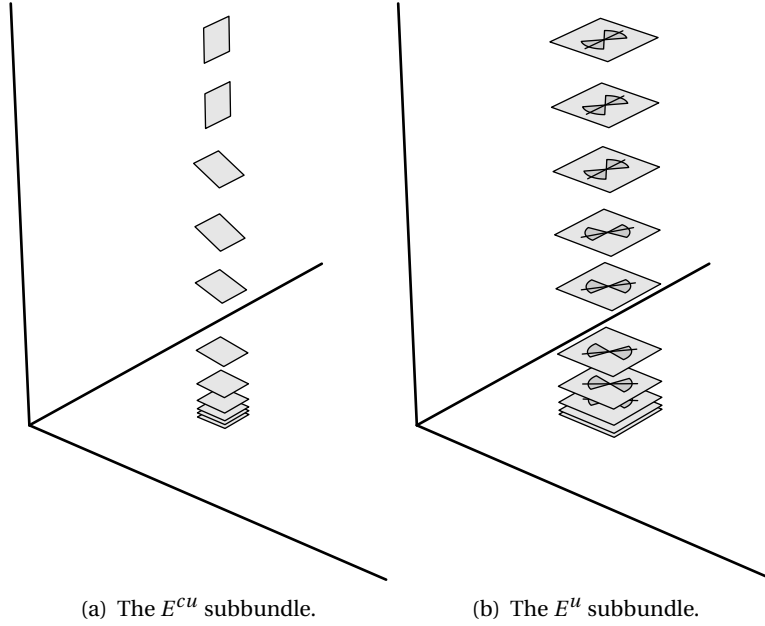


FIGURE 1. A depiction of the  $E^{cu}$  and  $E^u$  subbundles in the construction given in this section. Consider a point  $(x, s) \in \mathbb{T}^2 \times (e, \epsilon)$  and its forward orbit  $(x_n, s_n) := f^n(x, s)$ . For simplicity, we assume the sequence  $\{x_n\}$  is constant. The construction of  $f$  ensures that  $\{s_n\}$  decreases towards 0. Subfigure (a) shows, for each  $n \geq 0$ , the two-dimensional subbundle  $E^{cu}(x_n, s_n)$ . When  $s_n > d$ , the  $E^{cu}$  subbundle is vertical. When  $c < s_n < d$ , a shearing effect in the dynamics pushes the  $E^{cu}$  planes to be closer to horizontal. When  $n$  is large and therefore  $0 < s_n < a$ , the strong vertical contraction means the slopes of these planes tend to zero as  $n$  tends to  $+\infty$ . Subfigure (b) shows, for each  $n \geq 0$ , the one-dimensional subbundle  $E^u(x_n, s_n)$  lying inside the horizontal plane  $T_{x_n} \mathbb{T}^2 \times 0$ . It also depicts the cone field  $\mathcal{C}(x_n, s_n)$  determined by proposition 4.3. Both the horizontal planes and  $E^u$  are unaffected by the shearing. In the region  $\mathbb{T}^2 \times [a, b]$ , the  $E^u$  direction moves around as different horizontal maps  $g(\cdot, t)$  are applied. However, the  $E^u$  direction always stays within the cone field  $\mathcal{C}$ .

Extend  $f$  to all of  $\mathbb{T}^2 \times [-\epsilon, \epsilon]$  by the requirement that  $(y, t) = f(x, s)$  if and only if  $(y, -t) = f(x, -s)$ . Finally, set  $f(x, s) = (A(x), s)$  for all  $(x, s) \notin \mathbb{T}^2 \times [-\epsilon, \epsilon]$ .

We now consider the effect of  $Df$  on vectors of the tangent bundle. The identity  $\mathbb{T}^3 = \mathbb{T}^2 \times S^1$  means that, for a point  $p = (x, s)$ , a tangent vector  $u \in T_p\mathbb{T}^3$  may be decomposed as  $u = (v, w)$  with  $v \in T_x\mathbb{T}^2$  and  $w \in T_sS^1$ . During the proof, we will routinely write vectors this way and refer to  $v$  and  $w$  as the horizontal and vertical components of  $u$ . The linear toral automorphism  $A$  gives a linear splitting  $T_x\mathbb{T}^2 = E_A^u(x) \oplus E_A^s(x)$  which further defines subspaces  $E_A^u(x) \times 0$  and  $E_A^s(x) \times 0$  of  $T_p\mathbb{T}^3$ . Also, if  $\mathcal{C}(p) = \mathcal{C}(x, s) \subset T_x\mathbb{T}^2$  is the cone given by proposition 4.3, then  $\mathcal{C}(p) \times 0$  may be considered as a subset of  $T_p\mathbb{T}^3$ .

**Lemma 5.1.** *If  $p = (x, s) \in \mathbb{T}^2 \times [c, \epsilon]$  and  $y \in \mathbb{T}^2$  is such that  $Df(p) = (y, h(s))$ , then  $Df_p(E_A^u(x) \times 0) = E_A^u(y) \times 0$ .*

*Proof.* In this region,  $f$  is given by  $f(x, s) = (A(x) + \rho(s)z, h(s))$  and both  $A$  and the translation  $x \mapsto x + \rho(s)z$  leave the linear unstable foliation of  $A$  invariant.  $\square$

**Lemma 5.2.** *If  $p \in \mathbb{T}^2 \times [0, c]$ , then  $Df(\mathcal{C}(p) \times 0) \subset \mathcal{C}(f(p)) \times 0$ .*

*Proof.* This follows directly from the use of  $\eta > 0$  in the definition of  $f$ .  $\square$

**Lemma 5.3.**  *$f$  has a dominated splitting of the form  $E^u \oplus_{>} E^{cs}$  with  $\dim E^u = 1$ .*

*Proof.* We will apply theorem 3.5 with  $Y = \mathbb{T}^2 \times [0, e]$  and  $Z = \mathbb{T}^2 \times \{0, e\}$ . Note that  $Z$  has a well-defined partially hyperbolic splitting. If  $p = (x, e) \in \mathbb{T}^2 \times e$ , then  $E_f^u(p) = E_A^u(x) \times 0$ . If  $p = (x, 0) \in \mathbb{T}^2 \times 0$ , then  $E_f^u(p) = E_{g_0}^u(x) \times 0$ .

