ACCESSIBILITY OF DERIVED-FROM-ANOSOV SYSTEMS

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ABSTRACT. This paper shows any non-Anosov partially hyperbolic diffeomorphism on the 3-torus which is homotopic to Anosov must be accessible. It implies if the non-wandering set of such diffeomorphisms is the whole 3-torus, then the diffeomorphism is transitive.

1. INTRODUCTION

Let M be a closed Riemannian manifold, and $\mathrm{Diff}^r(M)$ be the space which consists of all C^r-diffeomorphisms ($r \geq 1$) of M and is endowed with C^r-topology. We say $f \in \mathrm{Diff}^r(M)$ is partially hyperbolic if there exist a continuous Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ and two continuous functions $\sigma, \mu : M \to \mathbb{R}$, such that $0 < \sigma < 1 < \mu$ and

$$\|Df(v^s)\| < \sigma(x) < \|Df(v^c)\| < \mu(x) < \|Df(v^u)\|$$

for every $x \in M$ and unit vector $v^* \in E^*(x)$, for $* = s, c, u$. Let $\mathrm{PH}^r(M)$ denote the set consisting of all partially hyperbolic diffeomorphisms on M. It is obvious that $\mathrm{PH}^r(M)$ is an open set in $\mathrm{Diff}^r(M)$ within the C^r-topology.

Robust transitivity is an important hallmark of chaotic dynamics. We say $f \in \mathrm{Diff}^1(M)$ is *robustly transitive* if f admits a C^1-neighborhood $\mathscr{U}$ such that every $g \in \mathscr{U}$ is robustly transitive. Due to the structural stability of Anosov systems, a transitive Anosov diffeomorphism is robustly transitive. In fact, for a long time, Anosov diffeomorphisms were the only known examples of robustly transitive diffeomorphisms until the discovery of counterexamples by M. Shub [Chi71, Page 39] and R. Mañé [Mn78].

R. Mañé constructed a non-hyperbolic robustly transitive diffeomorphism on $\mathbb{T}^3$. His example, now known as Mañé's example, was partially hyperbolic and homotopic to an Anosov automorphism. We say $f \in \mathrm{PH}^r(\mathbb{T}^3)$ is a *derived-from-Anosov diffeomorphism* or a *DA-diffeomorphism*, if it is homotopic to an Anosov automorphism. Here the Anosov automorphism is given by the linear part $f_* : \pi_1(\mathbb{T}^3) \to \pi_1(\mathbb{T}^3)$ of f.

Partially hyperbolic dynamics in dimension 3 and particularly on the 3-torus is now very well understood. Derived-from-Anosov diffeomorphisms have been studied extensively. For instance, every DA-diffeomorphism is dynamically coherent and leaf conjugate to its linear part [BBI09, Ham11, Pot15]. There exists

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an open set of conservative DA-diffeomorphisms whose center Lyapunov exponents have a different sign than their linear part [PT14], and for such examples the center foliation is “minimal yet measurable” [PTVa14]. The disintegration of measure along the center foliation has been studied in detail [VY17]. Every conservative partially hyperbolic DA-diffeomorphism is ergodic [HU14, GS19]. In certain settings, it is further known to be Bernoulli [PTVa18].

However, a major question of transitivity for derived-from-Anosov diffeomorphisms remains open.

Question 1.1. *Is every partially hyperbolic derived-from-Anosov diffeomorphism on $\mathbb{T}^3$ transitive?*

Since DA-diffeomorphisms consist an open subset of $\mathrm{PH}^1(\mathbb{T}^3)$, if every DA-diffeomorphism is transitive, then they are all robustly transitive as well.

For partially hyperbolic diffeomorphisms, one of the major tools for establishing transitivity is due to M. Brin [Bri75], showing that if every point is non-wandering and the system is accessible, then it is transitive. Here, we say $f \in \mathrm{PH}^r(M)$ is *accessible* if for every $x, y \in M$, there exists a piecewise smooth curve from x to y, such that each smooth piece is contained in a leaf of either stable or unstable foliations of f.

In light of Brin’s result [Bri75], one approach to construct a non-transitive example of DA-diffeomorphisms might be to start with a linear Anosov automorphism, which is not accessible as it has an invariant us-foliation, and to do a pitchfork bifurcation on a periodic point producing a non-Anosov example. One might also hope that one could do this deformation in such a way that the deformation "blows air" into the leaves of the original us-lamination. This DA-diffeomorphism would then not be accessible and in fact would have an invariant us-lamination. However in this paper, we prove a result showing that such an approach is impossible.

Theorem 1.2. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a $C^{1+\alpha}$ partially hyperbolic derived-from-Anosov diffeomorphism. If f is not Anosov, then it is accessible.*

This theorem shows that any version of constructing Mañé’s example [Mn78] must be accessible. Moreover, since every Anosov diffeomorphism on $\mathbb{T}^3$ is transitive, Brin’s work [Bri75] has the following corollary.

Corollary 1.3. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a $C^{1+\alpha}$ partially hyperbolic derived-from-Anosov diffeomorphism. If the non-wandering set of f is $\mathbb{T}^3$, then f is transitive.*

We recommend [Pot14a] for more discussions about the transitivity of partially hyperbolic derived-from-Anosov diffeomorphisms. Applying Theorem 5.1 of [GS19], Theorem 1.2 has the following corollary.

Corollary 1.4. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a $C^{1+\alpha}$ partially hyperbolic derived-from-Anosov diffeomorphism, then either f is accessible, or it is Anosov and every periodic point of f has the same center Lyapunov exponent with its linear part f_*.*

Another major motivation for Theorem 1.2 is the study of accessibility in its own right. M. Grayson, C. Pugh, and M. Shub [GPS94] first introduced the concept of accessibility in the volume-preserving setting as a tool for establishing ergodicity. However, accessibility has many applications in the non-volume-preserving setting such as the result of Brin [Bri75] mentioned above. C. Pugh and M. Shub conjectured that accessibility is an open and dense property among partially hyperbolic diffeomorphisms, volume preserving or not. This property has been established in a number of settings, including the case of one-dimensional center [BHH$^+$08] (which always holds when the diffeomorphism is defined on a three-dimensional manifold).

F. Rodriguez Hertz, J. Rodriguez Hertz, and R. Ures [RHRHU08b] conjectured that the existence of a torus tangent to the us-direction is the unique obstruction to accessibility of partially hyperbolic diffeomorphisms in dimension 3. Theorem 1.2 handles the last unknown case for accesssibility on the 3-torus. In fact, when combined with results in [Ham17], it gives a complete description of all possible accessibility classes for any 3-manifold with solvable fundamental group.

Corollary 1.5. *Let f be a partially hyperbolic diffeomorphism on a 3-manifold with solvable fundamental group, then exactly one of the following holds:*

(1) f is accessible;
(2) f has a minimal invariant us-foliation and is an Anosov diffeomorphism on $\mathbb{T}^3$;
(3) f has a 2-torus tangent to $E^s \oplus E^u$.

See [Ham17] for further details on the accessibility classes in the case of the third item above. This corollary gives a positive answer in the case of solvable fundamental group to the following question which still remains open for general 3-manifolds.

Question 1.6. *Suppose f is a partially hyperbolic diffeomorphism with one-dimensional center and $\emptyset \neq \Gamma(f) \neq M$ is an f-invariant lamination tangent to $E^s \oplus E^u$. Does $\Gamma(f)$ have a compact leaf?*

The proof of Theorem 1.2 extends techniques first developed in [HU14] and [GS19] to prove ergodicity for DA-diffeomorphisms in the volume-preserving setting. However, there is a key difference in the overall approach to the proof. For any DA-diffeomorphism, there is a semiconjugacy, discovered by Franks [Fra70], from the non-linear system to its linear part. One of the first steps in [HU14] is to use the volume-preserving assumption to show that this semiconjugacy is in fact a true conjugacy; that is, a homeomorphism. In the setting of the current paper, we do not assume the system is volume preserving and so we must always handle the possibility that the semiconjugacy is not bijective. This significantly complicates a number of parts of the proof. For instance, see proposition 2.1 and its use in the proof of proposition 2.6 below.

Organization of the paper: In Section 2, we study the semiconjugacy and the lift of invariant foliations to the universal cover $\mathbb{R}^3$. In Section 3, we prove a series of properties of the minimal invariant us-laminations on $\mathbb{T}^3$. In Section 4, we prove the main theorem.

2. Laminations on the universal cover

This section analyzes the dynamics when lifted to the universal cover $\mathbb{R}^3$. Later sections will use the results proved here in order to analyse the original system on the 3-torus. Before looking at the partially hyperbolic system, we first establish a property for curves on $\mathbb{R}^2$ which will be of use later.

Proposition 2.1. *Let S be a collection of curves in $\mathbb{R}^2$ and X be a dense subset of $\mathbb{R}^2$ with the following properties:*

(1) each curve in S is the graph $\operatorname{graph}(g) = \{(x, g(x)) : x \in \mathbb{R}\}$ *of a continuous function $g : \mathbb{R} \to \mathbb{R}$;*
(2) no two curves topologically cross; and
(3) if γ is a curve in S and $(x_0, y_0) \in X$, then the translation $\gamma + (x_0, y_0)$ is also a curve in S.

Then the curves in S are straight lines and all have the same slope.

Remark. If $\operatorname{graph}(g_1)$, $\operatorname{graph}(g_2)$ are curves in S, then the condition of no topological crossings implies that either $g_1(x) \le g_2(x)$ holds for all $x \in \mathbb{R}$ or $g_2(x) \le g_1(x)$ holds for all $x \in \mathbb{R}$.

Proof. Consider the closure of S in the compact-open topology. That is, the graph of a function $g : \mathbb{R} \to \mathbb{R}$ belongs to $\bar{S}$ if and only if there is a sequence $\{\operatorname{graph}(g_n)\}$ in S such that $g_n|_K$ converges uniformly to $g|_K$ on every compact subset K of $\mathbb{R}$. One can show that the no crossing condition on S implies a no crossing condition on $\bar{S}$. If $\gamma_n \to \gamma$ in the compact-open topology and $(x_0, y_0) \in X$, then the translates $\gamma_n + (x_0, y_0)$ converge to $\gamma + (x_0, y_0)$ in the compact-open topology. Hence, $\bar{S}$ is invariant under all translates $(x_0, y_0) \in X$. For $\gamma \in \bar{S}$ and any point $(x_0, y_0) \in \mathbb{R}^2$, let (x_n, y_n) be a sequence in X converging to (x_0, y_0). Then $\gamma + (x_n, y_n)$ converges to $\gamma + (x_0, y_0)$ in the compact-open topology and so $\gamma + (x_0, y_0) \in \bar{S}$.

Now knowing that $\bar{S}$ is invariant under *all* translations in $\mathbb{R}^2$, we can show it is linear. Indeed, let $\operatorname{graph}(g)$ be a curve in $\bar{S}$ passing though the origin so that $g(0) = 0$. For $a, b \in \mathbb{R}$, define functions $g_n : \mathbb{R} \to \mathbb{R}$ by $g_n(x) = g(x - a) + g(a) + \frac{1}{n}$. Then $g_n(a) > g(a)$ and the no crossing condition imply that

$$g(a) + g(b) + \tfrac{1}{n} = g_n(a + b) \ge g(a + b)$$

for all n. A similar argument shows that $g(a)+g(b)-\frac{1}{n} \le g(a+b)$ for all n, and so $g(a) + g(b) = g(a + b)$. As g is continuous and additive, it is linear. The translates of $\operatorname{graph}(g)$ produce a linear foliation on all of $\mathbb{R}^2$ that no other curve of $\bar{S}$ can cross. This implies that every curve of $\bar{S}$ is linear and of the same slope. $\square$

With proposition 2.1 established, we now consider the dynamics. Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be partially hyperbolic and homotopic to an Anosov diffeomorphism. Lift f to a map on the universal cover. All of the analysis in this section will be on $\mathbb{R}^3$, and so we again use f to denote the lifted map. This lift $f : \mathbb{R}^3 \to \mathbb{R}^3$ is at finite distance from a hyperbolic linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$. Here $A = f_*$ is the linear part of f. Up to replacing f by its inverse, we may assume the center direction of A is contracting. That is, the logarithms of the eigenvalues of A satisfy

$$\lambda^s(A) < \lambda^c(A) < 0 < \lambda^u(A).$$

Many properties have been established for f, first in the absolutely partially hyperbolic case [BI08, BBI09, Ham13] and then extended to the case of pointwise partial hyperbolicity [HP14].

There are unique invariant foliations tangent to the bundles E^u, E^s, E^{cs}, E^{cu}, and E^c of f. Denote these foliations by $\mathscr{F}^u$, $\mathscr{F}^s$, $\mathscr{F}^{cs}$, $\mathscr{F}^{cu}$, and $\mathscr{F}^c$. For the linear map A, we adopt the notation used in [HP15, HP18] and write $\mathscr{A}^u$, $\mathscr{A}^s$, $\mathscr{A}^{cs}$, $\mathscr{A}^{cu}$, $\mathscr{A}^c$, and $\mathscr{A}^{us}$ for the invariant linear foliations of A. For both f and A, all of these foliations have quasi-isometrically embedded leaves [HP14, Theorem 3.5] [Ham13, Proposition 2.6]. That is, there is $Q > 1$ such that if x and y lie on the same leaf of the foliation, then $d_{\mathscr{F}}(x, y) < Q\|x - y\| + Q$ where $\|x - y\|$ is the usual distance in $\mathbb{R}^3$ and $d_{\mathscr{F}}(x, y)$ is distance measured along the leaf. In this section, we use d_u, d_c, d_s to denote distance measured along leaves associated to the non-linear system f.

The foliations have *global product structure* [Ham13, Proposition 2.15]. That is, for $x, y \in \mathbb{R}^3$, the following pairs of sets intersect in a unique point:

(1) $\mathscr{F}^{cs}(x)$ with $\mathscr{F}^u(y)$,
(2) $\mathscr{F}^{cu}(x)$ with $\mathscr{F}^c(y)$,
(3) $\mathscr{F}^c(x)$ with $\mathscr{F}^u(y)$ if $x \in \mathscr{F}^{cu}(y)$, and
(4) $\mathscr{F}^c(x)$ with $\mathscr{F}^s(y)$ if $x \in \mathscr{F}^{cs}(y)$.

There is a *semiconjugacy* [Fra70], a continuous surjective map $h : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies $h(f(x)) = A(h(x))$ and $h(x+z) = h(x)+z$ for all $x \in \mathbb{R}^3$ and $z \in \mathbb{Z}^3$. Further, h is a finite distance from the identity map on $\mathbb{R}^3$.

For the center, center-stable, and center-unstable foliations, h defines a bijection on the spaces of leaves [Ham13, §3]. That is, if $x \in \mathbb{R}^3$ and $v = h(x)$, then $h(\mathscr{F}^c(x)) = \mathscr{A}^c(v)$ and $h^{-1}(\mathscr{A}^c(v)) = \mathscr{F}^c(x)$. Similar equalities hold for cs and cu in place of c. The restriction $h|_{\mathscr{F}^c(x)} : \mathscr{F}^c(x) \to \mathscr{A}^c(v)$ is a continuous surjective map, but in general it is not a homeomorphism. Throughout this section, we use the letter v to denote a point in $\mathbb{R}^3$ associated to the linear dynamics of A and we use x and y to denote points associated to the non-linear dynamics of f.

We assume all of the foliations have been given an orientation. For points x and y on a one-dimensional leaf, write $x = y$, $x < y$, or $x > y$ to denote their relative positions with respect to this orientation.

Proposition 2.2. *The semiconjugacy is monotonic along center leaves. That is, the orientations may be chosen so that if $L \in \mathscr{F}^c$ and $h(L) \in \mathscr{A}^c$ is its image, then $x \le y$ implies $h(x) \le h(y)$ for all points $x, y \in L$.*

This result is well known, but a proof does not appear to be given anywhere in the prior literature.

Proof. Suppose h is not monotonic along a center leaf L. Then there are points $x < y < z$ along L such that $h(x) = h(z) \ne h(y)$. As A^{-1} expands the linear center direction, $\|A^{-n}h(x) - A^{-n}h(y)\| = \|hf^{-n}(x) - hf^{-n}(y)\| \to \infty$ as $n \to \infty$. As h is a finite distance from the identity, it follows that $d_c(f^{-n}(x), f^{-n}(y)) \to \infty$. The same analysis shows $d_c(f^{-n}(y), f^{-n}(z)) \to \infty$ and since the points along the leaf have the ordering $f^{-n}(x) < f^{-n}(y) < f^{-n}(z)$, it follows that $d_c(f^{-n}(x), f^{-n}(z)) \to \infty$ as well. Using that center leaves are quasi-isometrically embedded, one can show that $\|A^{-n}h(x) - A^{-n}h(z)\| \to \infty$ which contradicts the fact that $h(x) = h(z)$. □

Corollary 2.3. *For each $v \in \mathbb{R}^3$, the preimage $h^{-1}(v)$ consists either of a single point or a compact segment inside a center leaf.*

Proof. As h is surjective, $h^{-1}(v)$ is non-empty. Let $L \in \mathscr{F}^c$ be such that $h(L)$ contains v. As h is a bijection on the spaces of center leaves, it follows that $h^{-1}(v) \subset L$. As L is properly embedded and h is a finite distance from the identity, $h^{-1}(v)$ is a compact subset of L. As $h|_L$ is monotonic, $h^{-1}(v)$ is connected. □

We now show h also defines a bijection between the spaces of unstable leaves.