Consider an orbit  $\{f^n(p)\}_{n \in \mathbb{Z}}$  where  $p \in \mathbb{T}^2 \times (0, e)$ . Up to shifting along the orbit, one may assume  $p = (x, s)$  with  $s \in [h(c), c]$ . Define  $V_p \subset T_p\mathbb{T}^3$  by  $V_p = E_A^u(x) \times 0$  and let  $u$  be a non-zero vector in  $V_p$ . Write  $p_n = (x_n, s_n) = f^n(p)$  for all  $n \in \mathbb{Z}$ . First, consider the backwards orbit of  $u$ . By lemma 5.1,  $u^{-m} \in E_A^u(x_{-m}) \times 0$  for all  $m > 0$ . For a subsequence  $\{m_j\}$ , if  $p_{-m_j}$  converges to a point  $p_- = (x_-, e)$ , then  $u^{-m_j}$  converges to a vector in  $E_A^u(x_-) \times 0 = E_f^u(p_-)$ .

Now consider the forward orbit of  $u$ . By lemma 5.2,  $u^n \in \mathcal{C}(x_n, s_n) \times 0$  for all  $n > 0$ . If a subsequence  $\{p_{n_j}\}$  converges to a point  $p_+ = (x_+, 0)$  and  $v^{n_j}$  converges to a vector  $v_+ \in T_x\mathbb{T}^2 \times 0$ , then  $v_+ \in \mathcal{C}(p_+) \times 0$ . In particular,  $v_+$  does not lie in  $E_A^s(x_+) \times 0$ .

This shows that the conditions of theorem 3.5 are satisfied and a dominated splitting exists on all of  $\mathbb{T}^2 \times [0, e]$ . By symmetry, a dominated splitting exists on  $\mathbb{T}^2 \times [-e, 0]$ . Since  $f$  is linear outside of  $\mathbb{T}^2 \times [-e, e]$ , there is a global dominated splitting on all of  $\mathbb{T}^3$ .  $\square$

For a non-zero vector  $u \in T\mathbb{T}^3$  with horizontal component  $v \in T\mathbb{T}^2$  and vertical component  $w \in TS^1$ , define the *slope* of  $u$  by

$$\text{slope}(u) = \frac{\|w\|}{\|v\|} \in [0, \infty].$$

Note that  $f$  maps a horizontal torus  $\mathbb{T}^2 \times s$  to a horizontal torus  $\mathbb{T}^2 \times h(s)$  and therefore  $\text{slope}(u) = 0$  implies that  $\text{slope} Df(u) = 0$ .

**Lemma 5.4.** *If  $p \in \mathbb{T}^2 \times [0, \frac{\epsilon}{2}]$  and  $u \in T_p \mathbb{T}^3$  with  $\text{slope}(u) < \infty$ , then*

$$\text{slope} Df(u) < \frac{1}{2} \text{slope}(u).$$

*Proof.* This follows from the choice of  $\lambda$  at the start of the section.  $\square$

**Lemma 5.5.** *There is  $k \geq 1$  and  $\delta > 0$  such that if  $p \in \mathbb{T}^2 \times [h^3(d), h^2(d)]$  and  $u \in T_p \mathbb{T}^3$  with  $\text{slope}(u) < \delta$ , then  $f^k(p) \in \mathbb{T}^2 \times [0, \frac{\epsilon}{2}]$  and  $\text{slope} Df^k(u) < 1$ .*

*Proof.* Since  $h(s) < s$  for all  $s \in (0, e)$ , there is  $k \geq 1$  so that  $s < h^2(d)$  implies  $h^k(s) < \epsilon/2$ . Let  $K$  be the compact set of all unit vectors based at points in  $\mathbb{T}^2 \times [h^3(d), h^2(d)]$ , and let  $K_0 \subset K$  be those vectors with slope zero. Define

$$\gamma : K \rightarrow [0, \infty], v \mapsto \text{slope} Df^k(v).$$

Since  $\gamma(K_0) = \{0\}$  and  $\gamma$  is uniformly continuous, one may find  $\delta > 0$  as desired.  $\square$

Since  $h^2(d) < c$ , the choice of  $z \in \mathbb{Z}^2$  does not affect the definition of  $f$  in the region  $\mathbb{T}^2 \times [0, h^2(d)]$ . Hence, the values  $k$  and  $\delta$  may be determined before specifying  $z$ . The next lemma, however, does rely on this choice and the conditions on  $z$  are given in the lemma's proof.

**Lemma 5.6.** *For any  $\delta > 0$ , the  $z \in \mathbb{Z}^2$  used in the definition of  $f$  may be chosen such that the following property holds:*

*If  $p = (x, s) \in \mathbb{T}^2 \times [h(d), d]$  and  $u \in E_A^u(x) \times T_s S^1 \subset T_p \mathbb{T}^3$ , then  $\text{slope} Df^2(u) < \delta$ .*

*Proof.* Write  $u = (v, w)$  as before. If  $w = 0$ , then  $\text{slope} Df^2(u) = 0$ . Therefore, one need only consider the case where  $w$  is non-zero. Up to rescaling the vector  $u$ , assume  $w$  is a unit vector pointing in the ‘‘up’’ direction of  $S^1$ . That is, pointing in the direction of increasing  $s$ . By calculating the derivative of

$$f^2(x, s) = (A^2(x) + \rho(s) \cdot A(z) + (\rho \circ h)(s) \cdot z, h^2(s))$$

one can show that

$$Df^2(v, w) = (A^2(v) + \rho'(s) \cdot A(z) + (\rho \circ h)'(s) \cdot z, Dh^2(w)).$$

Define

$$\alpha := \min\{\rho'(s) + (\rho \circ h)'(s) : s \in [h(d), d]\}$$

and

$$\beta := \max\{(h^2)'(s) : s \in [h(d), d]\}$$

and note that  $\alpha > 0$ . For some  $C > 1$ , if  $z$  is given by lemma 4.4, then

$$\begin{aligned} \|A^2(v) + \rho'(s) \cdot A(z) + (\rho \circ h)'(s) \cdot z\| &\geq \\ \text{dist}(\rho'(s) \cdot A(z) + (\rho \circ h)'(s) \cdot z, E_A^u(0)) &> C\alpha \end{aligned}$$

and therefore  $\text{slope} Df^2(u) < \beta/C\alpha$ . Take  $C$  large enough that  $\beta/C\alpha < \delta$ .  $\square$



For the remainder of the proof, assume  $z$  was chosen so that lemma 5.6 holds with  $\delta > 0$  given by lemma 5.5. The last three lemmas then combine to show the following.

**Corollary 5.7.** *If  $p = (x, s) \in \mathbb{T}^2 \times [h(d), d]$  and  $u \in E_A^u(x) \times 0 \subset T_p \mathbb{T}^3$ , then*

$$\lim_{n \rightarrow +\infty} \text{slope } Df^n(u) = 0.$$

**Lemma 5.8.**  *$f$  has a dominated splitting of the form  $E^{cu} \oplus_{>} E^s$  with  $\dim E^{cu} = 2$ .*

*Proof.* This proof follows the same general outline as the proof of lemma 5.3. Let  $Y$  and  $Z$  be as in that proof. If  $p = (x, e) \in \mathbb{T}^2 \times e$ , then  $E_f^{cu}(p) = E_A^u(x) \times T_e S^1$ . If  $p = (x, 0) \in \mathbb{T}^2 \times 0$ , then  $E_f^u(p) = T_x \mathbb{T}^2 \times 0$ .

Now, consider an orbit  $\{f^n(p)\}_{n \in \mathbb{Z}}$  where  $p \in \mathbb{T}^2 \times (0, e)$ . Up to shifting along the orbit, one may assume  $p = (x, s)$  with  $s \in [h(d), d]$ . Define  $V_p \subset T_p \mathbb{T}^3$  by  $V_p = E_A^u(x) \times T_s S^1$  and let  $u$  be a non-zero vector in  $V_p$ . Write  $p_n = (x_n, s_n) = f^n(p)$  for all  $n \in \mathbb{Z}$ . First, consider the backwards orbit of  $u$ . Note that

$$Df^n(V_p) = E_A^u(x_n) \times T_{s_n} S^1$$

for all  $n < 0$ . Hence, if  $\{u^{-m_j}\}$  is a convergent subsequence, then  $p_{-m_j}$  converges to a point  $p_- \in \mathbb{T}^2 \times e$ , and  $u^{-m_j}$  converges to a vector in  $E_f^{cu}(p_-)$ . In the other direction, corollary 5.7 implies that  $\text{slope}(u^n)$  tends to 0 as  $n \rightarrow \infty$ . If  $\{u^{n_j}\}$  is a convergent subsequence, then  $p_{n_j}$  converges to a point  $p_+ \in \mathbb{T}^2 \times 0$ , and  $u^{n_j}$  converges to a vector in  $E_f^{cu}(p_+)$ . One may then use theorem 3.5 to show that the dominated splitting extends to all of  $\mathbb{T}^3$ .  $\square$

Now that the global invariant dominated splittings  $E^u \oplus E^{cs}$  and  $E^{cu} \oplus E^s$  are known to exist, corollary 2.3 implies that  $f$  is strongly partially hyperbolic on all of  $\mathbb{T}^3$ .

## 6. FURTHER CONSTRUCTIONS

Rodriguez Hertz, Rodriguez Hertz and Ures gave two different constructions of a system on the 3-torus with an invariant center-unstable 2-torus [RHRHU16]. In the first of these constructions, the system is not dynamically coherent as there is no invariant foliation tangent to  $E^c$ . In the second of their constructions, the center bundle  $E^c$  is integrable, but not uniquely integrable. The construction we gave in section 5 corresponds to the first of these cases.

**Proposition 6.1.** *The construction of  $f$  given in section 5 is not dynamically coherent.*

*Proof.* The diffeomorphism  $f$  leaves the foliation of horizontal planes invariant. Therefore, if a vector  $u$  in the tangent bundle  $T\mathbb{T}^3$  has a non-zero vertical component, then  $Df(u)$  also has a non-zero vertical component. If  $p \in \mathbb{T}^2 \times [h(d), e]$ , and  $u$  is a unit vector in  $E_f^c(p)$ , then  $u$  has a non-zero vertical component. By iterating forward, one sees that the same property holds for any  $p \in \mathbb{T}^2 \times (0, e]$ .

Hence, one may choose an orientation for the line bundle  $E_f^c$  on  $\mathbb{T}^2 \times (0, e]$  so that the center direction always points in the direction of decreasing  $s$ . That is, the orientation always points towards  $\mathbb{T}^2 \times 0$ .

This choice extends continuously to  $\mathbb{T}^2 \times [0, e]$ . Further, by the symmetry of the construction, the center orientation may be extended to  $\mathbb{T}^2 \times [-e, e]$ , and on both sides, the center orientation points towards  $\mathbb{T}^2 \times 0$ . This means that any parameterized curve  $\gamma : [0, +\infty) \rightarrow \mathbb{T}^3$  that starts in  $\mathbb{T}^2 \times 0$ , stays tangent to  $E^c$ , and agrees with the orientation of  $E^c$ , must remain for all time inside of  $\mathbb{T}^2 \times 0$ .

The constructed  $f$  is homotopic to  $A$  times the identity map on  $S^1$ . If  $f$  were dynamically coherent, then by the leaf conjugacy given in [HP14, Theorem 1.3], there would be a circle tangent to  $E_f^c$  through every point in  $\mathbb{T}^3$ . In particular, there would be an invariant foliation of center circles lying in  $\mathbb{T}^2 \times 0$ . As the dynamics  $g_0$  on  $\mathbb{T}^2 \times 0$  is homotopic to a hyperbolic toral automorphism, this is not possible and gives a contradiction.  $\square$

We now look at ways in which the construction in the previous section may be modified. The definition of  $f$  may be stated piecewise as

$$f(x, s) = \begin{cases} (g(x, s) + \rho(s) \cdot z, h(s)), & \text{if } s \in [0, \epsilon] \\ (g(x, -s) + \rho(-s) \cdot z, -h(-s)), & \text{if } s \in [-\epsilon, 0] \\ (Ax, s), & \text{if } s \notin [-\epsilon, \epsilon]. \end{cases}$$

Recall that  $z \in \mathbb{Z}^2$  was chosen to satisfy the conclusions of lemma 4.4. If  $k$  is any non-zero integer, then the product  $k \cdot z \in \mathbb{Z}^2$  also satisfies those same conclusions. Thus, for any choice of non-zero integers  $k_1$  and  $k_2$ , one may show that the function defined by

$$(x, s) \mapsto \begin{cases} (g(x, s) + k_1 \rho(s) \cdot z, h(s)), & \text{if } s \in [0, \epsilon] \\ (g(x, -s) + k_2 \rho(-s) \cdot z, -h(-s)), & \text{if } s \in [-\epsilon, 0] \\ (Ax, s), & \text{if } s \notin [-\epsilon, \epsilon] \end{cases}$$

is partially hyperbolic with a  $cu$ -torus at  $\mathbb{T}^2 \times 0$ .

The choices of sign for  $k_1$  and  $k_2$  give four different ways to realize  $g_0$  as the dynamics on an invariant  $cu$ -torus. These correspond to the two different ways the center bundle can approach a horizontal direction on either side of  $\mathbb{T}^2 \times 0$  and are depicted in fig. 2. The cases (a) and (b) in the figure are not dynamically coherent, as may be shown by the argument in the proof of proposition 6.1. From the figure, it appears that the dynamics depicted in each of cases (c) and (d) has an invariant center foliation with leaves which topologically cross the torus. Rigorously proving the existence of this center foliation will require a sophisticated analysis of the Franks semiconjugacy of the system and its relation to the branching foliations of Brin, Burago and Ivanov. This work is left to a future paper.