Proposition 2.4. *For an unstable leaf $L \in \mathscr{F}^u$ of f, the image $h(L)$ is an unstable leaf of A, and $h|_L$ is a homeomorphism.*

Proof. Suppose $x, y \in L$. Then $\|f^{-n}(x) - f^{-n}(y)\| \to 0$ as $n \to \infty$. As h is uniformly continuous and a semiconjugacy, it follows that $\|A^{-n}h(x) - A^{-n}h(y)\| \to 0$ which is only possible if $h(x)$ and $h(y)$ are on the same linear unstable leaf. If $x \ne y$, then $\|f^n(x) - f^n(y)\| \to \infty$ from which one can show that $h(x) \ne h(y)$. As an injective proper map from one copy of $\mathbb{R}$ to another, $h|_L$ must be a homeomorphism. □

Proposition 2.5. *If $L \in \mathscr{F}^s$ is a stable leaf of f, then $h(L)$ is a continuous curve embedded in a center stable leaf of A. In this linear center-stable leaf, each linear center leaf intersects $h(L)$ exactly once.*

Proof. The stable leaf L lies in a center-stable leaf of f and by global product structure L intersects every center subleaf exactly once. As h is a bijection on the spaces of center and center-stable leaves, the result follows. □

We now assume that f is not accessible. Then there is a non-empty lamination $\Gamma \subset \mathbb{R}^3$ consisting of the non-open accessibility classes [RHRHU08a]. Call these the *us-leaves* of f. By global product structure, each us-leaf is bifoliated by stable and unstable leaves and intersects every center leaf of f exactly once.

Under this assumption, a stable analogue of proposition 2.4 holds.

Proposition 2.6. *For a stable leaf L^s of f, the image $h(L^s)$ is a (strong) stable leaf of A, and $h|_{L^s}$ is a homeomorphism.*

Proof. This is an adaptation of the argument given in [HU14, §6]. The key idea is to intersect the us-leaves with one fixed center-stable leaf, apply h to the resulting collection of stable leaves and show that the images satisfy the hypotheses of proposition 2.1. We now give the details.

Consider the linear center-stable leaf $\mathscr{A}^{cs}(0)$ passing through the origin in $\mathbb{R}^3$. The pre-image $h^{-1}(\mathscr{A}^{cs}(0))$ is a center-stable leaf of f. For any us-leaf $L \in \Gamma$, the intersection $L \cap h^{-1}(\mathscr{A}^{cs}(0))$ is a stable leaf and so its image $h(L) \cap \mathscr{A}^{cs}(0)$ satisfies the conclusions of proposition 2.5. Define a set of curves in $\mathscr{A}^{cs}(0)$ by

$$S = \left\{ h(L) \cap \mathscr{A}^{cs}(0) \,:\, L \in \Gamma \right\}.$$

Since h is monotonic along center leaves, the curves in S do not topologically cross. For $z \in \mathbb{Z}^3$, define a translation $\tau_z : \mathscr{A}^{cs}(0) \to \mathscr{A}^{cs}(0)$ by setting $\tau_z(v)$ to the unique intersection of $\mathscr{A}^u(v+z)$ with $\mathscr{A}^{cs}(0)$. If $\gamma = h(L) \cap \mathscr{A}^{cs}(0)$ is a curve in S, proposition 2.4 shows that $\tau_z(\gamma) = h(L+z) \cap \mathscr{A}^{cs}(0)$ is also a curve in S. As the unstable foliation of a linear Anosov map on $\mathbb{T}^3$ is minimal [Fra70], the set of translations $\{\tau_z : z \in \mathbb{Z}^3\}$ is dense in the set of all rigid translations of $\mathscr{A}^{cs}(0)$.

The collection of curves S satisfies the hypotheses of proposition 2.1 where $\mathscr{A}^{cs}(0)$ is identified with $\mathbb{R}^2$ and the linear center foliation is identified with vertical lines on $\mathbb{R}^2$. All curves in S are thus linear. Since the collection S is invariant under A, these curves must be aligned with the linear stable direction.

We have shown that for a us-leaf L of f, the image $h(L)$ is a us-leaf for the linear map A. On $\mathbb{T}^3$, such linear us-leaves are dense in $\mathbb{T}^3$ and so on the universal cover the image of the closed set Γ must be $h(\Gamma) = \mathbb{R}^3$. Consider now any stable leaf $L^s \in \mathscr{F}^s$. Since L^s does not cross through any us-leaf of f, the monotonicity of h implies that the image $h(L^s)$ does not topologically cross any leaf of the linear us-foliation. Hence, $h(L^s)$ must lie in a single linear us-leaf. It also lies in a single linear cs-leaf and so it is a linear stable leaf. That $h|_{L^s}$ is a homeomorphism is proved similarly to proposition 2.4. $\square$

We now consider the set $Y \subset \mathbb{R}^3$ consisting of all points where h is injective. That is, $y \in Y$ if and only if $h^{-1}(h(y)) = \{y\}$.

Proposition 2.7. *The set Y is a union of us-leaves.*

Proof. We first show that the complement is us-saturated. If $x \in \mathbb{R}^3 \setminus Y$, then x lies on a compact interval $J = h^{-1}(h(x))$. Using stable and unstable holonomies, we can map J to a compact interval on any other center leaf and propositions 2.4 and 2.6 show that this other interval is also mapped to a point by h.

If $U \subset \mathbb{R}^3$ is an open accessibility class, then $h(U)$ is a single linear us-leaf. Any center segment contained in U maps to a single point under h and so U and Y are disjoint. This shows that Y is a subset of Γ. $\square$

Proposition 2.8. *For any linear center leaf $\mathscr{A}^c(v)$, the set $\mathscr{A}^c(v) \setminus h(Y)$ is countable.*

Proof. Let x be such that h maps $\mathscr{F}^c(x)$ to $\mathscr{A}^c(v)$. Then any point of $\mathscr{A}^c(v) \setminus h(Y)$ is the image of an interval of positive length in $\mathscr{F}^c(x)$ and there can only be countably many disjoint intervals of this form. □

Proposition 2.9. *For a point $x \in \mathbb{R}^3$, the following are equivalent:*

(1) x is in the closure of Y;
(2) $h|_{\mathscr{F}^c(x)}$ is not locally constant at x;
(3) either x lies in Y or x is an endpoint of the interval $h(h^{-1}(x))$.

Proof. Since $h|_{\mathscr{F}^c(x)} : \mathscr{F}^c(x) \to \mathscr{A}^c(h(x))$ is continuous, surjective, and monotonic, it is straightforward to show (2) ⇔ (3). We show (1) ⇔ (2). Suppose $x \in \bar{Y} \setminus Y$. As Y is us-saturated, there are points $y_n \in \mathscr{F}^c(x) \cap Y$ converging to x. The images $h(y_n)$ are distinct from each other and so $h|_{\mathscr{F}^c(x)}$ is not locally constant at x. Conversely, if $h|^c_{\mathscr{F}}(x)$ is not locally constant at x, then for any neighbourhood $x \in J \subset \mathscr{F}^c(x)$, the image $h(J)$ has positive length and by proposition 2.8 there is $v \in h(J) \cap h(Y)$ and so $h^{-1}(v) \in J \cap Y$. □

Proposition 2.10. *For any us-leaf $L \in \Gamma$, the closure of $\bigcup_{z \in \mathbb{Z}^3}(L+z)$ contains $\bar{Y}$.*

Remark. This shows that $\bar{Y}$ when projected down to a subset of $\mathbb{T}^3$ yields a minimal us-lamination. Moreover, this is the unique minimal us-lamination.