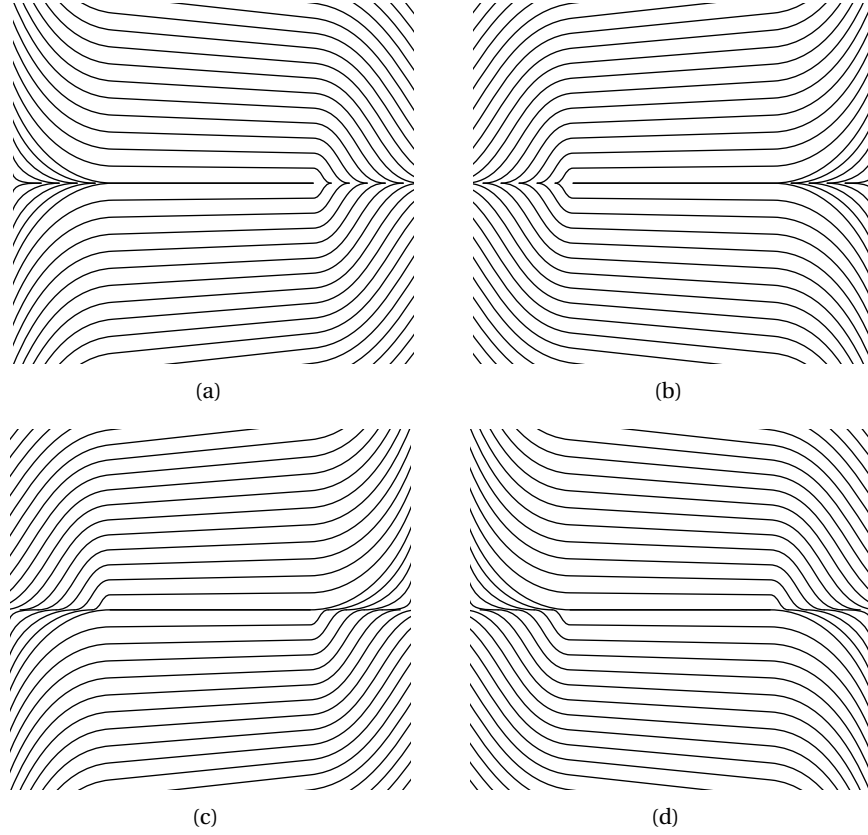


FIGURE 2. Four possible ways in which the center bundle may behave near a center-unstable torus with derived-from-Anosov dynamics. Shown here are lines tangent to the  $E^c$  direction inside a  $cs$ -leaf. In each subfigure, the  $cs$ -leaf intersects the  $cu$ -torus in a horizontal line passing through the middle of the subfigure. In this example, the middle of this line intersects the basin of repulsion of a repelling fixed point inside the  $cu$ -torus so that there are no cusps here.

The above modifications to the construction suggest a way to prove theorem 1.1 in the case where  $g_0$  reverses the orientation of  $E^c$ .

*Proof of theorem 1.1.* Let  $g_0$  be weakly partially hyperbolic with a splitting of the form  $E^c \oplus E^u$ . The case where  $g_0$  preserves the orientation of  $E^c$  was already handled in section 5, so assume here that  $g_0$  reverses the center orientation. Then  $g_0$  is homotopic to a hyperbolic toral automorphism  $A$  which reverses the orientation of its stable bundle  $E_A^s$ . Analogously to lemma 4.4, for any  $C > 1$ , there is  $z \in \mathbb{Z}^2$  such that  $\text{dist}(\zeta \cdot A(z) - \xi \cdot z, E_A^u(0)) \geq C(\zeta + \xi)$  for all  $\zeta, \xi \geq 0$ . (Note now the minus sign before  $\xi \cdot z$ .)

Our constructed diffeomorphism on  $\mathbb{T}^3$  will be the result of modifying the linear map  $A \times (-\text{id})$  defined on  $\mathbb{T}^2 \times S^1$ . Fix a small  $\epsilon > 0$  and define  $h : [0, \epsilon] \rightarrow [0, \epsilon]$  and  $\rho : [0, \epsilon] \rightarrow [0, 1]$  with the properties as listed in section 5. Define  $f$  by

$$f(x, s) = \begin{cases} (g(x, s) + \rho(s) \cdot z, -h(s)), & \text{if } s \in [0, \epsilon] \\ (g(x, -s) - \rho(-s) \cdot z, h(-s)), & \text{if } s \in [-\epsilon, 0] \\ (Ax, -s), & \text{if } s \notin [-\epsilon, \epsilon]. \end{cases}$$

If  $s \in [h(d), d]$ , then

$$f^2(x, s) = (A^2(x) + \rho(s) \cdot A(z) - (\rho \circ h)(s) \cdot z, h^2(s)).$$

The above analogue of lemma 4.4 then establishes an analogue of lemma 5.6 in this context. The other parts of the proof in section 5 are also easily adapted and one may show that  $f$  is strongly partially hyperbolic.  $\square$

For simplicity, the previous section constructed a diffeomorphism on  $\mathbb{T}^3$ . It is a simple matter to apply the same techniques to a 3-manifold defined by the suspension of either an Anosov map or “minus the identity” on  $\mathbb{T}^2$ . The important condition in each case is that there is a strongly partially hyperbolic map and an invariant subset of the manifold homeomorphic to  $\mathbb{T}^2 \times [-\epsilon, \epsilon]$  where the dynamics is given by  $A \times \text{id}$ . As shown in [RHRHU11], these are the only orientable 3-manifolds which allow a torus tangent to  $E^{cu}$  or  $E^{cs}$ .

As explored in [BW05, Section 4] and [HP15a, Appendix A], it is possible to define partially hyperbolic diffeomorphisms on non-orientable manifolds which are double covered by the 3-torus. A similar construction works in the current setting to define one-sided center-stable and center-unstable tori.

**Proposition 6.2.** *For any weakly partially hyperbolic diffeomorphism  $g_0 : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  which preserves its center orientation, there is a non-orientable 3-manifold  $M$ , an embedding  $i : \mathbb{T}^2 \rightarrow M$  and a strongly partially hyperbolic diffeomorphism  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  such that the one-sided torus  $i(\mathbb{T}^2)$  is tangent either to  $E_f^{cs}$  or  $E_f^{cu}$  and  $i^{-1} \circ f \circ i = g_0$ .*

*Proof.* Assume  $g_0$  has a splitting of the form  $E^u \oplus E^c$  and construct  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  as in section 5. Assume  $\mathbb{T}^3$  is defined as  $\mathbb{R}^3 / \mathbb{Z}^3$  and lift  $f$  to a map  $\tilde{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $\tilde{f}(\mathbb{R}^2 \times 0) = \mathbb{R}^2 \times 0$ . Construct a new closed 3-manifold by quotienting  $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$  by the group generated by the translations  $(v, s) \mapsto (v, s + 1)$  and  $(v, s) \mapsto (v + (0, 1), s)$  and the isometry  $(v, s) \mapsto (v + (1, 0), -s)$ .  $\square$

This concludes our construction of examples in dimension 3. The rest of paper handles constructions in higher dimension.

## 7. COMPACT CENTER-STABLE MANIFOLDS OF HIGHER DIMENSION

This section proves theorem 1.4. In fact, we will prove the following restatement of the theorem which gives more technical details about the nature of the constructed diffeomorphism  $F$ .

**Proposition 7.1.** *Let  $f_0 : M \rightarrow M$  be a diffeomorphism, let  $X \subset M$  be a finite invariant set such that every  $x \in X$  is either a periodic source or sink, and let  $U$  be a neighborhood of  $X$ . Then, there are a diffeomorphism  $f : M \rightarrow M$ , a toral automorphism  $A : \mathbb{T}^D \rightarrow \mathbb{T}^D$ , a smooth map  $h : M \rightarrow \mathbb{T}^D$ , and a diffeomorphism  $F : M \times \mathbb{T}^D \rightarrow M \times \mathbb{T}^D$  defined by*

$$F(x, v) = (f(x), Av + h(x))$$

such that:

- (1)  $F$  is strongly partially hyperbolic;
- (2)  $A$  is a linear Anosov diffeomorphism with  $\dim E_A^s = \dim E_A^u = \dim M$ ;
- (3)  $f(x) = f_0(x)$  and  $h(x) = 0$  for all  $x \in M \setminus U$ ;
- (4) if  $x \in NW(f) \setminus X$  and  $v \in \mathbb{T}^D$ , then

$$E_F^s(x, v) = 0 \oplus E_A^s(v), \quad E_F^c(x, v) = T_x M \oplus 0, \quad \text{and} \quad E_F^u(x, v) = 0 \oplus E_A^u(v);$$

- (5) if  $x \in X$  is a sink and  $v \in \mathbb{T}^D$ , then

$$E_F^s(x, v) = T_x M \oplus 0, \quad E_F^c(x, v) = 0 \oplus E_A^s(v), \quad \text{and} \quad E_F^u(x, v) = 0 \oplus E_A^u(v);$$

- (6) if  $x \in X$  is a source and  $v \in \mathbb{T}^D$ , then

$$E_F^s(x, v) = 0 \oplus E_A^s(v), \quad E_F^c(x, v) = 0 \oplus E_A^u(v), \quad \text{and} \quad E_F^u(x, v) = T_x M \oplus 0.$$

Note that the notation and, in particular, the functions  $f$ ,  $g$ , and  $h$  play very different roles here than in previous sections.

The basic idea of the construction is to replace the possibly non-linear behaviour of  $f_0$  in a neighbourhood of a point  $x \in X$  with a simple linear contraction or expansion. Then, both  $f$  and  $A$  are linear maps and there are exactly three rates of contraction or expansion given by  $f$  and the stable and unstable directions of  $A$ . This allows us to restrict our consideration to the case of a linear map

$$F(w, x, y) = (\lambda^{-1}w, bx, \lambda y)$$

defined on  $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$  and where  $0 < \lambda < b < 1$ . We deform this map so that a  $d$ -dimensional subspace which lies roughly in the direction of  $0 \times \mathbb{R}^d \times 0$  converges to the subspace  $0 \times 0 \times \mathbb{R}^d$  under application of the derivative  $DF^n$  as  $n \rightarrow +\infty$ . This provides the effect of pushing the center direction into the stable direction of  $A$ .

The first step is to establish the following.

**Lemma 7.2.** *For  $0 < \lambda < b < 1$  and  $C > 1$ , there is a diffeomorphism  $f$  of  $\mathbb{R}^d$  and a smooth map  $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that the diffeomorphism  $F$  of  $\mathbb{R}^d \times \mathbb{R}^d$  defined by*

$$F(x, y) = (f(x), \lambda y + h(x))$$

has the following properties. If  $p = (x, y) \in \mathbb{R}^d \times \mathbb{R}^d$  with  $b \leq \|x\| \leq 1$ , then

- (1)  $f(x) = bx$  and  $h(x) = 0$ ; and
- (2) if  $V \subset \mathbb{R}^d \times \mathbb{R}^d$  is the graph of a linear map  $L : \mathbb{R}^d \rightarrow \mathbb{R}^d$  with  $\|L\| < C$ , then  $DF_p^n(V)$  tends to  $0 \times \mathbb{R}^d$  as  $n$  tends to  $+\infty$ .

As an aid in proving lemma 7.2, we first introduce a notion of the “quality” of a square matrix. This is closely related to the idea of a row diagonally dominated matrix, however we use different wording here in order to avoid potential confusion between different notions of domination.

Let  $A$  be a  $d \times d$  matrix with entries  $a_{ij}$ . Define the *quality* of the matrix as

$$q(A) := \frac{\min\{a_{ii} : 1 \leq i \leq d\}}{\sum\{|a_{ij}| : 1 \leq i, j \leq d, i \neq j\}}.$$

To have positive quality, a matrix must have positive diagonal entries. We allow  $q(A) = +\infty$  which occurs if and only if  $A$  is diagonal and positive definite.