Proof. For a point $x \in \bar{Y}$, consider a short center segment $x \in J \subset \mathscr{F}^c(x)$ such that $h(J)$ has positive length. As $\bigcup_{z \in \mathbb{Z}^3} h(L+z)$ is a dense union of linear us-leaves, there is $z \in \mathbb{Z}^3$ such that $h(L+z)$ intersects the interior of $h(J)$. Then $h^{-1}(v)$ is contained in J and so $L+z$ intersects J. □

Proposition 2.11. *For any open set U which intersects $\bar{Y}$, there is $x \in \bar{Y} \cap U$, $k \ge 1$, and $z \in \mathbb{Z}^3$ such that $f^k(x) = x + z$. That is, x projects down to a periodic point on $\mathbb{T}^3$.*

Proof. Using proposition 2.9, one can show that $h(U)$ has non-empty interior. This interior contains a point v which projects down a periodic point for the linear Anosov diffeomorphism on $\mathbb{T}^3$; that is, there are $k \ge 1$ and $z \in \mathbb{Z}^3$ such that $A^k(v) = v + z$. By the leaf conjugacy, $f^k(h^{-1}(v)) + z = h^{-1}(v)$ and as v is in the interior of $h(U)$, it follows that $h^{-1}(v)$ is contained in U. If $h^{-1}(v)$ is a singleton set, it is the desired point x. Otherwise, we may take either endpoint of the interval $h^{-1}(v)$ to be x. □

3. MINIMAL LAMINATION AND SEMICONJUGACY ON $\mathbb{T}^3$

We have now finished working on the universal cover. All of the results from now until the end of the paper will be for the original partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$. In proving theorem 1.2 we may freely lift f to a finite cover and replace it by an iterate; therefore, we assume the invariant bundles

E^c, E^u, and E^s are oriented and that f preserves these orientations. The diffeomorphism is homotopic to a linear Anosov diffeomorphism $A: \mathbb{T}^3 \to \mathbb{T}^3$ and the logarithms of the eigenvalues of A satisfy

$$\lambda^s(A) < \lambda^c(A) < 0 < \lambda^u(A).$$

As before, assume that f is non-accessible. Then there is a lamination $\Gamma \subset \mathbb{T}^3$ consisting of the non-open accessibility classes of f. By the work of Potrie, the lamination Γ contains a unique minimal sublamination [Pot14b]. This can also be seen directly from proposition 2.10 above. Let $\mathscr{F}^u$, $\mathscr{F}^s$, $\mathscr{F}^{cs}$, $\mathscr{F}^{cu}$, and $\mathscr{F}^c$ denote the invariant foliations of f considered as foliations defined on $\mathbb{T}^3$, and similarly let $\mathscr{A}^u$, $\mathscr{A}^s$, $\mathscr{A}^{cs}$, $\mathscr{A}^{cu}$, $\mathscr{A}^c$, and $\mathscr{A}^{us}$ denote the invariant linear foliations of the toral automorphism A. We fix an orientation of $\mathscr{F}^c$ and of $\mathscr{A}^c$. Then each center leaf $\mathscr{F}^c(q)$ splits into two half-leaves

$$\mathscr{F}^c(q) \setminus \{q\} = \mathscr{F}^c_+(q) \cup \mathscr{F}^c_-(q),$$

where $+$ and $-$ are determined by the orientation of $\mathscr{F}^c$ in $\mathbb{T}^3$. For every $y \in \mathscr{F}^c_+(x)$, we let $[x,y]^c$ and $(x,y)^c$ denote the closed and open segments contained in $\mathscr{F}^c(x)$ with endpoints x and y respectively.

Let $h: \mathbb{T}^3 \to \mathbb{T}^3$ be the Franks semiconjugacy, now considered as a homeomorphism on the 3-torus. That is, h is isotopic to the identity and $h(f(x)) = A(h(x))$ for all $x \in \mathbb{T}^3$. Corollary 2.3 implies an analogous result for the semiconjugacy on the 3-torus: for each $v \in \mathbb{T}^3$, the preimage $h^{-1}(v)$ consists either of a single point or a compact segment inside a center leaf.

If $x \in \mathbb{T}^3$ is such that $h^{-1}(h(x)) = \{x\}$, we define $x_+ = x_- = x$. If instead $h^{-1}(h(x))$ is a positive length interval, we define x_+ and x_- to be the two endpoints of $h^{-1}(h(x))$ with corresponding orientation. That is, $x_+ \in \mathscr{F}^c_+(x_-)$ and $x_- \in \mathscr{F}^c_-(x_+)$. Define

$$\Lambda = \bigcup_{x \in \mathbb{T}^3} \{x_+, x_-\}.$$

Proposition 3.1. *The set Λ and semiconjugacy h satisfy the following properties:*

(1) The set Λ is a us-saturated minimal set, i.e. if $x \in \Lambda$, then

$$\mathscr{F}^{us}(x) \subset \Lambda \qquad \text{and} \qquad \overline{\mathscr{F}^{us}(x)} = \Lambda.$$

(2) The periodic points of $f|_\Lambda$ are dense in Λ.

(3) For every $x \in \Lambda$, if $x = x_+$, then $y = y_+$ for every $y \in \mathscr{F}^{us}(x) \subset \Lambda$; if $x = x_-$, then $y = y_-$ for every $y \in \mathscr{F}^{us}(x) \subset \Lambda$.

(4) For $\sigma = u, s, us$, the semiconjugacy $h|_\Lambda$ maps a leaf of the σ-foliation of f to a leaf of the σ-foliation of A:

$$h(\mathscr{F}^\sigma(x)) = \mathscr{A}^\sigma(h(x)), \qquad \forall x \in \Lambda.$$

Moreover, $h: \mathscr{F}^\sigma(x) \to \mathscr{A}^\sigma(h(x))$ is a homeomorphism.

Proof. These are all consequences of the results in section 2. Item (1) follows from propositions 2.9 and 2.10. Item (2) follows from proposition 2.11. Item (3) follows from proposition 2.7. Item (4) follows from propositions 2.4 and 2.6 and global product structure. □

Proposition 3.2. *There exist constants $C_1 > 1$ and $0 < \alpha < 1$, such that for every $\sigma = u, s, us$, the homeomorphism $h|_{\mathscr{F}^\sigma(x)} : \mathscr{F}^\sigma(x) \to \mathscr{A}^\sigma(h(x))$ is bi-Hölder continuous, i.e. for every $x_1, x_2 \in \mathscr{F}^\sigma(x)$, we have*

$$d_{\mathscr{F}^\sigma}(x_1, x_2) \le C_1 \cdot d_{\mathscr{A}^\sigma}(h(x_1), h(x_2))^\alpha, \quad d_{\mathscr{A}^\sigma}(h(x_1), h(x_2)) \le C_1 \cdot d_{\mathscr{F}^\sigma}(x_1, x_2)^\alpha.$$

Proof. We first prove the case $\sigma = u$ and $d_{\mathscr{F}^u}(x_1, x_2) \le C \cdot d_{\mathscr{A}^u}(h(x_1), h(x_2))^\alpha$. There exists $\delta_0 > 0$, such that for every $x_1, x_2 \in \mathscr{F}^u(x) \subset \Lambda$, if $d_{\mathscr{A}^u}(h(x_1), h(x_2)) < \delta_0$, then $d_{\mathscr{F}^u}(x_1, x_2) < 1$. Otherwise, there exist $x_1^n, x_2^n \in \mathscr{F}^u(x^n) \subset \Lambda$ such that $d_{\mathscr{F}^u}(x_1^n, x_2^n) = 1$ and $d_{\mathscr{A}^u}(h(x_1^n), h(x_2^n)) \to 0$. Taking a subsequence if necessary, we have $x_1^n \to y_1$ and $x_2^n \to y_2$ with $y_2 \in \mathscr{F}^u(y_1)$, $d_{\mathscr{F}^u}(y_1, y_2) = 1$, and $h(y_1) = h(y_2)$. This contradicts the fact that h is a homeomorphism on $\mathscr{F}^u(y_1)$.

Now we assume that $d_{\mathscr{A}^u}(h(x_1), h(x_2)) \ll \delta_0$. Let k be the largest positive number such that

$$d_{\mathscr{A}^u}(A^k \circ h(x_1), A^k \circ h(x_2)) < \delta_0.$$

Then we have

$$d_{\mathscr{A}^u}(h(x_1), h(x_2)) \ge \exp\left(-(k+1) \cdot \lambda^u(A)\right) \cdot \delta_0.$$

On the other hand, from the semiconjugacy and $d_{\mathscr{A}^u}(A^k \circ h(x_1), A^k \circ h(x_2)) < \delta_0$, we have $d_{\mathscr{F}^u}(f^k(x_1), f^k(x_2)) < 1$. This implies

$$d_{\mathscr{F}^u}(x_1, x_2) < \mu^{-k},$$

where $\mu = \inf_{z \in \mathbb{T}^3} m(Df|_{E^u(z)}) > 1$.