**Lemma 7.3.** *If  $q(A) > 2$ , then  $A$  is invertible and the operator norm of the inverse satisfies*

$$\|A^{-1}\| \leq \max\left\{\frac{2d}{a_{ii}} : 1 \leq i \leq d\right\}.$$

*Proof.* This is a variation on the Gershgorin circle theorem. Suppose  $v \in \mathbb{R}^d$  is non-zero and let  $i$  be an index such that  $|v_i| \geq |v_j|$  for all  $j$ . Then,

$$\left|\sum_{j=1}^d a_{ij} v_j\right| \geq \left(a_{ii} - \sum_{j \neq i} |a_{ij}|\right) |v_i| \geq \frac{1}{2} a_{ii} |v_i|$$

which implies that  $\|Av\| \geq \frac{1}{2d} a_{ii} \|v\|$ .  $\square$

**Lemma 7.4.** *If  $A$  is a  $d \times d$  matrix with  $q(A) > 0$  and  $B$  is a positive definite diagonal matrix with entries  $b_{ij}$ , then*

$$q(AB) \geq q(A) \min\left\{\frac{b_{ii}}{b_{jj}} : 1 \leq i, j \leq d\right\}.$$

*Proof.* Multiply  $A$  and  $B$  and check.  $\square$

*Proof of lemma 7.2.* We prove lemma 7.2 in the specific case where

$$\frac{b-\lambda}{b-1} < \lambda.$$

Showing that the general case of  $\lambda < b < 1$  may be proved from this special case is left to the reader. With this assumption added, there is a constant  $0 < a < \lambda$  such that

$$\frac{b-a}{b-1} < a.$$

Define a function  $g_0 : [0, \infty) \rightarrow [a, b]$  such that

- (1)  $g_0(t) = a$  for  $t \leq b$ ,
- (2)  $g_0(t) = b$  for  $t \geq 1$ , and
- (3)  $0 \leq t g_0'(t) < a$  for all  $t \geq 0$ .

Define a smooth bump function  $\rho : [0, \infty) \rightarrow [0, 1]$  with  $\rho(t) = 0$  for  $t \geq b$ , and  $\rho(t) = 1$  for  $t \leq b^2$ . Define  $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$  by  $h(x) = \rho(\|x\|)x$ .

Before defining  $f$ , we first consider the behaviour of  $\hat{F}(x, y) := (bx, \lambda y + h(x))$  under iteration. Let  $p = (x, y)$ ,  $V$ , and  $L$  be as in item (2) of the statement of the

lemma being proved. In particular,  $b \leq \|x\| \leq 1$ . For  $n \geq 0$ , define  $\hat{V}_n := D\hat{F}_p^n(V)$  and let  $\hat{L}_n : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the linear map such that  $\text{graph}(\hat{L}_n) = \hat{V}_n$ . The definition of  $\hat{F}$  implies that

$$\hat{L}_{n+1} = \frac{\lambda}{b}\hat{L}_n + \frac{1}{b}Dh$$

where the derivative  $Dh$  is evaluated at  $b^n x$ . If  $n > 2$ , then  $Dh$  is the identity map,  $I$ , and

$$\hat{L}_{n+1} = \frac{\lambda}{b}\hat{L}_n + \frac{1}{b}I.$$

It follows that  $\hat{L}_n$  converges exponentially fast to  $(b - \lambda)^{-1}I$ . When viewed as a matrix,  $(b - \lambda)^{-1}I$  is diagonal and positive definite and so its “quality,” as defined above, is  $q((b - \lambda)^{-1}I) = +\infty$ . Therefore, there is  $N > 2$  such that  $q(\hat{L}_n) > 4$  for all  $n \geq N$ . By compactness, one may find a uniform value of  $N$  such that this lower bound on  $q(\hat{L}_n)$  holds for any starting  $p = (x, y)$ ,  $V$ , and  $L$  with  $\|L\| < C$ .

With  $N$  now fixed, define  $g : [0, \infty) \rightarrow [a, b]$  by  $g(t) := g_0(b^{-N}t)$  and observe that

- (1)  $g(t) = a$  for  $t \leq b^{N+1}$ ,
- (2)  $g(t) = b$  for  $t \geq b^N$ , and
- (3)  $0 \leq t g'(t) < a$  for all  $t \geq 0$ .

Define  $f$  by  $f(x) = g(\|x\|)x$ . With  $f$  and  $h$  now defined, we show that  $F(x, y) = (f(x), \lambda y + h(x))$  satisfies the conclusions of the lemma.

This definition of  $F$  has a form of radial symmetry: if  $R$  is a rigid rotation about the origin in  $\mathbb{R}^d$ , then  $f \circ R = R \circ f$ ,  $h \circ R = R \circ h$ , and  $F \circ (R \times R) = (R \times R) \circ F$ . Further, any one-dimensional subspace in  $\mathbb{R}^d$  is invariant under  $f$ . Because of this symmetry, when analysing orbits of  $F$ , we need only consider points of the form  $p = (x, y)$  where  $x \in \mathbb{R} \times 0$ . That is, if  $x$  is written in coordinates as  $x = (x_1, x_2, \dots, x_d)$ , then  $x_2 = x_3 = \dots = x_d = 0$ .

The partial derivatives of  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  are given by

$$\frac{\partial f_i}{\partial x_j} = g(\|x\|)\delta_{ij} + \frac{x_i x_j}{\|x\|} g'(\|x\|).$$

Since we are assuming  $x \in \mathbb{R} \times 0$ , the terms  $x_i x_j$  all evaluate to 0 except for the term  $x_1 x_1$ . Therefore

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= g(\|x\|) + \|x\| g'(\|x\|) \\ \frac{\partial f_i}{\partial x_i} &= g(\|x\|) && \text{if } i > 1, \text{ and} \\ \frac{\partial f_i}{\partial x_j} &= 0 && \text{if } i \neq j. \end{aligned}$$

Further  $g'(\|x\|)$  is non-zero only when  $b^{N+1} < \|x\| < b^N$  and one may show that

$$g(\|x\|) \leq g(\|x\|) + \|x\| g'(\|x\|) \leq 2g(\|x\|).$$

In other words, the Jacobian of  $f$  is a diagonal matrix where no entry is more than twice as large as any other.

Let  $p = (x, y)$  with  $x \in \mathbb{R} \times 0$  and  $b \leq \|x\| \leq 1$ . Let  $V$  and  $L$  be as in item (2) of the statement of the lemma. For  $n \geq 0$ , define  $V_n := DF_p^n(V)$  and  $L_n : \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that  $\text{graph}(L_n) = V_n$ . We now analyze  $L_n$  as  $n$  tends to  $+\infty$ . First, if  $n < N$ , then  $\|f^n(x)\| \geq b^N$  and the functions  $F^n$  and  $\hat{F}^n$  are equal in a neighborhood of  $p$ . Therefore  $L_N = \hat{L}_N$  and in particular  $q(L_N) > 4$ .

For the case  $n = N$ , the equality  $\text{graph}(L_{N+1}) = DF(\text{graph}(L_N))$  may be written as

$$\{(u, L_{N+1}(u)) : u \in \mathbb{R}^d\} = \{(Df(v), \lambda L_N(v) + v) : v \in \mathbb{R}^d\}$$

showing that  $L_{N+1} = (\lambda L_N + I) \circ Df^{-1}$  where  $Df$  is evaluated at  $f^N(x)$ . Lemma 7.4, along with the above remark about the Jacobian of  $f$ , shows that

$$q((\lambda L_N + I) \circ Df^{-1}) \geq \frac{1}{2} q(\lambda L_N + I).$$

and this implies that  $q(L_{N+1}) \geq \frac{1}{2} q(L_N) > 2$ .