If $\mu \ge \exp \lambda^u(A)$, then we have

$$d_{\mathscr{F}^u}(x_1, x_2) < \frac{\exp \lambda^u(A)}{\delta_0} \cdot d_{\mathscr{A}^u}(h(x_1), h(x_2)).$$

Otherwise, we take $0 < \alpha < 1$ such that $\exp(\alpha \lambda^u(A)) < \mu$. Then we have

$$d_{\mathscr{F}^u}(x_1, x_2) < \mu^{-k} < \exp\left(-k\alpha\lambda^u(A)\right) \le \frac{\exp(-\alpha\lambda^u(A))}{\delta_0^\alpha} \cdot d_{\mathscr{A}^u}(h(x_1), h(x_2))^\alpha.$$

The proof of the other inequality and the case $\sigma = s$ are the same. If $h|_\Lambda$ is bi-Hölder continuous on every leaf of $\mathscr{F}^s$ and $\mathscr{F}^u$, then it is bi-Hölder continuous on every leaf of $\mathscr{F}^{us}$. □

Proposition 3.3. *Let $x, y \in \Lambda$ and a sequence of points $x_n \in \mathscr{F}^{us}(x) \cap \mathscr{F}^c_+(y)$ such that*

$$x_{n+1} \in (y, x_n)^c \qquad \textit{and} \qquad \lim_{n\to\infty} d_{\mathscr{F}^c}(x_n, y) = 0.$$

For every $z \in \mathscr{F}^c(x) \cap \Lambda$ satisfying $h(z) \ne h(x)$, there exists $\delta_z > 0$, such that if we denote $h^{us}_{x,x_n} : \mathscr{F}^c(x) \to \mathscr{F}^c(x_n)$ the holonomy map induced by $\mathscr{F}^{us}$ in Λ from x to x_n, and $z_n = h^{us}_{x,x_n}(z)$, then

$$d_{\mathscr{F}^c}(x_n, z_n) \ge \delta_z.$$

The same conclusion holds for the sequence of points $x_n \in \mathscr{F}^{us}(x) \cap \mathscr{F}^c_-(y)$.

Proof. We denote $\delta_1 = d_{\mathscr{A}^c}(h(x), h(x)) > 0$ and assume $x = x_+$. If $z \in \mathscr{F}^c_+(x)$, then $h(z) \in \mathscr{A}^c_+(h(x))$. Since $\lim_{n\to\infty} d_{\mathscr{F}^c}(x_n, y) = 0$, we have $y = y_+$. There exists $w_1 \in \mathscr{F}^c_+(y)$, such that $d_{\mathscr{A}^c}(h(y), h(w_1)) = \delta_1$. Moreover, there exists $N_1 > 0$, such that $d_{\mathscr{A}^c}(h(x_{N_1}), h(y)) \le \delta_1/2$. This implies $w_1 \in (x_n, z_n)$ and $[x_{N_1}, w_1]^c \subset [x_n, z_n]^c$ for every $n \ge N_1$. So we define

$$\delta_z = \min\left\{d_{\mathscr{F}^c}(x_1, z_1), \cdots\cdots, d_{\mathscr{F}^c}(x_{N_1-1}, z_{N_1-1}), d_{\mathscr{F}^c}(x_{N_1}, w_1)\right\}.$$

If $z \in \mathscr{F}^c_-(x)$, then $h(z) \in \mathscr{A}^c_-(h(x))$. Since $\lim_{n\to\infty} d_{\mathscr{F}^c}(x_n, y) = 0$, there exists $N_2 > 0$, such that $d_{\mathscr{A}^c}(h(x_{N_2}), h(y)) \le \delta_1/2$. This implies $y \in (z_n, x_n)^c$ and $[z_{N_2}, y]^c \subset [z_n, x_n]^c$ for every $n \ge N_2$. So we define

$$\delta_z = \min\left\{d_{\mathscr{F}^c}(x_1, z_1), \cdots\cdots, d_{\mathscr{F}^c}(x_{N_2-1}, z_{N_2-1}), d_{\mathscr{F}^c}(z_{N_2}, y)\right\}.$$

This proves the case $x = x_+$. The proof for $x = x_-$ is the same. □

Define the real number $\lambda^- = \inf\{\lambda^c(p) : p \in \mathrm{Per}(f|_\Lambda)\}$.

Lemma 3.4. *If $p \in \mathrm{Per}(f|_\Lambda)$, then $\lambda^c(p) \le 0$. This implies $\lambda^- \le 0$.*

Proof. Let $p \in \mathrm{Per}(f|_\Lambda)$ with period π. Since $\Lambda = \bar{Z} = \bigcap_{x\in\mathbb{T}^3} \partial^c h^{-1}(h(x))$ and us-saturated, there exists a sequence of points $x_n \in Z$, such that $x_n \in \mathscr{F}^c(p)$ and x_n converge to p in $\mathscr{F}^c(p)$. Moreover, for every n, there exists $k_n > 0$, such that $A^{k_n}(h(x_1))$ is between $h(x_n)$ and $h(p)$ in $\mathscr{A}^c(h(p))$. From the semiconjugacy and $x_n \to p$ in $\mathscr{F}^c(p)$, we have

$$\lim_{k\to+\infty} d(f^{k\pi}(x_1), p) = 0.$$

This implies $\lambda^c(p) \le 0$ for every $p \in \mathrm{Per}(f|_\Lambda)$. □

Proposition 3.5. *If μ is an ergodic measure supported on Λ and $\lambda^c(\mu) < 0$, then there exists a sequence of periodic points $q_n \in \Lambda$, such that*

$$\lim_{n\to\infty} \lambda^c(q_n) = \lambda^c(\mu).$$

In particular, if μ is an ergodic measure supported on Λ, then its central Lyapunov exponent satisfies $\lambda^c(\mu) \ge \lambda^-$.

Proof. Assume μ is an ergodic measure supported on Λ with $\lambda^c(\mu) < 0$. Take a μ-typical point x. This point is recurrent and has a Pesin stable manifold. The shadowing lemma of Pesin theory leads to a periodic point q_n of period $\pi(q_n)$ with center Lyapunov exponent satisfying

$$|\lambda^c(q) - \lambda^c(\mu)| < \min\left\{\frac{1}{n}, \frac{|\lambda^c(\mu)|}{2}\right\}.$$

Moreover, the Pesin stable manifold of q_n transversely intersects the unstable manifolds $\mathscr{F}^u(x)$ of x. If we denote y_n this intersecting point, then we have

$$y \in \mathscr{F}^u(x) \subset \mathscr{F}^{us}(x) \subset \Lambda.$$

Since Λ is a compact invariant set, and $f^{k\pi(q_n)}(y) \to q_n$ as $k \to +\infty$, we have $q_n \in \Lambda$. □

Proposition 3.6. *The inequality $\lambda^- \le \lambda^c(A) < 0$ holds.*

Proof. We only need to show that there exists an ergodic measure μ supported on Λ, such that $\lambda^c(\mu) < 0$. Actually, we consider the measure μ_0 of maximal entropy of f, then [Ure12] shows that its support $supp(\mu_0) \subset \Lambda$.

Again, [Ure12] shows that $\lambda^c(\mu_0) \le \lambda^c(A) < 0$. This proves that $\lambda^- \le \lambda^c(A) < 0$. □

4. Rigidity of center Lyapunov exponents

In this section, we prove Proposition 4.3, which states that all periodic points in Λ have the same center Lyapunov exponent. This implies that Λ is hyperbolic. We then use this hyperbolicity to show that $\Lambda = \mathbb{T}^3$.

Lemma 4.1. *For every $\epsilon > 0$, up to changing the metric, there is a point $p \in \mathrm{Per}(f|_\Lambda)$ such that*

$$\log \|Df|_{E^c(x)}\| > \lambda^c(p) - \epsilon$$

holds for all $x \in \Lambda$.

Proof. We have proved that $\lambda^- \le \lambda^c(\mu)$ for every ergodic measure μ supported on Λ. From the definition of λ^-, there exists a sequence of periodic points $p_n \in \mathrm{Per}(f|_\Lambda)$ such that $\lim_{n\to\infty} \lambda^c(p_n) = \lambda^- < 0$.

For every $\epsilon > 0$, up to changing the metric, we have

$$\log \|Df|_{E^c(x)}\| > \lambda^- - \frac{\epsilon}{2}$$

for all $x \in \Lambda$. Then we take $p = p_n$ for n large enough which proves the lemma. □

Lemma 4.2. *There exist two constants $C_2 > 0$ and $0 < \beta < 1$, such that for every two periodic points $p, q \in \Lambda$, there exist two sequence of points $x_n \in \mathscr{F}^s(p)$, $y_n \in \mathscr{F}^u(x_n)$ with $y_n \in \mathscr{F}^c(q)$, such that*

$$\lim_{n\to\infty} d_{\mathscr{F}^c}(y_n, q) = 0, \quad \text{and} \quad d_{\mathscr{F}^u}(x_n, y_n) \le \frac{C_2}{D_n^\beta}, \quad \text{where} \quad D_n = d_{\mathscr{F}^s}(p, x_n).$$

Moreover, for every $\eta > 0$, there exists $N_\eta > 0$, such that for every $n > N_\eta$,

$$d_{\mathscr{F}^c}(f^k(y_n), f^k(q)) \le \eta, \qquad \forall k \ge 0.$$

Proof. Since $\Lambda = \bigcup_{x\in\mathbb{T}^3} \partial^c h^{-1}(h(x))$ is a us-minimal set, at least one branch $\mathscr{F}^c_+(q)$ or $\mathscr{F}^c_-(q)$ contains a sequence of points $z_m \in \Lambda$, such that z_m converges to q in $\mathscr{F}^c(q)$. We assume $z_m \in \mathscr{F}^c_+(q) \cap \Lambda$ for every m. Then we have $h(z_m) \ne h(q)$ and $h(z_m)$ converges to $h(q)$ in $\mathscr{A}^c_+(h(q))$.

The points $h(p)$ and $h(q)$ are periodic points A. Since $\mathscr{A}^s$ is an linear foliation with algebraic irrational rotation vector on $\mathbb{T}^3$, there exist $C_2' > 0$ and two sequences of points $x_n' \in \mathscr{A}^s(h(p))$, $y_n' \in \mathscr{A}^u(x_n')$ with $y_n' \in \mathscr{A}^c_+(h(q))$, such that

$$d_{\mathscr{A}^c}(h(q), y_n') \le \frac{C_2'}{\sqrt{D_n'}} \qquad \text{and} \qquad d_{\mathscr{A}^u}(y_n', x_n') \le \frac{C_2'}{\sqrt{D_n'}},$$

where $D'_n = d_{\mathscr{A}^s}(h(p), x'_n) \to +\infty$ as $n \to +\infty$.

If we have $p = h^{-1}(h(p))$, then for every n, $h^{-1}(x'_n)$ and $h^{-1}(y'_n)$ are single points, we denote

$$x_n = h^{-1}(x'_n) \in \mathscr{F}^s(p), \qquad \text{and} \qquad y_n = h^{-1}(y'_n) \in \mathscr{F}^u(x_n).$$

Otherwise, the set $\partial^c h^{-1}(h(p)) = \Lambda \cap h^{-1}(h(p))$ consists of two periodic points, and p is one of them. This implies $h^{-1}(h(\mathscr{F}^{us}(p))) \cap \Lambda$ contains two us-leaves, and one of them is $\mathscr{F}^{us}(p)$. So there exist unique a pair of points

$$x_n = h^{-1}(x'_n) \cap \mathscr{F}^s(p), \qquad \text{and} \qquad y_n = h^{-1}(y'_n) \cap \mathscr{F}^u(x_n).$$

In both cases, we have $y_n \in \mathscr{F}^c_+(q)$. Moreover, for every $h(z_m) \in \mathscr{A}^c_+(h(q))$, there exists $N_m > 0$, such that y'_n is contained in the open interval with endpoints $h(q)$ and $h(z_m)$ in $\mathscr{A}^c_+(h(q))$ for every $n \geq N_m$. This implies y_n is between q and z_m in $\mathscr{F}^c_+(q)$. Thus $\lim_{n\to\infty} d_{\mathscr{F}^c}(z_m, q) = 0$ implies $\lim_{n\to\infty} d_{\mathscr{F}^c}(y_n, q) = 0$.

Finally, let $H : \mathbb{R}^3 \to \mathbb{R}^3$ be the lifting map of the semiconjugacy $h : \mathbb{T}^3 \to \mathbb{T}^3$. Then there exists a constant $K > 0$, such that $d(H, \mathrm{Id}) < K$. For $D'_n = d_{\mathscr{A}^s}(h(p), x'_n)$, we have

$$D_n = d_{\mathscr{F}^s}(p, x_n) > d_{\mathscr{A}^s}(h(p), x'_n) - 2K = D'_n - 2K.$$

So for n large enough, we have $D_n > D'_n/2$. On the other hand, from $d_{\mathscr{A}^u}(y'_n, x'_n) \leq C'_2/\sqrt{D'_n}$ and Propisition 3.2, we have

$$d_{\mathscr{F}^u}(x_n, y_n) \leq C_1 \cdot d_{\mathscr{A}^u}(x_n, y_n)^\alpha \leq C_1 \cdot (2C'_2)^\alpha \cdot D_n^{-\frac{\alpha}{2}}.$$

Thus we set $C_2 = C_1 \cdot (2C'_2)^\alpha$ and $\beta = \alpha/2$.

Finally, since $y_n \in \mathscr{F}^{us}(p) \subset \Lambda$ converges to q in $\mathscr{F}^c_+(q)$, this implies for every n,

$$\lim_{k\to\infty} d_{\mathscr{F}^c}(f^k(y_n), f^k(q)) = 0.$$

So for $n = 1$, there exists $K_0 > 0$, such that $d_{\mathscr{F}^c}(f^k(y_n), f^k(q)) < \eta$ for every $k \geq K_0$. Let $\pi(q)$ be the period of q, and set $N_\eta > 0$, where y_{N_η} is contained in $(q, f^{K_0\pi(q)}(y_1))$. Then for every $n > N_\eta$,

$$d_{\mathscr{F}^c}(f^k(y_n), f^k(q)) \leq \eta, \qquad \forall k \geq 0.$$

$\square$

Remark. Since $\lim_{n\to\infty} d_{\mathscr{F}^c}(y_n, q) = 0$, we can assume $y_{n+1} \in (q, y_n)^c$ by taking subsequence. This allows us to apply Proposition 3.3.

Now we fix the constant

$$\theta = -\frac{1}{2} \cdot \frac{\beta \cdot \log\left(\sup_{x\in\mathbb{T}^3} \|Df|_{E^s(x)}\|\right)}{\log\left(\inf_{x\in\mathbb{T}^3} m(Df|_{E^u(x)})\right)} \in (0,1).$$

We can state the main result of this section, which implies that Λ is a hyperbolic set and equal to $\mathbb{T}^3$. The proof of this proposition is similar to Proposition 4.1 of [GS19].

Proposition 4.3. *All periodic points in Λ have the same center Lyapunov exponent and satisfies*

$$\lambda^c(p) \le \lambda^c(A) < 0, \qquad \forall p \in \mathrm{Per}(f|_\Lambda).$$

Proof. Lemma 3.4 proved that $\lambda^c(p) \le 0$ for every $p \in \mathrm{Per}(f|_\Lambda)$. Proposition 3.5 and 3.6 proved that there exists a sequence of periodic points $q_n \in \mathrm{Per}(f|_\Lambda)$, such that $\lim_{n\to\infty} \lambda^c(q_n) = \lambda^- \le \lambda^c(A) < 0$.

Assume there exists two periodic points $r, q \in \mathrm{Per}(f|_\Lambda)$ such that $0 < \lambda^c(r) < \lambda^c(q) \le 0$. Denote

$$\delta_0 = \frac{\theta}{4}\left(\lambda^c(q) - \lambda^c(r)\right) > 0.$$

From Lemma 4.1, there exists a periodic point $p \in \mathrm{Per}(f|_\Lambda)$, such that

$$\lambda^c(p) \le \lambda^c(r), \qquad \text{and} \qquad \lambda^c(p) < \log\|Df|_{E^c(x)}\| + \delta_0, \quad \forall x \in \Lambda.$$

Denote n_0 be the minimal common period of p and q.

Let $\eta_0 > 0$, such that for every $z_1, z_2 \in \mathbb{T}^3$ satisfying $d(z_1, z_2) \le 3\eta_0$, we have

$$\left|\log\|Df|_{E^c(z_1)}\| - \log\|Df|_{E^c(z_1)}\|\right| < \delta_0.$$

Since $p \in \mathrm{Per}(f|_\Lambda)$ and $\lambda^c(p) < 0$, there exists $z \in \mathscr{F}^c(p) \cap \Lambda$, such that

$$h(z) \ne h(p), \quad \text{and} \quad d_{\mathscr{F}^c}(f^k(p), f^k(z)) \le \eta_0, \qquad \forall k \ge 0.$$

We assume $x \in \mathscr{F}^c_+(p)$.