Finally, for  $n > N$ , the point  $f^n(x)$  satisfies  $\|f^n(x)\| \leq b^{N+1}$ . For points in this region,  $Df = aI$  and so  $L_{N+1} = \frac{\lambda}{a} L_N + \frac{1}{a} I$ , which implies that  $q(L_{n+1}) > q(L_n) > 2$  for all large  $n$ . Since  $\frac{\lambda}{a} > 1$ , the linear map  $L_n$  when viewed as a matrix has positive entries on its diagonal and these entries tend to  $+\infty$  as  $n$  tends to  $+\infty$ . Lemma 7.3 implies that  $\|L_n^{-1}\|$  tends to zero as  $n \rightarrow +\infty$  and therefore the sequence of subspaces  $V_n$  tends to  $0 \times \mathbb{R}^d$ .  $\square$

The next result simply adds an expanding direction to lemma 7.2.

**Corollary 7.5.** *For  $0 < \lambda < b < 1$  and  $C > 1$ , there is a diffeomorphism  $f$  of  $\mathbb{R}^d$  and a smooth map  $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that the diffeomorphism  $F$  of  $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$  defined by*

$$F(w, x, y) = (\lambda^{-1}w, f(x), \lambda y + h(x))$$

*has the following properties. If  $p = (w, x, y) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$  with  $b \leq \|x\| \leq 1$ , then*

- (1)  $f(x) = bx$  and  $h(x) = 0$ ;
- (2) *if  $V \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$  is the graph of a linear map  $L : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  with  $\|L\| < C$ , then  $DF_p^n(V)$  tends to  $\mathbb{R}^d \times 0 \times 0$  as  $n$  tends to  $+\infty$ ; and*
- (3) *if  $V \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$  is the graph of a linear map  $L : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  with  $\|L\| < C$ , then  $DF_p^n(V)$  tends to  $\mathbb{R}^d \times 0 \times \mathbb{R}^d$  as  $n$  tends to  $+\infty$ .*

*Proof.* Use the same  $f$  and  $h$  as in lemma 7.2.  $\square$

With this established, we now consider diffeomorphisms defined on closed manifolds. For a closed manifold  $M$  and a hyperbolic toral automorphism  $A : \mathbb{T}^D \rightarrow \mathbb{T}^D$ , an  $A$ -map is a map  $F : M \times \mathbb{T}^D \rightarrow M \times \mathbb{T}^D$  of the form

$$F(x, v) = (f(x), Av + h(x)).$$

See [GORH15] for a more general definition and further details. If  $F$  is also a (strongly) partially hyperbolic diffeomorphism, we call it a partially hyperbolic  $A$ -map. Note that we do not a priori assume that the partially hyperbolic splitting has any relation to the fibers of the torus bundle.



There is a small subtlety in proving proposition 7.1 in the case where the basin of a sink overlaps the basin of a source. To handle this, we will prove proposition 7.1 by induction and keep track of a property we call being “graph like” for the splitting at a point.

For a partially hyperbolic  $A$ -map and a point  $x \in M$ , the subbundle  $E^u$  is *graph like* at  $x$  if, for all  $v \in \mathbb{T}^D$ ,  $E^u(x, v)$  is the graph of a linear function from  $E_A^u(v)$  to  $E_A^s(v) \oplus T_x M$ . Similarly,  $E^{cu}$ ,  $E^{cs}$ , and  $E^s$  are graph like at  $x$  if they are graphs of linear functions

$$T_x M \oplus E_A^u(v) \rightarrow E_A^s(v), \quad T_x M \oplus E_A^s(v) \rightarrow E_A^u(v), \quad \text{and} \quad E_A^s(v) \rightarrow E_A^u(v) \oplus T_x M$$

respectively. If all of  $E^u$ ,  $E^{cs}$ ,  $E^{cu}$ , and  $E^s$  are graph like at  $x$ , we say the splitting is graph like at  $x$ .

Since the bundles in the splitting are continuous and  $DF$ -invariant the following is easily verified.

**Lemma 7.6.** *Let  $F$  be a partially hyperbolic  $A$ -map with base map  $f : M \rightarrow M$ . For a bundle  $E \in \{E^u, E^{cu}, E^{cs}, E^s\}$ , the set of graph-like points is open and  $f$ -invariant.*

Next, we consider a normally attracting fiber.

**Lemma 7.7.** *For a partially hyperbolic  $A$ -map  $F$  with base map  $f : M \rightarrow M$ , if  $x \in M$  is a periodic sink for  $f$  and  $x \times \mathbb{T}^D$  is tangent to  $E^{cu}$ , then  $E^s$  and  $E^{cs}$  are graph like for every point in the basin of  $x$ .*

*Proof.* Since  $E_F^s$  is transverse to  $x \times \mathbb{T}^D$ , it is graph like at  $x$ . By the uniqueness of the dominated splitting on  $x \times \mathbb{T}^D$ ,  $E_F^s(x, v) = E_A^s(v)$  for all  $v \in \mathbb{T}^D$ . Therefore,  $E_F^{cs}$  is also graph like at  $x$ . By the previous lemma, being graph like at  $x$  extends to being graph like on the basin of  $x$ .  $\square$

The next lemma allows us to replace non-linear sinks with linear ones.

**Lemma 7.8.** *Let  $f_0 : M \rightarrow M$  be a diffeomorphism with a periodic sink  $x_0 = f_0^k(x_0)$  and let  $\epsilon > 0$  and  $0 < b < 1$ . Then there is a diffeomorphism  $f : M \rightarrow M$  and a coordinate chart  $\varphi : [-1, 1]^d \rightarrow M$  such that*

- (1) *if  $\text{dist}(x, x_0) > \epsilon$ , then  $f(x) = f_0(x)$ ,*
- (2)  *$f$  and  $f_0$  have the same non-wandering set,*
- (3)  *$\varphi(0) = x_0$ , and*
- (4)  *$\varphi^{-1} \circ f^k \circ \varphi(y) = by$  for all  $y \in [-1, 1]^d$ .*

*Proof.* This follows from standard methods of pasting diffeomorphisms [Wil72]. First, one may make a  $C^1$  small perturbation in order to assume that  $\varphi^{-1} \circ f^k \circ \varphi$  is linear in a neighborhood of 0. Then, deform the linear map inside that neighborhood to get the desired homothety.  $\square$

Now we state what will be the inductive step in proving proposition 7.1.