We apply Lemma 4.2 to p and q, there exists $x_n \in \mathscr{F}^s(p)$, $y_n \in \mathscr{F}^u(x_n)$ with $y_n \in \mathscr{F}^c(q)$ such that

$$\lim_{n\to\infty} d_{\mathscr{F}^c}(y_n, q) = 0, \quad \textit{and} \quad d_{\mathscr{F}^u}(x_n, y_n) \le \frac{C_2}{D_n^\beta}, \quad \textit{where} \quad D_n = d_{\mathscr{F}^s}(p, x_n).$$

By taking subsequence, we can assume $y_{n+1} \in (q, y_n)^c$. Moreover, there exists $N_0 > 0$, such that for every $n > N_0$,

$$d_{\mathscr{F}^c}(f^k(y_n), f^k q) \le \eta_0.$$

Denote $h^{us}_{p,y_n} : \mathscr{F}^c(p) \to \mathscr{F}^c(y_n)$ the holonomy map induced by $\mathscr{F}^{us}$ in Λ from p to y_n, and $z_n = h^{us}_{p,y_n}(z)$. Proposition 3.3 shows that there exists $\delta_z > 0$, such that

$$d_{\mathscr{F}^c}(y_n, z_n) \ge \delta_z.$$

Claim 4.4. *If we denote $h^s_{p,x_n} : \mathscr{F}^c(p) \to \mathscr{F}^c(x_n)$ the holonomy map induced by $\mathscr{F}^s$ in $\mathscr{F}^{cs}(p)$ from p to x_n, and $w_n = h^s_{p,x_n}(z)$, then there exists $N_1 > 0$, such that for every $n > N_1$, it satisfies*

$$d_{\mathscr{F}^c}(x_n, w_n) \ge \delta_z/2.$$

Proof of the claim. Consider the holonomy map $h^u_{y_n,x_n} : \mathscr{F}^c(y_n) \to \mathscr{F}^c(x_n)$ induced by $\mathscr{F}^u$ in $\mathscr{F}^{cu}(q)$ for y_n to x_n. We denote $w_n = h^u_{y_n,x_n}(x_n)$. From Theorem B of [PSW97], $h^u_{y_n,x_n}$ is C^{1+} continuous. Moreover, since $d_{\mathscr{F}^u}(x_n, y_n) \to 0$ as $n \to \infty$, the derivative of $h^u_{y_n,x_n}$ converges to 1 as $n \to \infty$. So there exists $N_1 > 0$,

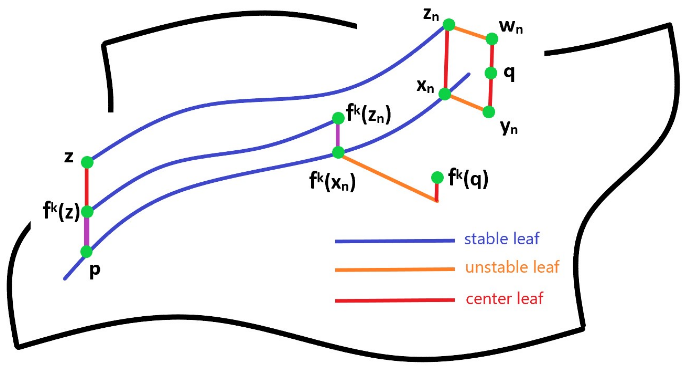

FIGURE 1. Holonomy maps of stable foliations

such that for every $n > N_1$, it satisfies

$$d_{\mathcal{F}^c}(x_n, w_n) \geq \delta_z/2.$$

□

For every n, we denote

- m_n be the smallest positive integer where $d_{\mathcal{F}^s}(p, f^{k_n n_0}(x_n)) \leq 1$.
- k_n be the largest positive integer where $d_{\mathcal{F}^u}(f^{k_n n_0}(x_n), f^{k_n n_0}(y_n)) \leq \eta_0$.

Claim 4.5. *There exists $N_2 > 0$, such that for every $n > N_2$, we have*

$$\frac{k_n}{m_n} > \theta.$$

Proof of the claim. From the definition of m_n and k_n, they satisfy

- m_n satisfies $D_n \cdot \left(\sup_{x\in\mathbb{T}^3} \|Df|_{E^s(x)}\|\right)^{(m_n-1)n_0} > 1$, this implies
$$m_n < -\frac{\log D_n}{n_0 \cdot \log\left[\sup_{x\in\mathbb{T}^3} \|Df|_{E^s(x)}\|\right]} + 1.$$
- k_n satisfies $(C_2/D_n^\beta) \cdot \left(\inf_{x\in\mathbb{T}^3} m(Df|_{E^u(x)})\right)^{k_n n_0} < \eta_0$, this implies
$$k_n > \frac{\beta \cdot \log D_n + \log \eta_0 - \log C_2}{n_0 \cdot \log\left[\inf_{x\in\mathbb{T}^3} m(Df|_{E^u(x)})\right]}.$$

Since $D_n \to +\infty$ as $n \to \infty$, we also have $m_n \to \infty$ as $n \to \infty$. Thus for

$$\theta = -\frac{1}{2} \cdot \frac{\beta \cdot \log\left(\sup_{x\in\mathbb{T}^3} \|Df|_{E^s(x)}\|\right)}{\log\left(\inf_{x\in\mathbb{T}^3} m(Df|_{E^u(x)})\right)},$$

there exists $N_2 > 0$, such that if $n > N$, then $k_n/m_n > \theta$. □

Claim 4.6. *For every $n > \max\{N_0, N_1, N_2\}$, we have*

$$d_{\mathcal{F}^c}(f^{m_n n_0}(z), p) \leq \exp\left[m_n n_0 \cdot (\lambda^c(p) + \delta_0)\right],$$

and

$$d_{\mathcal{F}^c}(f^{k_n n_0}(x_n), f^{k_n n_0}(w_n)) > \exp\left[m_n n_0 \cdot \left(\lambda^c(p) + 2\delta_0\right)\right] \cdot \min\left\{\eta_0, \delta_z/2\right\}.$$

Proof of the claim. Now we let $n > \max\{N_0, N_1, N_2\}$. The length of $(m_n n_0)$-iteration of the segment $[p,z]^c$ satisfies

$$d_{\mathcal{F}^c}(f^{m_n n_0}(z), p) \leq \exp\left[m_n n_0 \cdot (\lambda^c(p) + \delta_0)\right].$$

On the other hand, since $d(x_n, q) \leq 2\eta_0$, for every $1 \leq k \leq k_n n_0$, either

$$d_{\mathcal{F}^c}(f^{k-1}(x_n), f^{k-1}(w_n)) > \eta_0;$$

or

$$d_{\mathcal{F}^c}(f^k(x_n), f^k(w_n)) > \|Df|_{E^c(f^{k-1}(q))}\| \cdot \exp(-\delta_0) \cdot d_{\mathcal{F}^c}(f^{k-1}(x_n), f^{k-1}(w_n)).$$

Recall that $d_{\mathcal{F}^c}(x_n, w_n) > \delta_z/2$, this implies that

$$d_{\mathcal{F}^c}(f^{k_n n_0}(x_n), f^{k_n n_0}(w_n)) > \exp\left[k_n n_0 \cdot (\lambda^c(q) - \delta_0)\right] \cdot \min\left\{\eta_0, \delta_z/2\right\}.$$

Moreover, for every $k_n n_0 < k \leq m_n n_0$, either

$$d_{\mathcal{F}^c}(f^{k-1}(x_n), f^{k-1}(w_n)) > \eta_0;$$

or

$$\begin{aligned} d_{\mathcal{F}^c}(f^k(x_n), f^k(w_n)) &> \|Df|_{E^c(f^{k-1}(x_n))}\| \cdot \exp(-\delta_0) \cdot d_{\mathcal{F}^c}(f^{k-1}(x_n), f^{k-1}(w_n)) \\ &> \exp\left(\lambda^c(p) - 2\delta_0\right) \cdot d_{\mathcal{F}^c}(f^{k-1}(x_n), f^{k-1}(w_n)). \end{aligned}$$