**Proposition 7.9.** *Let  $A$  be a hyperbolic toral automorphism of  $\mathbb{T}^D$  with eigenvalues  $\lambda < 1$  and  $\lambda^{-1} > 1$ , each of multiplicity  $d = \frac{1}{2}D$ . Suppose  $F_0$  is a partially hyperbolic  $A$ -map having a base map  $f_0 : M \rightarrow M$  with  $\dim M = d$  and  $x_0$  is a periodic sink such that the splitting is graph like at  $x_0$ . For any  $\epsilon > 0$ , there is a partially hyperbolic  $A$ -map  $F$  such that*

- (1) *if  $\text{dist}(x, x_0) > \epsilon$ , then  $F(x, v) = F_0(x, v)$  for all  $v \in \mathbb{T}^D$ ;*
- (2) *if the splitting for  $F_0$  is graph like at  $x \neq x_0$ , then the splitting for  $F$  is also graph like at  $x$ ; and*
- (3)  *$x_0 \times \mathbb{T}^D$  is an  $F$ -periodic submanifold tangent to  $E_F^{cu}$ .*

*Proof.* This proof breaks into two steps. First, we deform  $F_0$  to produce a partially hyperbolic map  $F_1$  which is linear in a neighborhood of  $x_0 \times \mathbb{T}^D$ , but which still has a graph like splitting at  $x_0$ . Then, we paste in the dynamics given by corollary 7.5, to produce a partially hyperbolic map  $F$  for which  $E_F^{cu}$  is tangent to  $x_0 \times \mathbb{T}^D$ .

Let  $U$  be a neighborhood of the orbit of  $x_0$  such that  $\overline{U}$  is contained in the basin of attraction and  $f_0(\overline{U}) \subset U$ . Define a smooth function  $h_1 : M \rightarrow \mathbb{T}^D$  such that  $h_1(x) = h_0(x)$  for all  $x \in M \setminus U$  and  $h_1(x) = 0$  for all  $x \in f(U)$ .

Fix  $b$  such that  $\lambda < b < 1$  where  $\lambda$  is the stable eigenvalue of  $A$ . Let  $k$  denote the period of  $x_0$ . By lemma 7.8, there is a coordinate chart  $\varphi : [-1, 1]^d \rightarrow M$  and a diffeomorphism  $f_1 : M \rightarrow M$  such that  $\varphi^{-1} \circ f_1^k \circ \varphi(x) = bx$  for all  $x \in [-1, 1]^d$ . Moreover, we may freely assume that  $\varphi([-1, 1]^d) \subset f_0(U)$  and that  $f_1(x) = f_0(x)$  for all  $x \in M \setminus f_0(U)$ . By abuse of notation, we identify  $[-1, 1]^d$  with its image and regard  $[-1, 1]^d$  as a subset of  $M$ .

Define a diffeomorphism  $F_1$  of  $M \times \mathbb{T}^D$  by  $F_1(x, v) = (f_1(x, v), Av + h_1(x))$ . If  $x \in \overline{U} \setminus f_1(U)$  and  $v \in \mathbb{T}^D$ , define  $E_{F_1}^u(x, v) := E_{F_0}^u(x, v)$ . Using theorem 3.5, one may then establish the existence of a dominated splitting  $E_{F_1}^u \oplus E_{F_1}^{cs}$  on all of  $M \times \mathbb{T}^D$ . Similarly, if  $x \in \overline{U} \setminus f_1(U)$  and  $v \in \mathbb{T}^D$ , define  $E_{F_1}^{cu}(x, v) := E_{F_0}^{cu}(x, v)$  and apply the same reasoning to establish a dominated splitting of the form  $E_{F_1}^{cu} \oplus E_{F_1}^s$  on all of  $M \times \mathbb{T}^D$ . From this, one may show that  $F_1$  is partially hyperbolic and that the splitting of  $F_1$  is graph like at a point  $x$  if and only if the original  $F_0$  was graph like at  $x$ .

Since  $E_{F_1}^u$  is continuous and graph like on  $U$ , there is a uniform constant  $C > 1$  such that if  $x \in [-1, 1]^d \subset M$  with  $b \leq \|x\| \leq 1$  and  $v \in \mathbb{T}^D$  then  $E_{F_1}^u(x, v)$  is the graph of a linear function  $L : E_A^u(x, v) \rightarrow T_x M \oplus E_A^s(x, v)$  with  $\|L\| < C$ . A similar bound also holds when  $E_{F_1}^{cu}(x, v)$  is expressed as the graph of a linear function. By corollary 7.5, there are functions  $f : M \rightarrow M$  and  $h : M \rightarrow \mathbb{T}^D$  such that  $F$  defined by  $F(x, v) = (f(x), Av + h(x))$  satisfies the following properties.

- (1) If either  $x \in M \setminus [-1, 1]^d$  or  $x \in [-1, 1]^d$  with  $\|x\| > 1$ , then  $f(x) = f_1(x)$  and  $h(x) = h_1(x)$ .

- (2) If  $x \in [-1, 1]^d$  with  $b \leq \|x\| \leq 1$ , then  $f(x) = bx$ . Further, if  $\{n_j\} \subset \mathbb{N}$  is such that  $F^{n_j}(x, v)$  converges to a point  $(x_0, v_0)$  in  $x_0 \times \mathbb{T}^D$ , then  $DF^{n_j}(E_{F_1}^u(x, v))$  converges to  $0 \times E_A^u(v_0)$  and  $DF^{n_j}(E_{F_1}^{cu}(x, v))$  converges to  $0 \times T_{v_0} \mathbb{T}^D$ .

Then theorem 3.5 shows that  $F$  is partially hyperbolic with  $x_0 \times \mathbb{T}^D$  tangent to  $E_F^{cu}$ .  $\square$

With proposition 7.9 established, proposition 7.1 easily follows.

*Proof of proposition 7.1.* Given  $f_0$ , define a hyperbolic toral automorphism  $A : \mathbb{T}^D \rightarrow \mathbb{T}^D$  such that  $F_0 := f_0 \times A$  is partially hyperbolic. For instance,  $A$  can be the direct product of  $d$  copies of a high iterate of the cat map. Clearly,  $F_0$  is a partially hyperbolic  $A$ -map and the splitting is graph like at all points. Let  $x_0$  be any point in  $X$  and apply proposition 7.9 to  $F_0$  and  $x_0$  to produce a map  $F_1$  where  $x_0 \times \mathbb{T}^D$  is tangent either to  $E^{cs}$  and  $E^{cu}$ . If  $X$  contains a point  $x_1$  which is not in the orbit of  $x_0$ , then apply proposition 7.9 to  $F_1$  and  $x_1$  to produce a map  $F_2$ . After a finite number of steps of this form, the desired map  $F$  in proposition 7.1 is constructed.  $\square$

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