This implies

$$\begin{aligned} &d_{\mathcal{F}^c}(f^{k_n n_0}(x_n), f^{k_n n_0}(w_n)) \\ &> \exp\left[k_n n_0 \cdot (\lambda^c(q) - \delta_0) + (m_n - k_n) n_0 \cdot (\lambda^c(p) - 2\delta_0)\right] \cdot \min\left\{\eta_0, \delta_z/2\right\} \\ &> \exp\left[m_n n_0 \cdot \left(\theta\lambda^c(q) + (1-\theta)\lambda^c(p) - 2\delta_0\right)\right] \cdot \min\left\{\eta_0, \delta_z/2\right\} \end{aligned}$$

Since $\delta_0 = \frac{\theta}{4}\left(\lambda^c(q) - \lambda^c(r)\right)$, we have

$$d_{\mathcal{F}^c}(f^{k_n n_0}(x_n), f^{k_n n_0}(w_n)) > \exp\left[m_n n_0 \cdot \left(\lambda^c(p) + 2\delta_0\right)\right] \cdot \min\left\{\eta_0, \delta_z/2\right\}.$$

This proves the claim. □

Since $m_n \to +\infty$ as $n \to \infty$, this claim implies

$$\lim_{n\to\infty} \frac{d_{\mathcal{F}^c}(f^{k_n n_0}(x_n), f^{k_n n_0}(w_n))}{d_{\mathcal{F}^c}(f^{m_n n_0}(z), p)} \longrightarrow +\infty$$

as $n \to \infty$.

However, $d_{\mathcal{F}^s}(p, f^{k_n n_0}(x_n)) \leq 1$, this contradicts that the holonomy maps stable foliation in a center-stable leaf is C^{1+}, see Theorem B of [PSW97]. This proves

$$\lambda^c(p) = \lambda^c(q), \qquad \forall\, p, q \in \mathrm{Per}(f|_\Lambda).$$

□

Corollary 4.7. Λ *is a uniformly hyperbolic attractor.*

Proof. For every $p \in \mathrm{Per}(f|_\Lambda)$, we have $p = p_+$ or $p = p_-$. If $p = p_+$, then $\mathscr{F}^c_+(p)$ is contained in the stable manifold of p. If $p = p_-$, then $\mathscr{F}^c_+(p)$ is contained in the stable manifold of p. From the global product structure of $\mathscr{F}^{cs}$ and $\mathscr{F}^u$, this implies every pair of periodic points $p, q \in \Lambda$ are homoclinic related. Moreover, every periodic point $r \notin \Lambda$ satisfies $r \neq r_+$ and $r \neq r_-$. This implies its stable manifold

$$W^s(r) \subset \bigcup_{x \in (r_-, r_+)^c} \mathscr{F}^s(x).$$

This implies

$$W^s(r) \cap \mathscr{F}^{us}(r_-) = \emptyset, \qquad \text{and} \qquad W^s(r) \cap \mathscr{F}^{us}(r_+) = \emptyset.$$

Thus r is not homoclinic related to r_+ and r_-. For every $p \in \mathrm{Per}(f|_\Lambda)$, its homoclinic class $H(p) \subseteq \Lambda$. Proposition 3.1 shows that $\Lambda = \overline{\mathrm{Per}(f|_\Lambda)}$, this implies

$$\Lambda = H(p), \qquad \forall p \in \mathrm{Per}(f|_\Lambda).$$

Proposition 4.3 shows that $\lambda^c(p) \le \lambda^c(A) < 0$ for every $p \in \mathrm{Per}(f|_\Lambda)$. We apply the Main Theorem of [BGY09], which shows that Λ is a hyperbolic set. Moreover, the unstable manifold

$$W^u(x) = \mathscr{F}^u(x) \subset \Lambda, \qquad \forall x \in \Lambda.$$

This implies Λ is a hyperbolic attractor. □

The following proposition finishes the proof of Theorem 1.2.

Proposition 4.8. *The hyperbolic attractor Λ is the whole of $\mathbb{T}^3$ and so f is an Anosov diffeomorphism.*

Proof. We have $\Lambda = \mathbb{T}^3$ if $x = h^{-1}(h(x))$ for every $x \in \mathbb{T}^3$. Assume there exists $x \in \mathbb{T}^3$ such that $x \neq h^{-1}(h(x))$. Then for every $y \in \mathscr{A}^{us}(h(x))$, $h^{-1}(y)$ is a non-trivial center segment.

Claim 4.9. *There exists $z' \in \mathscr{A}^{us}(h(x))$, such that the set*

$$B_1^{us}(z') = \left\{ y' \in \mathscr{A}^{us}(z') : d_{\mathscr{A}^{us}}(y', z') \le 1 \right\}$$

satisfies

$$A^{-k}\left(B_1^{us}(z')\right) \cap B_1^{us}(z') = \emptyset, \qquad \forall k > 0.$$

Proof of the claim. There are two possibilities, either $\mathscr{A}^{us}(h(x))$ contains no periodic points of A, or $\mathscr{A}^{us}(h(x))$ contains a periodic point.

If $\mathscr{A}^{us}(h(x))$ contains no periodic points of A, then

$$A^{-k}(\mathscr{A}^{us}(h(x))) \cap \mathscr{A}^{us}(h(x)) = \emptyset, \qquad \forall k > 0$$

Since $B_1^{us}(h(x)) \subset \mathscr{A}^{us}(h(x))$, we only need to choose $z' = h(x)$.

If $\mathscr{A}^{us}(h(x))$ contains a periodic point p with period π, then $A^\pi : \mathscr{A}^{us}(h(x)) \to \mathscr{A}^{us}(h(x))$ is a linear Anosov action on the plane. So there exists $z' \in \mathscr{A}^s(p) \setminus \{p\} \subset \mathscr{A}^{us}(x)$ sufficiently far from p in $\mathscr{A}^s(p)$, which satisfies

$$A^{-k\pi}\left(B_1^{us}(z')\right) \cap B_1^{us}(z') = \emptyset, \qquad \forall k > 0.$$

This implies $A^{-k}\left(B_1^{us}(z')\right) \cap B_1^{us}(z') = \emptyset$ for every $k > 0$ $\square$

Let z be the center point of the non-trivial segment $h^{-1}(z')$, and $\delta > 0$ satisfying

$$B_{3\delta}(z) \cap \Lambda = \emptyset.$$

The semiconjugacy $h : \mathbb{T}^3 \to \mathbb{T}^3$ is a continuous map. There exists $\epsilon_0 > 0$, such that if $d(x_1, x_2) < \epsilon_0$ and $h(x_2) \in \mathscr{A}^{us}(h(x_1))$, then $d_{us}(h(x_1), h(x_2)) < 1$. Here $d_{us}(\cdot,\cdot)$ is the distance in each leaf of $\mathscr{A}^{us}$.

Since Λ is a hyperbolic attractor, there exists a constant $0 < \epsilon < \min\{\delta, \epsilon_0\}$, such that the ϵ-neighborhood $B_\epsilon(\Lambda)$ satisfies

$$f^k(B_\epsilon(\Lambda)) \subset B_\delta(\Lambda), \qquad \forall k > 0.$$

This implies

$$f^{-k}(B_\delta(z)) \cap B_\epsilon(\Lambda) = \emptyset, \qquad \forall k > 0.$$

Otherwise, we have $B_\delta(z) \cap B_\delta(\Lambda) \neq \emptyset$, which contradicts to $B_{3\delta}(z) \cap \Lambda = \emptyset$. Thus for every $k > 0$, we have

$$B_\epsilon(f^{-k}(z)) \cap \Lambda = \emptyset, \qquad \text{and} \qquad h\Big(B_\epsilon(f^{-k}(z))\Big) \subset \mathscr{A}^{us}(A^{-k}(z')).$$

Moveover, since $0 < \epsilon < \epsilon_0$, for every $y \in B_\epsilon(f^{-k}(z))$, we have

$$d_{us}\Big(h(y), A^{-k}(z')\Big) < 1.$$

This implies

$$h\Big(B_\epsilon(f^{-k}(z))\Big) \subset B_1^{us}(A^{-k}(z')), \qquad \forall k > 0.$$

Since $A^{-k}\left(B_1^{us}(z')\right) \cap B_1^{us}(z') = \emptyset$ for every $k > 0$, the semiconjugacy property implies the sequence of balls

$$\Big\{B_\epsilon(f^{-k}(z)) : k > 0\Big\}$$

are mutually disjoint. This is absurd since the volume of each $B_\epsilon(f^{-k}(z))$ has a lower bound. Thus h is injective everywhere and $\Lambda = \mathbb{T}^3$. $\square$

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